Commutative Algebra and Noncommutative Algebraic Geometry, II
MSRI Publications
Volume **68**, 2015

Modules for elementary abelian groups and hypersurface singularities

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This paper is a version of the lecture I gave at the conference on "Representation Theory, Homological Algebra and Free Resolutions" at MSRI in February 2013, expanded to include proofs. My goals in this lecture were to explain to an audience of commutative algebraists why a finite group representation theorist might be interested in zero dimensional complete intersections, and to give a version of the Orlov correspondence in this context that is well suited to computation. In the context of modular representation theory, this gives an equivalence between the derived category of an elementary abelian p-group of rank r, and the category of (graded) reduced matrix factorisations of the polynomial $y_1 X_1^p + \cdots + y_r X_r^p$. Finally, I explain the relevance to some recent joint work with Julia Pevtsova on realisation of vector bundles on projective space from modular representations of constant Jordan type.

This material is based upon work supported by the National Science Foundation under grant no. 0932078 000, while the author was in residence at the Mathematical Science Research Institute (MSRI) in Berkeley, California, during the Spring semester of 2013.

1. Introduction

My goal here is to explain why a finite group representation theorist might be interested in commutative algebra, and in particular the Orlov correspondence [Orlov 2006]. I will then give an exposition of the Orlov correspondence for an arbitrary zero-dimensional complete intersection. Rather than go down the same route as Orlov, my description will be better suited to computation and will have the added advantage of giving a lift of this correspondence from the stable category to the derived category. Finally I shall explain the relevance to some recent joint work with Julia Pevtsova [Benson and Pevtsova 2012] on realisation of vector bundles on projective space from modular representations of constant Jordan type.

I should point out that Theorem 2.4, the main theorem of this paper, is a special case of Theorem 7.5 of Burke and Stevenson [2015]; even the functors realising the equivalences in the theorem are the same. The proof presented here uses a minimum of heavy machinery, taking advantage of the special situation in hand to reduce to an explicit computation involving the "bidirectional Koszul complex", introduced in Section 5.

Let G be a finite group and k a field of characteristic p. Recall that many features of the representation theory and cohomology are controlled by elementary abelian subgroups of G, that is, subgroups that are isomorphic to a direct product $E \cong (\mathbb{Z}/p)^r$ of cyclic groups of order p. The number r of copies of $\mathbb{Z}/p$ is called the *rank* of E.

For example, Chouinard's theorem [1976] states that a kG-module is projective if and only if its restriction to every elementary abelian p-subgroup E of G is projective.

A theorem of Quillen [1971a; 1971b] states that mod p cohomology of G is detected up to F*-isomorphism* by the elementary abelian p-subgroups of G. More precisely, the map

$$H^*(G,k) \to \varprojlim_E H^*(E,k)$$

is an F-isomorphism, where the inverse limit is taken over the category whose objects are the elementary abelian p-subgroups E of G and the maps are given by conjugations in G followed by inclusions. To say that a map of $\mathbb{F}_p$-algebras is an F-isomorphism means that the kernel is nilpotent, and given any element of the target, some p-power power of it is in the image. This is equivalent to the statement that the corresponding map of prime ideal spectra in the opposite direction is a homeomorphism in the Zariski topology. The role of the cohomology ring in the representation theory of G has been investigated extensively.

We refer the reader in particular to [Alperin 1987; Benson et al. 1997; 2011; Linckelmann 1999].

So our goal will be to understand the *stable module category* $\mathsf{stmod}(kG)$. This category has as its objects the finitely generated kG-modules, and its arrows are given by

$$\underline{\mathrm{Hom}}_{kG}(M, N) = \mathrm{Hom}_{kG}(M, N)/\mathrm{PHom}_{kG}(M, N),$$

where $\mathrm{PHom}_{kG}(M, N)$ denotes the linear subspace consisting of those homomorphism that factor through some projective kG-module. Note that the algebra kG is *self-injective*, meaning that the projective and injective kG-modules coincide.

The category $\mathsf{stmod}(kG)$ is a not an abelian category, but rather a *triangulated category*. This is true for any finite dimensional self-injective algebra, or more generally for the stable category of any Frobenius category. The details can be found in [Happel 1988]. This triangulated category is closely related to the *bounded derived category* $\mathsf{D}^b(kG)$. Let $\mathsf{perf}(kG)$ be the thick subcategory of $\mathsf{D}^b(kG)$ consisting of the *perfect complexes*, namely those complexes that are isomorphic in $\mathsf{D}^b(kG)$ to finite complexes of finitely generated projective kG-modules.

Theorem 1.1. *There is a canonical equivalence between the quotient*

$$\mathsf{D}^b(kG)/\mathsf{perf}(kG)$$

and the stable module category $\mathsf{stmod}(kG)$.

This theorem appeared in the late 1980s in the work of several people and in several contexts; see, for example, [Buchweitz 1986; Keller and Vossieck 1987; Rickard 1989, Theorem 2.1]. It motivated the following definition for any ring R.

Definition 1.2. Let R be a ring, and let $\mathsf{D}^b(R)$ be the bounded derived category of finitely generated R-modules. Then the *singularity category* of R is the Verdier quotient

$$\mathsf{D}_{\mathrm{sg}}(R) = \mathsf{D}^b(R)/\mathsf{perf}(R).$$

Likewise, if R is a graded ring, we denote by $\mathsf{D}^b(R)$ the bounded derived category of finitely generated graded R-modules and $\mathsf{D}_{\mathrm{sg}}(R)$ the quotient by the perfect complexes of graded modules.

Warning 1.3. In commutative algebra, this definition is much better behaved for Gorenstein rings than for more general commutative Noetherian rings. For a Gorenstein ring, the singularity category is equivalent to the stable category of maximal Cohen–Macaulay modules [Buchweitz 1986], but the following example is typical of the behaviour for non-Gorenstein rings.

Example 1.4. Let R be the ring
$$k[X, Y]/(X^2, XY, Y^2).$$
Then the radical of R is isomorphic to $k \oplus k$, so there is a short exact sequence of R-modules
$$0 \to k \oplus k \to R \to k \to 0.$$
This means that in $\mathsf{D}_{\mathrm{sg}}(R)$ the connecting homomorphism of this short exact sequence gives an isomorphism $k \cong k[1] \oplus k[1]$. We have
$$k \cong k[1]^{\oplus 2} \cong k[2]^{\oplus 4} \cong k[3]^{\oplus 8} \cong \cdots,$$
and so k is an infinitely divisible module. Its endomorphism ring $\mathrm{End}_{\mathsf{D}_{\mathrm{sg}}(R)}(k)$ is the colimit of
$$k \to \mathrm{Mat}_2(k) \to \mathrm{Mat}_4(k) \to \mathrm{Mat}_8(k) \to \cdots,$$
where each matrix ring is embedded diagonally into a product of two copies, sitting in the next matrix ring. In fact, this endomorphism ring is an example of a von Neumann regular ring. For a generalisation of this example to finite dimensional algebras with radical square zero, see [Chen 2011].

The reason why $\mathsf{D}_{\mathrm{sg}}(R)$ is called the "singularity category" is that it only "sees" the singular locus of R.

Example 1.5. If R is a regular ring then R has finite global dimension. So $\mathsf{D}^b(R) = \mathsf{perf}(R)$ and hence $\mathsf{D}_{\mathrm{sg}}(R) = 0$.

More generally, we have the following.

Definition 1.6. Let R be a [graded] Noetherian commutative ring. Then the *singular locus* of R is the set of [homogeneous] prime ideals $\mathfrak{p}$ of R such that the [homogeneous] localisation $R_{\mathfrak{p}}$ is not regular.

Remark 1.7. Provided that R satisfies a mild technical condition known as "excellence", the singular locus is a Zariski closed set, so that it is of the form $V(I)$ for some [homogeneous] radical ideal I of R. Thus $a \in I$ if and only if $R[a^{-1}]$ is regular. Quotients of polynomial rings, for example, are excellent.

Theorem 1.8. *Let R be a [graded] Noetherian commutative ring of finite Krull dimension whose singular locus is a Zariski closed set. Then $\mathsf{D}_{\mathrm{sg}}(R)$ is generated by [the graded shifts of] the modules $R/\mathfrak{p}$ where $\mathfrak{p}$ is a [homogeneous] prime ideal in the singular locus of R.*

Remark 1.9. In the ungraded case, the theorem of Schoutens [2003] implies the above theorem, but it is stronger, and the proof is more complicated. Schoutens' theorem also holds in the graded case, with minor adjustments to the proof.

I'd like to thank Srikanth Iyengar for suggesting the simple proof presented here. The idea behind this argument also appears in Lemma 2.2 of Herzog and Popescu [1997]. This theorem will be used in the proof of Proposition 8.1.

Proof of Theorem 1.8. Let $d = \dim R$ and let $V(I)$ be the singular locus of R. It suffices to show that if M is a finitely generated [graded] R-module then M is in the thick subcategory of $\mathsf{D}^b(R)$ generated by [graded shifts of] R and of $R/\mathfrak{p}$ with $\mathfrak{p} \in V(I)$, that is, with $\mathfrak{p} \supseteq I$.

The first step is to replace M by its d-th syzygy $\Omega^d(M)$, that is, the d-th kernel in any [graded] resolution of M by finitely generated free [graded] R-modules (we allow free graded modules to be sums of *shifts* of R). Thus we may assume that M is a d-th syzygy.

We claim that if a is a [homogeneous] element of I then for some $n > 0$, a^n annihilates $\mathrm{Ext}^1_R(M, \Omega(M))$. This is because $R[a^{-1}]$ has global dimension at most d, so the fact that M is a d-th syzygy implies that $M[a^{-1}]$ is also a d-th syzygy and is hence projective as an $R[a^{-1}]$-module. So

$$\mathrm{Ext}^1_R(M, \Omega(M))[a^{-1}] = \mathrm{Ext}^1_{R[a^{-1}]}(M[a^{-1}], \Omega(M[a^{-1}])) = 0.$$

Apply this to the extension $0 \to \Omega(M) \to F \to M \to 0$, with F a finitely generated free [graded] R-module. Multiplying this extension by a^n amounts to forming the pullback X in the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega(M) & \longrightarrow & X & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \big\downarrow & & \big\downarrow{\scriptstyle a^n} & & \\
0 & \longrightarrow & \Omega(M) & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

The resulting extension splits, so $X \cong M \oplus \Omega(M)$. The snake lemma implies that the middle vertical arrow gives rise to an exact sequence

$$0 \to \mathsf{Ker}\,(a^n, M) \to X \to F \to M/a^n M \to 0.$$

Now $\mathsf{Ker}\,(a^n, M)$ and $M/a^n M$ are annihilated by a^n. So X, and hence M, is in the thick subcategory of $\mathsf{D}^b(R)$ generated by R and modules supported on R/aR. Now inducting on a finite set of generators for the ideal I, we see that M is in the thick subcategory of $\mathsf{D}^b(R)$ generated by R and modules supported on $V(I)$. The latter are in turn generated by the $R/\mathfrak{p}$ with $\mathfrak{p} \in V(I)$. $\square$

Example 1.10. Consider the graded ring $A = R/(f)$ where $R = k[X_1, \dots, X_n]$, each X_i is given some nonnegative degree, and f is some homogeneous element of positive degree. Then Buchweitz [1986] showed that

$$\mathsf{D}_{\mathrm{sg}}(A) \simeq \underline{\mathsf{MCM}}(A),$$

the stable category of (finitely generated, graded) maximal Cohen–Macaulay A-modules. Eisenbud [1980] showed that this category is equivalent to the category of *reduced matrix factorisations* of f over R.

If we let $Y = \mathsf{Proj}\, A$, the quasiprojective variety of homogeneous prime ideals of A, then the category $\mathsf{Coh}(Y)$ of coherent sheaves on Y is equivalent to the quotient of the category $\mathsf{mod}(A)$ of finitely generated graded A-modules by the Serre subcategory of modules which are only nonzero in a finite number of degrees. We write $\mathsf{D}_{\mathrm{sg}}(Y)$ for the corresponding singularity category, namely the quotient $\mathsf{D}^b(\mathsf{Coh}(Y))/\mathsf{perf}(Y)$, where $\mathsf{perf}(Y)$ denotes the perfect complexes. Thus we have

$$\mathsf{D}_{\mathrm{sg}}(Y) \simeq \underline{\underline{\underline{\mathsf{MCM}}}}(A),$$

where $\underline{\underline{\underline{\mathsf{MCM}}}}(A)$ is the quotient of $\underline{\mathsf{MCM}}(A)$ by the maximal Cohen–Macaulay approximations of modules which are only nonzero in a finite number of degrees.

Grading conventions. We grade everything homologically, so that the differential decreases degree. When we talk of complexes of graded modules, there are two subscripts. The first subscript gives the homological degree and the second gives the internal degree. If C is a complex of graded modules with components $C_{i,j}$ then we write $C[n]$ for the homological shift: $C[n]_{i,j} = C_{i+n,j}$, and $C(n)$ for the internal shift: $C(n)_{i,j} = C_{i,j+n}$.

2. The Orlov correspondence

In this section we give a version of the Orlov correspondence for a complete intersection of dimension zero. Let $C = k[X_1, \dots, X_r]/(f_1, \dots, f_r)$ where $f_1, \dots, f_r$ is a regular sequence contained in the square of the maximal ideal $(X_1, \dots, X_r)$.

Example 2.1. Let

$$E = \langle g_1, \dots, g_r \rangle \cong (\mathbb{Z}/p)^r$$

and let k be a field of characteristic p. Let kE be the group algebra of E over k, and let

$$X_i = g_i - 1 \in kE.$$

Then $kE = k[X_1, \dots, X_r]/(X_1^p, \dots, X_r^p)$ is a complete intersection of codimension r and dimension zero.

Let $R_0 = k[X_1, \dots, X_r]$ and let $R = k[y_1, \dots, y_r] \otimes_k R_0$. We regard R as a graded polynomial ring with the y_i in degree one and the X_i in degree zero. Let

$$f = y_1 f_1 + \cdots + y_r f_r \in R,$$

an element of degree one. Let

$$A = R/(f) \quad \text{and} \quad B = R/(f_1, \ldots, f_r) = k[y_1, \ldots, y_r] \otimes_k C.$$

We have a diagram

$$\begin{array}{ccccc} B & \twoheadleftarrow & A & \twoheadleftarrow & R \\ \uparrow & & & \nwarrow & \uparrow \\ C & \twoheadleftarrow & & & k[X_1, \ldots, X_r] \end{array}$$

Taking Proj of these graded rings, we get the diagram in Section 2 of [Orlov 2006]:

$$\begin{array}{ccccc} Z & \overset{i}{\hookrightarrow} & Y & \overset{u}{\hookrightarrow} & S' \\ {\scriptstyle p}\downarrow & & & \overset{\pi}{\searrow} & \downarrow{\scriptstyle q} \\ X & & \overset{j}{\hookrightarrow} & & S \end{array}$$

Theorem 2.2 (Orlov). *The functor* $\mathbb{R}i_* p^* \colon \mathsf{D}^b(X) \to \mathsf{D}^b(Y)$ *descends to an equivalence of categories* $\mathsf{stmod}(C) = \mathsf{D}_{\mathrm{sg}}(X) \to \mathsf{D}_{\mathrm{sg}}(Y)$. *The right adjoint*

$$\mathbb{R}p_* i^\flat = \mathbb{R}p_* \mathbb{L}i^*(- \otimes \omega_{Z/Y})[-r+1] \tag{2.3}$$

gives the inverse equivalence.

Our goal is to prove the following lift of the Orlov correspondence to the derived category.

Theorem 2.4. *There is an equivalence of categories* $\mathsf{D}^b(C) \simeq \mathsf{D}_{\mathrm{sg}}(A)$ *lifting the equivalence* $\mathsf{D}_{\mathrm{sg}}(X) \simeq \mathsf{D}_{\mathrm{sg}}(Y)$ *of Orlov.*

Remark 2.5. In Orlov's version, he makes use of local duality as described in Chapter III, Corollary 7.3 of [Hartshorne 1966] to identify the right adjoint (2.3). In our version, this is replaced by the self-duality (6.1) of a complex ${}_A\hat{\mathbf{\Delta}}_A$ giving a Tate resolution of A as an A-A-bimodule. The discrepancy between the shift of $-r+1$ in (2.3) and the shift of $-r$ in (6.1) is explained by the fact that in our situation the sheaf $\omega_{Z/Y}$ is just $\mathcal{O}(-1)$.

In the following corollary, we spell out the consequences for the modular representation theory of elementary abelian p-groups.

Corollary 2.6. *Let E be an elementary abelian p-group of rank r. Then the following triangulated categories are equivalent*:

(1) *the derived category* $\mathsf{D}^b(kE)$;

(2) *the singularity category of graded A-modules* $\mathsf{D}_{\mathrm{sg}}(A)$ *where*

$$A = R/(f), \quad R = k[y_1, \dots, y_r, X_1, \dots, X_r], \quad f = y_1 X_1^p + \dots + y_r X_r^p,$$

the X_i *have degree zero and the* y_i *have degree one*;

(3) *the stable category of maximal Cohen–Macaulay graded A-modules*;

(4) *the category of reduced graded matrix factorisations of f over R.*

We shall see that under the correspondence given by this Corollary, the image of the trivial kE-module k is a $2^{r-1} \times 2^{r-1}$ matrix factorisation given by taking the even and odd terms in a bidirectional Koszul complex. The perfect complexes correspond to the maximal Cohen–Macaulay approximations to the A-modules which are nonzero only in finitely many degrees, so that the equivalence descends to Orlov's equivalence

$$\mathsf{stmod}(kE) \simeq \mathsf{D}_{\mathrm{sg}}(\mathsf{Proj}\,(A)).$$

The elements y_i correspond to the basis for the primitive elements in $H^2(E, k)$ obtained by applying the Bockstein map to the basis of $H^1(E, k)$ dual to $X_1, \dots, X_r$.

3. The functors

First we describe the functor $\Phi \colon \mathsf{D}^b(C) \to \mathsf{D}_{\mathrm{sg}}(A)$. If M_* is a bounded complex of finitely generated C-modules then the tensor product

$$k[y_1, \dots, y_r] \otimes_k M_*$$

is a bounded complex of finitely generated B-modules, which may then be regarded as a bounded complex of finitely generated A-modules. Passing down to the singularity category $\mathsf{D}_{\mathrm{sg}}(A)$, we obtain $\Phi(M_*)$. Thinking of $\mathsf{D}_{\mathrm{sg}}(A)$ as equivalent to $\underline{\mathsf{MCM}}(A)$, we can view $\Phi(M_*)$ as a maximal Cohen–Macaulay approximation to $k[y_1, \dots, y_r] \otimes_k M_*$.

Next we describe the functor $\Psi \colon \mathsf{D}_{\mathrm{sg}}(A) \to \mathsf{D}^b(C)$. An object N in $\mathsf{D}_{\mathrm{sg}}(A)$ can be thought of as a maximal Cohen–Macaulay A-module. It is therefore represented by a reduced graded matrix factorisation of the polynomial f over R. Namely, we have a pair of finitely generated free R-modules F and F' and maps

$$F \xrightarrow{U} F' \xrightarrow{V} F(1),$$

such that UV and VU are both equal to f times the identity map. Then we have the following sequence of free C-modules:

$$\begin{array}{ccc} \cdots & & \cdots \\ & \swarrow^{V} & \\ C \otimes_{R_0} F_0 & \xrightarrow{U} & C \otimes_{R_0} F'_0 \\ & \swarrow^{V} & \\ C \otimes_{R_0} F_1 & \xrightarrow{U} & C \otimes_{R_0} F'_1 \\ & \swarrow^{V} & \\ \cdots & & \cdots \end{array}$$

Since the free modules are finitely generated, this is zero far enough up the page. We shall see in Lemma 3.1 below, that the resulting complex only has homology in a finite number of degrees. It is therefore a complex of free C-modules, bounded to the left, and whose homology is totally bounded. It is therefore a semi-injective resolution of a well defined object in $\mathsf{D}^b(C)$. We shift in degree so that the term $F_0 \otimes_{R_0} C$ appears in degree $-r$, and this is the object $\Psi(N)$ in $\mathsf{D}^b(C)$.

Another way of viewing the object $\Psi(N)$ is to take a complete resolution of N as an A-module:

$$\begin{array}{ccccccc} \cdots \longrightarrow P_1 \longrightarrow P_0 & & \longrightarrow & & P_{-1} \longrightarrow \cdots \\ & \searrow & & \nearrow & \\ & & N & & \\ & \nearrow & & \searrow & \\ 0 & & & & 0 \end{array}$$

and then $\Psi(N)$ is the complex $(B \otimes_A P_*)_0[-r]$, whose degree n term is $(B \otimes_A P_*)_{(n-r,0)}$.

Lemma 3.1. *If M is a maximal Cohen–Macaulay A-module then for all $j \geq 0$, $\mathrm{Tor}_j^A(B, M)$ is nonzero only in finitely many degrees.*

Proof. This follows from the fact that B is locally (but not globally) a complete intersection as an A-module. More explicitly, for $1 \leq i \leq r$, in the ring $R[y_i^{-1}]$ we have the following equation:

$$X_i^p = y_i^{-1}(f - y_1 f_1 - \cdots \overset{i}{\uparrow} \cdots - y_r f_r).$$

It follows that

$$B[y_i^{-1}] = A[y_i^{-1}]/(f_1, \cdots \overset{i}{\uparrow} \cdots, f_r)$$

is a complete intersection of codimension $r-1$ over $A[y_i^{-1}]$. Using a Koszul complex, it follows that for all A-modules M and all $j \geq r$ we have

$$\operatorname{Tor}_j^A(B, M)[y_i^{-1}] = \operatorname{Tor}_j^{A[y_i^{-1}]}(B[y_i^{-1}], M[y_i^{-1}]) = 0.$$

If M is a maximal Cohen–Macaulay module then the minimal resolution of M is periodic and so $\operatorname{Tor}_j^A(B, M)[y_i^{-1}] = 0$ for all $j \geq 0$. Since $\operatorname{Tor}_j^A(B, M)$ is finitely generated, it is annihilated by a high enough power of each y_i and is hence it is nonzero only in finitely many degrees. □

4. An example

Before delving into proofs, let us examine an example in detail. Let $E = (\mathbb{Z}/p)^2 = \langle g_1, g_2\rangle$, an elementary abelian group of rank two, and let k be a field of characteristic p. Then setting $X_1 = g_1 - 1$, $X_2 = g_2 - 1$, we have

$$\begin{aligned} C &= kE = k[X_1, X_2]/(X_1^p, X_2^p), \\ A &= k[y_1, y_2, X_1, X_2]/(y_1X_1^p + y_2X_2^p). \end{aligned}$$

Let us compute $\Phi(k)$, where k is the trivial kE-module. This means we should resolve $k[y_1, y_2]$ as an A-module, and look at the corresponding matrix factorisation. This minimal resolution has the form

$$A(-1) \oplus A(-1) \xrightarrow{\begin{pmatrix} y_2X_2^{p-1} & -y_1X_1^{p-1} \\ X_1 & X_2 \end{pmatrix}} A \oplus A(-1) \xrightarrow{\begin{pmatrix} X_2 & y_1X_1^{p-1} \\ -X_1 & y_2X_2^{p-1} \end{pmatrix}} A \oplus A \xrightarrow{\begin{pmatrix} X_1 & X_2 \end{pmatrix}} A \to k[y_1, y_2].$$

This pair of 2×2 matrices gives a matrix factorisation of the polynomial $y_1X_1^p + y_2X_2^p$, and it is the matrix factorisation corresponding to the trivial kE-module.

Applying the functor Ψ to this matrix factorisation, we obtain the minimal injective resolution of the trivial kE-module, shifted in degree by two. The elements y_1 and y_2 give the action of the degree two polynomial generators of $H^*(E, k)$ on the minimal resolution.

Similarly, we compute $\Phi(kE)$ using the following resolution:

$$A(-1) \oplus A(-1) \xrightarrow{\begin{pmatrix} y_2 & -y_1 \\ X_1^p & X_2^p \end{pmatrix}} A \oplus A(-1) \xrightarrow{\begin{pmatrix} X_2^p & y_1 \\ -X_1^p & y_2 \end{pmatrix}} A \oplus A \xrightarrow{\begin{pmatrix} X_1^p & X_2^p \end{pmatrix}} A \to kE \otimes_k k[y_1, y_2].$$

This should be compared with the resolution of $k[X_1, X_2]$:

$$A(-2) \oplus A(-2) \xrightarrow{\begin{pmatrix} y_2 & -y_1 \\ X_1^p & X_2^p \end{pmatrix}} A(-1) \oplus A(-2)$$
$$\xrightarrow{\begin{pmatrix} X_2^p & y_1 \\ -X_1^p & y_2 \end{pmatrix}} A(-1) \oplus A(-1) \xrightarrow{(y_2 \ \ -y_1)} A \to k[X_1, X_2].$$

This is eventually the same resolution as above, but shifted two places to the left. Thus $\Phi(kE)$ is a maximal Cohen–Macaulay approximation to a module concentrated in a single degree.

5. The bidirectional Koszul complex

In this section, we construct the bidirectional Koszul complex. This first appears in [Tate 1957], and reappears in many places. This allows us to describe the minimal resolution of B as an A-module. This computes the value of the functor Φ on the free C-module of rank one.

Let Λ_n $(0 \le n \le r)$ be the free R-module of rank $\binom{r}{n}$ on generators

$$e_{j_1} \wedge \cdots \wedge e_{j_n}$$

with $1 \le j_1 < \cdots < j_n \le r$. We use the convention that the wedge is alternating, in the sense that $e_i \wedge e_j = -e_j \wedge e_i$ and $e_i \wedge e_i = 0$, to give meaning to wedge products with indices out of order or repeated indices.

We give Λ_* a differential $d\colon \Lambda_n \to \Lambda_{n-1}$ described by

$$d(e_{j_1} \wedge \cdots \wedge e_{j_n}) = \sum_i (-1)^{i-1} f_{j_i} (e_{j_1} \wedge \cdots \overset{j_i}{\uparrow} \cdots \wedge e_{j_n}), \tag{5.1}$$

where the vertical arrow indicates a missing term. We also give Λ_* a differential in the other direction $\delta\colon \Lambda_n \to \Lambda_{n+1}(1)$ described by

$$\delta(e_{j_1} \wedge \cdots \wedge e_{j_n}) = \sum_j y_j e_j \wedge (e_{j_1} \wedge \cdots \wedge e_{j_n}). \tag{5.2}$$

We call the graded R-module Λ_* with these two differentials the *bidirectional Koszul complex* with respect to the pair of sequences $f_1, \dots, f_r$ and $y_1, \dots, y_r$:

$$\Lambda_r \underset{\underset{(1)}{\delta}}{\overset{d}{\rightleftarrows}} \Lambda_{r-1} \rightleftarrows \quad \cdots \quad \rightleftarrows \Lambda_1 \underset{\underset{(1)}{\delta}}{\overset{d}{\rightleftarrows}} \Lambda_0 \ .$$

Lemma 5.3. *The map $d\delta + \delta d\colon \Lambda_n \to \Lambda_n(1)$ is equal to multiplication by $f = \sum_i y_i f_i$.*

Proof. We have

$$\delta d(e_{j_1} \wedge \cdots \wedge e_{j_n}) = \sum_{i,j} (-1)^{i-1} y_j f_j e_j \wedge (e_{j_1} \wedge \cdots \overset{j_i}{\uparrow} \cdots \wedge e_{j_n}),$$

whereas

$$d\delta(e_{j_1} \wedge \cdots \wedge e_{j_n})$$
$$= \sum_{i,j} (-1)^{i-1} y_j f_j e_j \wedge (e_{j_1} \wedge \cdots \overset{j_i}{\uparrow} \cdots \wedge e_{j_n}) + \sum_i y_i f_i (e_{j_1} \wedge \cdots \wedge e_{j_n}). \square$$

Thus, taking even and odd parts of Λ_*, we see that

$$\bigoplus_{n=0}^{\lfloor \frac{r}{2} \rfloor} \Lambda_{2n}(n-1) \xrightarrow{d+\delta} \bigoplus_{n=1}^{\lfloor \frac{r+1}{2} \rfloor} \Lambda_{2n-1}(n-1) \xrightarrow{d+\delta} \bigoplus_{n=0}^{\lfloor \frac{r}{2} \rfloor} \Lambda_{2n}(n) \qquad (5.4)$$

is a matrix factorisation of f, called the *Koszul factorisation*. Note that the free R-modules in this matrix factorisation all have rank 2^{r-1}, because this is the sum of the even binomial coefficients as well as the sum of the odd binomial coefficients. For notation, we write

$$K_0(-1) \xrightarrow{d+\delta} K_1 \xrightarrow{d+\delta} K_0 \qquad (5.5)$$

for the Koszul factorisation, and we write K for the cokernel of $d+\delta\colon K_1 \to K_0$.

For example, if $r = 4$ we get the Koszul factorisation

$$\Lambda_4(1) \oplus \Lambda_2 \oplus \Lambda_0(-1) \xrightarrow{\begin{pmatrix} d & \delta & 0 \\ 0 & d & \delta \end{pmatrix}} \Lambda_3(1) \oplus \Lambda_1 \xrightarrow{\begin{pmatrix} \delta & 0 \\ d & \delta \\ 0 & d \end{pmatrix}} \Lambda_4(2) \oplus \Lambda_2(1) \oplus \Lambda_0.$$

The minimal resolution of the A-module B is obtained by applying $A \otimes_R -$ to

$$\cdots \to \Lambda_4 \oplus \Lambda_2(-1) \oplus \Lambda_0(-2) \to \Lambda_3 \oplus \Lambda_1(-1) \xrightarrow{d+\delta} \Lambda_2 \oplus \Lambda_0(-1) \xrightarrow{d+\delta} \Lambda_1 \xrightarrow{d} \Lambda_0. \qquad (5.6)$$

This takes r steps to settle down to the Koszul factorisation, but in large degrees (i.e., far enough to the left) it agrees with

$$\cdots \to K_0(-2) \xrightarrow{d+\delta} K_1(-1) \xrightarrow{d+\delta} K_0(-1) \xrightarrow{d+\delta} K_1 \xrightarrow{d+\delta} K_0. \qquad (5.7)$$

For notation, we write $\bar{\Lambda}_i = A \otimes_R \Lambda_i$, so that the minimal resolution of B as an A-module takes the form

$$\cdots \to \bar{\Lambda}_4 \oplus \bar{\Lambda}_2(-1) \oplus \bar{\Lambda}_0(-2) \to \bar{\Lambda}_3 \oplus \bar{\Lambda}_1(-1) \xrightarrow{d+\delta} \bar{\Lambda}_2 \oplus \bar{\Lambda}_0(-1) \xrightarrow{d+\delta} \bar{\Lambda}_1 \xrightarrow{d} \bar{\Lambda}_0. \qquad (5.8)$$

Let us write $\Delta_{i,j}$ for this complex. As usual, i denotes the homological degree and j the internal degree. Thus $\Delta \to B$ is a free resolution.

A complete resolution of B as an A-module is also easy to write down at this stage. Namely, we just continue (5.7) to the right in the obvious way. Let us write $\hat{\Delta}_{i,j}$ for this complete resolution. Then we notice a self-duality up to shift:

$$\mathrm{Hom}_A(\hat{\Delta}, A) \cong \hat{\Delta}[-r] \tag{5.9}$$

and a periodicity

$$\hat{\Delta}[2] \cong \hat{\Delta}(1).$$

Next observe that the minimal resolution of $A/(y_1, \dots, y_r) = k[X_1, \dots, X_r]$ as an A-module takes the form

$$\cdots \to \bar{\Lambda}_{r-3}(-3) \oplus \bar{\Lambda}_{r-1}(-2) \xrightarrow{d+\delta} \bar{\Lambda}_{r-2}(-2) \oplus \bar{\Lambda}_r(-1) \xrightarrow{d+\delta} \bar{\Lambda}_{r-1}(-1) \xrightarrow{\delta} \bar{\Lambda}_r. \tag{5.10}$$

This again takes r steps to settle down to the Koszul factorisation, but in large degrees (i.e., far enough to the left) it agrees with the result of applying $A \otimes_R -$ to

$$\cdots \to K_1(\lfloor\tfrac{-r-2}{2}\rfloor) \xrightarrow{d+\delta} K_0(\lfloor\tfrac{-r-1}{2}\rfloor) \xrightarrow{d+\delta} K_1(\lfloor\tfrac{-r}{2}\rfloor) \xrightarrow{d+\delta} K_0(\lfloor\tfrac{-r+1}{2}\rfloor). \tag{5.11}$$

Theorem 5.12. *The minimal resolutions over A of*

$$A/(y_1, \dots, y_r) = k[X_1, \dots, X_r]$$

after r steps and of B after $2r$ steps are equal.

Proof. Compare (5.11) with (5.7). □

Now let M_* be a bounded complex of C-modules, regarded as an object in $\mathsf{D}^b(C)$, and let $X_{i,j}$ be a free resolution of $k[y_1, \dots, y_r] \otimes_k M_*$ as an A-module. Thus for large positive homological degree i, this is a periodic complex corresponding to a matrix factorisation of f, namely $\Phi(M_*)$. The maps $\Delta_{*,*} \to B$ and $X_{*,*} \to k[y_1, \dots, y_r] \otimes_k M_*$ induce homotopy equivalences

$$B \otimes_A X_{*,*} \leftarrow \Delta_{*,*} \otimes_A X_{*,*} \to \Delta_{*,*} \otimes_A (k[y_1, \dots, y_r] \otimes_k M_*).$$

Now $f_1, \dots, f_r$ annihilate M_* and so act as zero in $\Delta_{*,*} \otimes_A (k[y_1, \dots, y_r] \otimes_k M_*)$. So the operator d in the complex $\Delta_{*,*}$ acts as zero in the tensor product, which therefore decomposes as a direct sum of pieces, each living in a finite set of degrees. To be more explicit, it decomposes as a sum of the following pieces tensored over A with $(k[y_1,\dots,y_r]\otimes_k M_*)$:

$$\begin{array}{ccccccc}
 & & & & & & \bar\Lambda_0 \\
 & & & & \bar\Lambda_0(-1) & \xrightarrow{\delta} & \bar\Lambda_1 \\
 & & \bar\Lambda_0(-2) & \xrightarrow{\delta} & \bar\Lambda_1(-1) & \xrightarrow{\delta} & \bar\Lambda_2 \\
\bar\Lambda_0(-3) & \xrightarrow{\delta} & \bar\Lambda_1(-2) & \xrightarrow{\delta} & \bar\Lambda_2(-1) & \xrightarrow{\delta} & \bar\Lambda_3 \\
\cdots & & \cdots & & & &
\end{array}$$

Eventually, this just consists of copies of the Koszul complex for parameters $y_1,\dots,y_r$ on $k[y_1,\dots,y_r]\otimes_k M_*$, shifted in degree by $(2n,-n+r)$. This Koszul complex is quasi-isomorphic to M_* shifted $(2n,-n+r)$.

It follows that if we take a *complete* resolution over A of $k[y_1,\dots,y_r]\otimes_k M_*$, apply $B\otimes_A -$ to it, and take the part with internal degree zero, we obtain a complex which is quasi-isomorphic to M_* shifted in degree by r. This process is exactly the functor Ψ applied to $\Phi(M_*)$. To summarise, we have proved the following:

Theorem 5.13. *The composite functor* $\Psi\circ\Phi\colon \mathsf{D}^b(C)\to\mathsf{D}^b(C)$ *is naturally isomorphic to the identity functor.* □

If we restrict just to C-modules rather than complexes, we have the following formulation:

Theorem 5.14. *Let M be a C-module. Then for $i\ge 0$ we have*

$$\begin{aligned}\mathrm{Tor}^A_{i+r,j}(B,k[y_1,\dots,y_r]\otimes_k M)&\cong\mathrm{Tor}^A_{i,j}(k[X_1,\dots,X_r],k[y_1,\dots,y_r]\otimes_k M)\\ &\cong\begin{cases}M & i=2j,\\ 0 & \textit{otherwise.}\end{cases}\end{aligned}$$

Proof. This follows from Theorems 5.12 and 5.13. □

6. A bimodule resolution

A similar bidirectional Koszul complex can be used to describe the minimal resolution of A as an A-A-bimodule. This works more generally for any hypersurface (or indeed with suitable modifications for any complete intersection; see Section 3 of [Wolffhardt 1972]), so we introduce it in that context. Let $S=k[u_1,\dots,u_n]$ where each u_i is a homogeneous variable of nonnegative degree. Let $\phi(u_1,\dots,u_n)\in S$ be a homogeneous polynomial of positive degree and let $H=S/(\phi)$ be the corresponding hypersurface. We write

$$\begin{aligned}S\otimes_k S&=k[u'_1,\dots,u'_n,u''_1,\dots,u''_n],\\ H\otimes_k H&=S\otimes_k S/(\phi(u'_1,\dots,u'_n),\phi(u''_1,\dots,u''_n)).\end{aligned}$$

Then we can form the bidirectional Koszul complex on the two sequences

$$u'_1 - u''_1, \dots, u'_n - u''_n$$

and

$$\begin{gathered}(\phi(u'_1, u'_2 \dots, u'_n) - \phi(u''_1, u'_2, \dots, u'_n))/(u'_1 - u''_1),\\ (\phi(u''_1, u'_2, u'_3, \dots, u'_n) - \phi(u''_1, u''_2, u'_3, \dots, u'_n))/(u'_2 - u''_2),\\ \dots\\ (\phi(u''_1, \dots, u''_{n-1}, u'_n) - \phi(u''_1, \dots, u''_{n-1}, u''_n))/(u'_n - u''_n).\end{gathered}$$

Note that the latter is indeed a sequence of polynomials, and that the sum of the products of corresponding terms in these two sequences gives

$$\phi(u'_1, \dots, u'_n) - \phi(u''_1, \dots, u''_n).$$

We therefore obtain a matrix factorisation of this difference over $S \otimes_k S$ looking much like (5.4). The construction corresponding to (5.8) in this situation gives a resolution of the module $S = (S \otimes_k S)/(u'_1 - u''_1, \dots, u'_r - u''_r)$ over the hypersurface $(S \otimes_k S)/(\phi' - \phi'')$. Since ϕ' is a non zero-divisor on both the module and the hypersurface, we can mod it out, retaining exactness, to obtain a resolution of $S/\phi = H$ as a module over $(S \otimes_k S)/(\phi' - \phi'', \phi') = H \otimes_k H$ (i.e., as an H-H-bimodule). We write ${}_H\mathbf{\Delta}_H$ for this resolution. It is eventually periodic with period two. The corresponding complete resolution is periodic with period two, and we denote it by ${}_H\hat{\mathbf{\Delta}}_H$.

Applying this in the particular case of A as an A-A-bimodule, we have

$$A \otimes_k = \frac{k[y'_1, \dots, y'_r, y''_1, \dots, y''_r, X'_1, \dots, X'_r, X''_1, \dots, X''_r]}{(y'_1 f'_1 + \cdots + y'_r f'_r, y''_1 f''_1 + \cdots + y''_r f''_r)}.$$

Let ${}_A\hat{\mathbf{\Delta}}_A$ be the corresponding complete resolution. Exactly as in (5.9) we have a self-duality up to shift:

$$\operatorname{Hom}_{A \otimes_k A}({}_A\hat{\mathbf{\Delta}}_A, A \otimes_k A) \cong {}_A\hat{\mathbf{\Delta}}_A[-r] \tag{6.1}$$

and a periodicity

$${}_A\hat{\mathbf{\Delta}}_A[2] \cong {}_A\hat{\mathbf{\Delta}}_A(1). \tag{6.2}$$

Now regarding B as an A-B-bimodule via the map $A \to B$, we have a free resolution given by ${}_A\mathbf{\Delta}_A \otimes_A {}_AB_B$. We write ${}_A\mathbf{\Delta}_B$ for this resolution, and ${}_A\hat{\mathbf{\Delta}}_B$ for the corresponding complete resolution ${}_A\hat{\mathbf{\Delta}}_A \otimes_A {}_AB_B$. Similarly, if we regard B as a B-A-bimodule, we have a free resolution ${}_B\mathbf{\Delta}_A = {}_BB_A \otimes_A {}_A\mathbf{\Delta}_A$ and a complete resolution ${}_B\hat{\mathbf{\Delta}}_A = {}_BB_A \otimes_A {}_A\hat{\mathbf{\Delta}}_A$. The duality (6.1) gives

$$\operatorname{Hom}_{A \otimes_k B}({}_A\hat{\mathbf{\Delta}}_B, A \otimes_k B) \cong {}_B\hat{\mathbf{\Delta}}_A[-r], \tag{6.3}$$

and the periodicity (6.2) gives

$$ {}_A\hat{\mathbf{\Delta}}_B[2] \cong {}_A\hat{\mathbf{\Delta}}_B(1). \tag{6.4}$$

Finally, for any left A-module N we have a free resolution ${}_A\mathbf{\Delta}_A \otimes_A N$ and a complete resolution ${}_A\hat{\mathbf{\Delta}}_A \otimes_A N$.

7. The adjunction

Proposition 7.1. *The functor* Ψ *is right adjoint to* Φ.

Proof. Let M_* be a bounded complex of C-modules. If N is a maximal Cohen–Macaulay A-module, then ${}_A\hat{\mathbf{\Delta}}_A \otimes_A N$ is a complete resolution of N as an A-module. Then

$$\Psi(N) = ({}_B B_A \otimes_A ({}_A\hat{\mathbf{\Delta}}_A \otimes_A N))_0[-r] \cong ({}_B\hat{\mathbf{\Delta}}_A \otimes_A N)_0[-r].$$

Write $\underline{\mathrm{Hom}}$ for homomorphisms of complexes modulo homotopy. Since

$$({}_B\hat{\mathbf{\Delta}}_A \otimes_A N)_0[-r]$$

is semi-injective, homomorphisms in $\mathsf{D}^b(C)$ from an object to it are just homotopy classes of maps of complexes. So using the duality (6.3), we have

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{D}^b(C)}(M_*, \Psi(N)) &= \underline{\mathrm{Hom}}^n_C(M_*, ({}_B\hat{\mathbf{\Delta}}_A \otimes_A N)_0[-r]) \\
&\cong \underline{\mathrm{Hom}}_B(k[y_1, \dots, y_r] \otimes_k M_*, {}_B\hat{\mathbf{\Delta}}_A \otimes_A N[-r]) \\
&\cong \underline{\mathrm{Hom}}_B(k[y_1, \dots, y_r] \otimes_k M_*, \mathrm{Hom}_A({}_A\hat{\mathbf{\Delta}}_B, N)) \\
&\cong \underline{\mathrm{Hom}}_A({}_A\hat{\mathbf{\Delta}}_B \otimes_B (k[y_1, \dots, y_r] \otimes_k M_*), N) \\
&\cong \underline{\mathrm{Hom}}_A(\Phi(M), N).
\end{aligned}$$

In the last line, we are using the fact that

$${}_A\hat{\mathbf{\Delta}}_B \otimes_B (k[y_1, \dots, y_r] \otimes_k M_*) = {}_A\hat{\mathbf{\Delta}}_A \otimes_A (k[y_1, \dots, y_r] \otimes_k M_*)$$

is a complete resolution over A of a maximal Cohen–Macaulay approximation to the complex $k[y_1, \dots, y_r] \otimes_k M_*$. □

8. The equivalence

Proposition 8.1. *The category* $\mathsf{D}_{\mathrm{sg}}(A)$ *is generated by the* A*-module* $k[y_1, \dots, y_r]$.

Proof. The singular locus of A is defined by the equations $\partial f/\partial y_i = 0$ and $\partial f/\partial X_i = 0$. The former give the equations $f_1 = 0, \dots, f_r = 0$; since C is a zero dimensional complete intersection, these equations define the same variety as $X_1 = 0, \dots, X_r = 0$. The latter give the equations $\sum_j y_j \partial f_j/\partial X_i = 0$. Since $f_1, \dots, f_r$ are in the square of the maximal ideal, $\partial f_j/\partial X_i$ has zero constant term and so no new conditions are imposed by these equations. So we have shown that the prime ideals in the singular locus are those containing $(X_1, \dots, X_r)$.

By Theorem 1.8, the singularity category $\mathsf{D}_{\mathrm{sg}}(A)$ is generated by the modules $A/\mathfrak{p}$ with $\mathfrak{p} \supseteq (X_1, \dots, X_r)$, namely the quotients of $k[y_1, \dots, y_r]$ by prime ideals. These in turn are generated by the single object $k[y_1, \dots, y_r]$, by the Hilbert syzygy theorem. □

Theorem 8.2. *The composite functor $\Phi \circ \Psi \colon \mathsf{D}_{\mathrm{sg}}(A) \to \mathsf{D}_{\mathrm{sg}}(A)$ is naturally isomorphic to the identity functor.*

Proof. Consider the adjunction of Proposition 7.1. By Theorem 5.13, the unit of this adjunction gives an isomorphism $k \to \Psi\Phi(k)$, where k is the residue field of C, regarded as an object in $\mathsf{D}^b(C)$ by putting it in degree zero. It follows that $\Phi\Psi(\Phi(k)) = \Phi(\Psi\Phi(k)) \cong \Phi(k)$. An easy diagram chase shows that this isomorphism is given by the counit of the adjunction. It follows that the counit of the adjunction is an isomorphism for every object in the thick subcategory of $\mathsf{D}_{\mathrm{sg}}(A)$ generated by $\Phi(k)$. Since $\Phi(k) \cong k[y_1, \dots, y_r]$, by Proposition 8.1 this is the whole of $\mathsf{D}_{\mathrm{sg}}(A)$. □

Theorem 8.3. *The functors $\Phi \colon \mathsf{D}^b(C) \to \mathsf{D}_{\mathrm{sg}}(A)$ and $\Psi \colon \mathsf{D}_{\mathrm{sg}}(A) \to \mathsf{D}^b(C)$ are inverse equivalences of categories.*

Proof. This follows from Theorems 5.13 and 8.2. □

Theorem 8.4. *The equivalence*

$$\mathsf{D}^b(C) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathsf{D}_{\mathrm{sg}}(A)$$

descends to an equivalence

$$\mathsf{stmod}(C) \underset{\bar{\Psi}}{\overset{\bar{\Phi}}{\rightleftarrows}} \underline{\underline{\mathsf{MCM}}}(A).$$

Proof. It follows from Theorem 5.12 that $\Phi(C)$ is a shift of $k[X_1, \dots, X_r]$. The theorem now follows, because $\mathsf{stmod}(C)$ is the quotient of $\mathsf{D}^b(C)$ by the thick subcategory generated by C and $\underline{\underline{\mathsf{MCM}}}(A)$ is the quotient of $\underline{\mathsf{MCM}}(A)$ by the thick subcategory generated by $k[X_1, \dots, X_r]$. □

9. The trivial module

So, in the case $C = kE$, where does the trivial kE-module k go to under the correspondence of Theorem 8.3? To answer this, we must find the minimal resolution of $k[y_1, \dots, y_r]$ as an A-module. This is again given in terms of a bidirectional Koszul complex, this time for the pair of sequences $X_1, \dots, X_r$ and

$y_1X_1^{p-1}, \dots, y_rX_r^{p-1}$. The sum of the products of corresponding terms in these sequences again gives $f = \sum_i y_iX_i^p$, and so we obtain a matrix factorisation of f by taking the even and odd parts of this bidirectional Koszul complex. Adding up the even and odd binomial coefficients, we see that this is a $2^{r-1} \times 2^{r-1}$ matrix factorisation of f. The minimal resolution of $k[y_1, \dots, y_r]$ as an A-module is given by (5.8) with respect to this version of the bidirectional Koszul complex.

In a similar way, we can find the image of any module of the form

$$kE/(X_1^{a_1}, \dots, X_r^{a_r})$$

under the correspondence of Theorem 8.3 by doing the same process with a bidirectional Koszul complex for the pair of sequences $X_1^{a_1}, \dots, X_r^{a_r}$ and $y_1X_1^{p-a_1}, \dots, y_rX_r^{p-a_r}$.

Let us look at some examples. First we look at the case $r = 1$. In this case E is cyclic of order p,

$$kE = k[X]/(X^p), \quad A = k[y, X]/(yX^p) \quad \text{and} \quad B = k[y, X]/(X^p).$$

The indecomposable kE-modules are the Jordan blocks

$$J_n = k[X]/(X^n) \quad (1 \le n \le p).$$

Resolving $J_n \otimes_k k[y]$ we get

$$\cdots \to A[-1] \xrightarrow{X^n} A[-1] \xrightarrow{yX^{p-n}} A \xrightarrow{X^n} A.$$

So the matrix factorisation corresponding to J_n is given by the 1×1 matrices

$$A[-1] \xrightarrow{(yX^{p-n})} A \xrightarrow{(X^n)} A.$$

Next let $r = 2$, so that $A = k[y_1, y_2, X_1, X_2]/(y_1X_1^p + y_2X_2^p)$. To find the matrix factorisation corresponding to the trivial module, we resolve the A-module $k[y_1, y_2] = A/(X_1, X_2)$. Using the construction in Section 5, we obtain the following minimal resolution:

$$A[-1] \oplus A[-1] \xrightarrow{\begin{pmatrix} y_2X_2^{p-1} & -y_1X_1^{p-1} \\ X_1 & X_2 \end{pmatrix}} A \oplus A[-1] \xrightarrow{\begin{pmatrix} X_2 & y_1X_1^{p-1} \\ -X_1 & y_2X_2^{p-1} \end{pmatrix}} A \oplus A \xrightarrow{\begin{pmatrix} X_1 & X_2 \end{pmatrix}} A.$$

The two square matrices in this resolution alternate, and so the matrix factorisation corresponding to the trivial module is as follows:

$$A \oplus A[-1] \xrightarrow{\begin{pmatrix} X_2 & y_1X_1^{p-1} \\ -X_1 & y_2X_2^{p-1} \end{pmatrix}} A \oplus A \xrightarrow{\begin{pmatrix} y_2X_2^{p-1} & -y_1X_1^{p-1} \\ X_1 & X_2 \end{pmatrix}} A[1] \oplus A.$$

In rank three, the minimal resolution of $k[y_1, y_2, y_3]$ as an A-module takes the following form:

$$A[-1]^{\oplus 3} \oplus A[-2] \xrightarrow{\begin{pmatrix} 0 & X_3 & -X_2 & y_1X_1^{p-1} \\ -X_3 & 0 & X_1 & y_2X_2^{p-1} \\ X_2 & -X_1 & 0 & y_3X_3^{p-1} \\ y_1X_1^{p-1} & y_2X_2^{p-2} & y_3X_3^{p-1} & 0 \end{pmatrix}} A[-1]^{\oplus 3} \oplus A$$

$$\xrightarrow{\begin{pmatrix} 0 & -y_3X_3^{p-1} & y_2X_2^{p-1} & X_1 \\ y_3X_3^{p-1} & 0 & -y_1X_1^{p-1} & X_2 \\ -y_2X_2^{p-1} & y_1X_1^{p-1} & 0 & X_3 \\ X_1 & X_2 & X_3 & 0 \end{pmatrix}} A^{\oplus 3} \oplus A[-1]$$

$$\xrightarrow{\begin{pmatrix} 0 & X_3 & -X_2 & y_1X_1^{p-1} \\ -X_3 & 0 & X_1 & y_2X_2^{p-1} \\ X_2 & -X_1 & 0 & y_3X_3^{p-1} \end{pmatrix}} A^{\oplus 3} \xrightarrow{\begin{pmatrix} X_1 & X_2 & X_3 \end{pmatrix}} A.$$

The left-hand pair of matrices therefore gives the matrix factorisation of f corresponding to the trivial kE-module.

10. Computer algebra

Here is some code in the computer algebra language Macaulay2 for computing the functor $\mathsf{mod}(kE) \hookrightarrow \mathsf{D}^b(kE) \xrightarrow{\Phi} \underline{\mathsf{RMF}}(f)$. It is given here for the trivial module for a rank two group with $p = 7$, but the code is easy to modify. The last two commands print out the sixth and seventh matrices in the minimal resolution, which in this case is easily far enough to give a matrix factorisation.

```
p=7
R = ZZ/p[X1,X2,y1,y2]
f = X1^p * y1 + X2^p * y2
A = R/(f)
U = cokernel matrix {{X1,X2}}
F = resolution (U,LengthLimit=>8)
F.dd_6
F.dd_7
```

To modify the code to work for other kE-modules M, the fifth line should be changed to give a presentation of

$$U = k[y_1, \dots, y_r] \otimes_k M$$

as an A-module. Don't forget the relations saying that X_i^p annihilates M. For example, if $M = \Omega(k)$ for the same rank two group above, then U has two generators and three relations; it is the cokernel of the matrix

$$\begin{pmatrix} X_1^p & X_2 & 0 \\ 0 & X_1 & X_2^p \end{pmatrix} : A^{\oplus 3} \to A^{\oplus 2}.$$

So the fifth line should be changed to

```
U = cokernel matrix {{X1^p,X2,0},{0,X1,X2^p}}.
```

11. Cohomology

For this section, we stick with the case $C = kE$. The elements $y_1, \dots, y_r \in A$ act on maximal Cohen–Macaulay modules N as maps $N \to N(1)$. Now $N(1)$ is isomorphic to $\Omega^{-2}(N)$, and Ω^{-1} is the shift functor in the triangulated category $\underline{\mathsf{MCM}}(A)$. It follows that under equivalence of categories of Theorem 8.3, these elements correspond to maps in $\mathsf{D}^b(kE)$ from M_* to $M_*[2]$. We claim that these elements act as the polynomial part of the cohomology ring, namely the subring generated by the Bocksteins of the degree one elements.

Recall that kE is a Hopf algebra, either via the group theoretic diagonal map defined by $\Delta(g_i) = g_i \otimes g_i$, $\Delta(X_i) = X_i \otimes 1 + 1 \otimes X_i + X_i \otimes X_i$ or via the restricted Lie algebra diagonal map defined by $\tilde{\Delta}(X_i) = X_i \otimes 1 + 1 \otimes X_i$. In both cases, we make A into a right kE-comodule via $\Delta, \tilde{\Delta} \colon A \to A \otimes kE$ defined by the same formula on X_i and via $\Delta(y_i) = \tilde{\Delta}(y_i) = y_i \otimes 1$. It is easy to check that this is a coaction: $(1 \otimes \Delta) \circ \Delta = (\Delta \otimes 1) \circ \Delta$ and $(1 \otimes \tilde{\Delta}) \circ \tilde{\Delta} = (\tilde{\Delta} \otimes 1) \circ \tilde{\Delta}$. We denote the corresponding tensor products $N \otimes_k M$ and $N \mathbin{\tilde{\otimes}_k} M$, where M is a kE-module and N and the tensor product are A-modules.

Lemma 11.1. *If N is a maximal Cohen–Macaulay A-module and M is a kE-module then $N \otimes_k M$ and $N \mathbin{\tilde{\otimes}_k} M$ are maximal Cohen–Macaulay A-modules.*

Proof. The kE-module M has a finite filtration in which the filtered quotients are copies of the trivial kE-module k. So the tensor products $N \otimes_k M$ and $N \mathbin{\tilde{\otimes}_k} M$ both have finite filtrations in which the quotients are isomorphic to N. The lemma now follows from the fact that every extension of maximal Cohen–Macaulay modules is maximal Cohen–Macaulay. □

Recall that the maximal Cohen–Macaulay module K corresponding to the Koszul factorisation (5.5) is the image of the trivial kE-module k under Φ.

Proposition 11.2. *If M is a kE-module then $\Phi(M)$ may be taken to be the maximal Cohen–Macaulay module $K \otimes_k M$ (or $K \mathbin{\widetilde{\otimes}}_k M$).*

Proof. It follows from Lemma 11.1 that $K \otimes_k M$, resp. $K \mathbin{\widetilde{\otimes}}_k M$ is a maximal Cohen–Macaulay approximation to $k[y_1, \dots, y_r] \otimes_k M$. □

Let V be the linear space spanned by $X_1, \dots, X_r$. Then we may use the polynomial f to identify the space spanned by $y_1, \dots, y_r$ with the Frobenius twist of the dual $F(V^*)$. There is an action of $GL(V)$ on kE induced by linear substitutions of the X_i, and this induces an action on the linear space $F(V^*)$ spanned by the y_i.

Theorem 11.3. (i) *The maps*

$$\Psi(y_1), \dots, \Psi(y_r) \colon k \to k[2]$$

form a vector space basis for the image of the Bockstein map

$$H^1(E, k) \to H^2(E, k).$$

(ii) *For any M in $\mathsf{stmod}(kE)$, the induced map*

$$\overline{\Psi}(y_i) \colon M \to \Omega^{-2}(M)$$

is equal to the map

$$\overline{\Psi}(y_i) \otimes 1 \colon k \otimes_k M \to \Omega^{-2}(k) \otimes M$$

and also to the map

$$\overline{\Psi}(y_i) \mathbin{\widetilde{\otimes}} 1 \colon k \mathbin{\widetilde{\otimes}}_k M \to \Omega^{-2}(k) \mathbin{\widetilde{\otimes}} M.$$

Proof. (i) Consider the action of $GL(V)$ on $H^2(E, k)$. For p odd, we have

$$H^2(E, k) \cong F(V^*) \oplus \Lambda^2(V^*).$$

For $p = 2$, we have $H^2(E, k) \cong S^2(V)$; in this case $S^2(V^*)$ has two composition factors as a $GL(V)$-module, given by the nonsplit short exact sequence

$$0 \to F(V^*) \to S^2(V^*) \to \Lambda^2(V^*) \to 0.$$

In both cases, there is a unique $GL(V)$-invariant subspace of $H^2(E, k)$ isomorphic to $F(V^*)$, and this is the image of the Bockstein map. It therefore suffices to prove that the $\overline{\Psi}(y_i)$ are not all equal to zero. To see this, take the Koszul complex for K with respect to the parameters $y_1, \dots, y_r$. Since K is a maximal Cohen–Macaulay approximation to $k[y_1, \dots, y_r]$, the homology of this complex disappears after applying $\overline{\Psi}$. If $\overline{\Psi}(y_i)$ were zero, this could not be the case. It follows that $\overline{\Psi}(y_i) \neq 0$ and the theorem is proved.

(ii) This follows from Proposition 11.2. □

12. Modules of constant Jordan type

The study of modules of constant Jordan type was initiated by Carlson, Friedlander and Pevtsova [Carlson et al. 2008]. The definition can be phrased as follows:

Definition 12.1. A finitely generated kE-module is said to have *constant Jordan type* if every nonzero linear combination of $X_1, \dots, X_r$ has the same Jordan canonical form on M.

It is a remarkable fact that if a module M satisfies this definition then every element of $J(kE) \setminus J^2(kE)$ has the same Jordan canonical form on M.

Let $\mathcal{O}$ be the structure sheaf of $\mathbb{P}^{r-1} = \mathsf{Proj}\, k[Y_1, \dots, Y_r]$. If M is a finitely generated kE-module, we write $\widetilde{M}$ for $M \otimes_k \mathcal{O}$, a trivial vector bundle of rank equal to $\dim_k(M)$. For each $j \in \mathbb{Z}$ we define a map

$$\theta \colon \widetilde{M}(j) \to \widetilde{M}(j+1)$$

via

$$\theta(m \otimes f) = \sum_{i=1}^{r} X_i m \otimes Y_i f.$$

We then define

$$\mathcal{F}_i(M) = \frac{\mathsf{Ker}\,\theta \cap \mathsf{Im}\,\theta^{i-1}}{\mathsf{Ker}\,\theta \cap \mathsf{Im}\,\theta^{i}} \quad (1 \le i \le p).$$

This is regarded as a subquotient of $\widetilde{M}$, giving a coherent sheaf of modules on $\mathbb{P}^{r-1}$. Namely, when we write $\mathsf{Ker}\,\theta$ we mean $\theta \colon \widetilde{M} \to \widetilde{M}(1)$, and for the images, $\theta^{i-1} \colon \widetilde{M}(-i+1) \to \widetilde{M}$, $\theta^i \colon \widetilde{M}(-i) \to \widetilde{M}$. The relationship between this definition and constant Jordan type is given by the following proposition.

Proposition 12.2. *The kE-module M has constant Jordan type if and only if $\mathcal{F}_i(M)$ is a vector bundle for each $1 \le i \le p$.*

Here, "vector bundle" should be interpreted as "locally free sheaf of $\mathcal{O}_{\mathbb{P}^{r-1}}$-modules". See, for example, Exercise II.5.18 of [Hartshorne 1977].

The following theorem appeared in [Benson and Pevtsova 2012].

Theorem 12.3. *Let $\mathcal{F}$ be a vector bundle on $\mathbb{P}^{r-1}$. Then there exists a finitely generated kE-module M of constant Jordan type, with all Jordan blocks of length one or p, such that*

(1) *if $p = 2$ then $\mathcal{F}_1(M) \cong \mathcal{F}$, and*

(2) *if p is odd, then $\mathcal{F}_1(M) \cong F^*(\mathcal{F})$, the inverse image of $\mathcal{F}$ along the Frobenius morphism $F \colon \mathbb{P}^{r-1} \to \mathbb{P}^{r-1}$.*

The proof of this theorem was somewhat elaborate, and relied on a construction in $\mathsf{D}^b(kE)$ that mimics a resolution of a module over a polynomial ring, followed by descent to $\mathsf{stmod}(kE)$. The case $p = 2$ is essentially the BGG correspondence. In a sense for p odd it may be regarded as a weak version of the BGG correspondence.

An alternative proof can be given using the equivalence in Theorem 8.3. Namely, given a vector bundle on $\mathbb{P}^{r-1}$, there is a corresponding graded module over $k[y_1, \dots, y_r]$. Make this into an A-module via the map $A \twoheadrightarrow k[y_1, \dots, y_r]$, and let N be a maximal Cohen–Macaulay approximation to this module. Now take $\Psi(N) \in \mathsf{D}^b(kE)$, and look at its image in $\mathsf{stmod}(kE)$. This is the required module M of constant Jordan type, in case p is odd. A careful analysis using Theorem 11.3 of the construction given in [Benson and Pevtsova 2012] shows that it gives an module isomorphic to the one produced using $\Psi(N)$.

Acknowledgments

I'd like to thank Mark Walker for giving a lecture on "Support for complete intersections via higher matrix factorizations" at the 2012 Seattle Conference, "Cohomology and Support in Representation Theory and Related Topics" that inspired my interest in the Orlov correspondence. I'd also like to thank Jesse Burke, Jon Carlson, Srikanth Iyengar, Claudia Miller, Julia Pevtsova and Greg Stevenson for enlightening conversations and feedback about this topic.

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