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Ideals generated by superstandard tableaux

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We investigate products J of ideals of "row initial" minors in the polynomial ring $K[X]$ defined by a generic $m \times n$-matrix. The defining "shape" of J determines a set of "row initial" standard bitableaux that we call superstandard. They form a Gröbner basis of J, and J has a linear minimal free resolution. These results are used to derive a new generating set for the Grothendieck group of finitely generated $T_m \times \mathrm{GL}_n(K)$-equivariant modules over $K[X]$. We employ the Knuth–Robinson–Schensted correspondence and a toric deformation of the multi-Rees algebra that parametrizes the ideals J.

1. Introduction

Let K be a field and X an $m \times n$ matrix of indeterminates x_{ij} over K. We write $R = K[X]$ for the polynomial ring in the x_{ij}. The group $\mathrm{GL}_m(K) \times \mathrm{GL}_n(K)$ acts on R with an action induced by the rule $(g, g') \cdot X = gXg'^{-1}$. The representation theory of R as a module for this group is intimately connected to the linear basis of R given by *bitableaux* [Bruns and Vetter 1988, Chapter 11; de Concini et al. 1980]. The bitableaux are products of minors which are indexed by pairs of tableaux of the same shape with strictly increasing rows and weakly increasing columns. We say that a bitableau is *superstandard* if its left factor tableau has column i filled with the number i. The left tableau determines the row indices of the minors whose product the bitableau represents.

For each i, $1 \le i \le m$, let $J_i \subset R$ denote the ideal generated by the size i minors of the first i rows of X. In the current work we study an arbitrary product of such ideals. For a decreasing sequence of positive integers $\min(m, n) \ge s_1 \ge \cdots \ge s_\nu$ we set $J_S = J_{s_1} \dots J_{s_\nu}$. It is a consequence of Theorem 2.2 that the ideals J_S are exactly those that are generated by superstandard bitableau of shape S.

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Our main results are Theorems 3.3 and 4.7, which we summarize here as follows.

Theorem. (1) *The collection of superstandard bitableaux of shape S in R forms a Gröbner basis for the ideal J_S with respect to a diagonal monomial order.*

(2) *The ideal J_S has a linear minimal free resolution.*

The theorem is supplemented by results on primary decompositions and integral closedness. Statement (1) will be demonstrated in two ways. The first is via the Knuth–Robinson–Schensted correspondence, and this approach, together with a brief introduction to standard bitableaux, the straightening law, and the KRS correspondence, will occupy Sections 2 and 3. The second proof of (1) and the proof of (2) are via Sagbi (or toric) deformations. It will take place in Section 4. The crucial point for (2) is that the multi-Rees algebra of the ideals $J_1, \ldots, J_m$ is a Koszul algebra, and, in its turn, this will be derived from the Koszul property of the initial algebra of the multi-Rees algebra.

The theorem should be viewed as occurring in the greater context of ideals generated by a family of bitableaux possessing natural Gröbner bases [Bruns and Conca 2001; 2003; Conca 1997; Sturmfels 1990]. Nevertheless, the fact that the standard bitableaux in a product ideal like J_S form a Gröbner basis, is a rare phenomenon associated with ideals generated by "maximal" minors. Statement (1) of the theorem is a direct generalization of Conca's result [1997] for rectangular shapes S.

In Section 5 we use statement (1) of the theorem to derive a new generating set for the Grothendieck group of finitely generated $T_m \times \mathrm{GL}_n(K)$-equivariant R-modules, where $T^m \subset \mathrm{GL}_m(K)$ is the torus of diagonal matrices. Having a basis for this group coming from structure sheaves of schemes was the original motivation for studying the class of ideals J_S.

2. The straightening law

Let K be a field and $X = (x_{ij})$ an $m \times n$ matrix of indeterminates x_{ij} over K. We will study determinantal ideals in the polynomial ring $R = K[X] = K[x_{ij} : i = 1, \ldots, m,\ j = 1, \ldots, n]$ generated by all the indeterminates x_{ij}.

Almost all of the approaches one can choose for the investigation of determinantal ideals use standard bitableaux and the straightening law. The principle governing this approach is to consider all the minors of X (and not just the 1-minors x_{ij}) as generators of the K-algebra R so that products of minors appear as "monomials". The price to be paid, of course, is that one has to choose a proper subset of all these "monomials" as a linearly independent K-basis: the standard bitableaux to be defined below are a natural choice for such a basis, and the straightening law tells us how to express an arbitrary product of minors

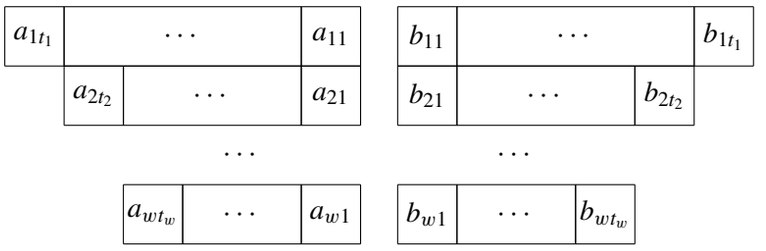

Figure 1. A bitableau.

as a K-linear combination of the basis elements. (In [Bruns and Vetter 1988] standard bitableaux were called *standard monomials*; however, we will have to consider the ordinary monomials in $K[X]$ so often that we reserve the term "monomial" for products of the x_{ij}.)

In the following,

$$[a_1, \dots, a_t \mid b_1, \dots, b_t]$$

stands for the determinant of the submatrix $(x_{a_i b_j} : i = 1, \dots, t,\ j = 1, \dots, t)$.

The letter Δ always denotes a product $\delta_1 \cdots \delta_w$ of minors, and we assume that the sizes $|\delta_i|$ (i. e., the number of rows of the submatrix X' of X such that $\delta_i = \det(X')$) are *descending*: $|\delta_1| \geq \cdots \geq |\delta_w|$. By convention, the empty minor $[\,|\,]$ denotes 1. The *shape* $|\Delta|$ of Δ is the sequence $(|\delta_1|, \dots, |\delta_w|)$. If necessary we may add factors $[\,|\,]$ at the right hand side of the products, and extend the shape accordingly.

A product of minors is also called a *bitableau*. The choice of this term "bitableau" is motivated by the graphical description of a product Δ as a pair of Young tableaux as in Figure 1: Every product of minors is represented by a bitableau and, conversely, every bitableau stands for a product of minors if the length of the rows is decreasing from top to bottom, the entries in each row are strictly increasing from the middle to the outmost box, the entries of the left tableau are in $\{1, \dots, m\}$ and those of the right tableau are in $\{1, \dots, n\}$. These conditions are always assumed to hold.

For formal correctness one should consider the bitableaux as purely combinatorial objects and distinguish them from the ring-theoretic objects represented by them, but since there is no real danger of confusion, we simply identify them.

Whether Δ is a standard bitableau is controlled by a partial order of the minors, namely,

$$[a_1, \dots, a_t \mid b_1, \dots, b_t] \leq [c_1, \dots, c_u \mid d_1, \dots, d_u]$$
$$\iff \quad t \geq u \quad \text{and} \quad a_i \leq c_i,\ b_i \leq d_i,\ i = 1, \dots, u.$$

A product $\Delta = \delta_1 \cdots \delta_w$ is called a *standard bitableau* if

$$\delta_1 \le \cdots \le \delta_w,$$

in other words, if in each column of the bitableau the indices are nondecreasing from top to bottom. The letter Σ is reserved for standard bitableaux.

The fundamental straightening law of Doubilet–Rota–Stein says that every element of R has a unique presentation as a K-linear combination of standard bitableaux (for example, see [Bruns and Vetter 1988]).

Theorem 2.1. (a) *The standard bitableaux are a K-vector space basis of $K[X]$.*

(b) *If the product $\delta_1\delta_2$ of minors is not a standard bitableau, then it has a representation*

$$\delta_1\delta_2 = \sum x_i \varepsilon_i \eta_i, \quad x_i \in K,\ x_i \neq 0,$$

where $\varepsilon_i\eta_i$ is a standard bitableau for all i and $\varepsilon_i < \delta_1, \delta_2 < \eta_i$ (here we must allow that $\eta_i = 1$).

(c) *The standard representation of an arbitrary bitableau Δ, i.e., its representation as a linear combination of standard bitableaux Σ, can be found by successive application of the straightening relations in* (b).

Let $e_1, \ldots, e_m$ and $f_1, \ldots, f_n$ denote the canonical $\mathbb{Z}$-bases of $\mathbb{Z}^m$ and $\mathbb{Z}^n$ respectively. Clearly $K[X]$ is a $\mathbb{Z}^m \oplus \mathbb{Z}^n$-graded algebra if we give x_{ij} the "vector bidegree" $e_i \oplus f_j$. All minors are homogeneous with respect to this grading. In a bitableau of bidegree

$$(c_1, \ldots, c_m, d_1, \ldots, d_n) \in \mathbb{Z}^m \oplus \mathbb{Z}^n,$$

row i appears with multiplicity c_i, and column j appears with multiplicity d_j, $i = 1, \ldots, m$, $j = 1, \ldots, n$. The straightening relations must therefore preserve these multiplicities, whose collection is often called the *content* of the bitableau.

We say that an ideal $I \subset R$ has a *standard basis* if I is the K-vector space spanned by the standard bitableaux $\Sigma \in I$.

Let $S = s_1, \ldots, s_v$ be weakly decreasing sequence of positive integers $s_i \le \min(m, n)$. In this article we investigate the ideal

$$J_S = J_{s_1} \cdots J_{s_v},$$

where J_t is the ideal generated by the t-minors of the first t rows of X. In other words, J_t is the ideal of maximal minors of the matrix X_t formed by the first t rows of X in $K[X_t]$ and extended to $K[X]$. We will see that the ideals J_S behave very much like the powers of ideals of maximal minors that they generalize in a natural way.

The bitableaux $\Delta = \delta_1 \cdots \delta_v$ with $\delta_i \in J_{s_i}$, $|\delta_i| = s_i$, are automatically standard on the left side (the tableau of row indices). We call them *row superstandard* and just *superstandard* if they are also standard on the right side. Note that in a (row) superstandard bitableau all indices a_{ij} are as small as possible, namely $a_{ij} = j$. In [Bruns and Vetter 1988] superstandard tableaux are called *row initial*, but we want to reserve the term "initial" for use in connection with monomial orders.

Let $\Delta = \delta_1 \cdots \delta_u$ and $\Delta' = \delta'_1 \cdots \delta'_w$ be bitableaux. We say that Δ' is a *subtableau* of Δ if $w \le u$, $|\delta'_i| \le |\delta_i|$ for $i = 1, \ldots, w$ and, with $s = |\delta_i|$, $t = |\delta'_i|$, and $\delta = [a_{i1} \ldots a_{is} \mid b_{i1} \ldots b_{is}]$ one has

$$\delta'_i = [a_{i1} \ldots a_{it} \mid b_{i1} \ldots b_{it}],$$

for $i = 1, \ldots, w$. Subtableaux of (super)standard bitableaux are evidently (super)standard.

Theorem 2.2. *The ideal J_S has a standard basis that is given by all standard bitableaux containing a superstandard tableau of shape S.*

Proof. As a vector space over K, J_S is certainly generated by all products

$$\delta_1 \cdots \delta_w, \quad w \ge v,$$

such that $\delta_i = [1 \ldots s_i \ldots \mid \ldots]$ for $i = 1, \ldots, v$. (We do not assume that the δ_i are ordered by size.) It is enough to show that this property is preserved by all products of minors that arise if we replace an incomparable subproduct $\delta_i \delta_j$ by the right hand side of the straightening relation.

Let

$$\delta_i = [1 \ldots s_i \ldots \mid \ldots] \quad \text{and} \quad \delta_j = [1 \ldots s_i \ldots \mid \ldots],$$

where we have set $s_j = 0$ if $j > v$. It is immediately clear that the first factor ε of each summand on the right hand side of the straightening relation must be of type $[1 \ldots s_i \ldots \mid \ldots]$ since $\varepsilon \le \delta_i$, and since no index is lost on the right hand side, the second factor satisfies $\eta = [1 \ldots s_j \ldots \mid \ldots]$.

After finitely many steps we arrive at a K-linear combination of standard bitableaux, each of which contains a superstandard tableau of shape S. □

The description of the standard basis yields the primary composition of the ideals J_S as an easy consequence:

Corollary 2.3. *Write $\{s_1, \ldots, s_v\} = \{t_1, \ldots, t_u\}$ with $t_1 > \cdots > t_u$ and set $e_i = \max\{j : s_j = t_i\}$. Then*

$$J_S = \bigcap_{i=1}^{u} J_{t_i}^{e_i}$$

is an irredundant primary decomposition, and J_S is an integrally closed ideal.

Proof. The ideals on both sides have a standard basis as follows from the theorem. Therefore it is enough to compare these. But a standard bitableau contains a superstandard bitableau of shape S if and only if it contains a rectangular superstandard bitableau with e_i rows of length t_i for every i, and the latter form the standard basis of $J_{t_i}^{e_i}$ by the theorem.

Comparing standard bases once more, we see that none of the $J_{t_i}^{e_i}$ is contained in the intersection of the others.

Finally, it remains to observe that the ideals $J_{t_i}^{e_i}$ are primary. But $J_{t_i}^{e_i}$ arises from $I_{t_i}(X_{t_i})^{e_i}$ by tensoring over K with the polynomial ring in the variables x_{kl} outside X_{t_i}, and such extensions preserve the property of being primary. That the powers of $I_{t_i}(X_{t_i})$ are primary is well-known; see [Bruns and Vetter 1988, 9.18].

For the last statement it is enough to note that the powers $J_{t_i}^{e_i}$ are not only primary, but also integrally closed. This follows from the normality of the Rees algebra $\mathcal{R}(J_{t_i})$ [Bruns and Vetter, 9.17]. □

The statement on integral closedness is equivalent to the normality of a multi-Rees algebra. We postpone this aspect until Theorem 4.7.

3. The Knuth–Robinson–Schensted correspondence

Let Σ be a standard bitableau. The Knuth–Robinson–Schensted correspondence KRS (see [Fulton 1997] or [Stanley 1999]) sets up a bijective correspondence between standard bitableaux and monomials in the ring $K[X]$. The treatment of KRS below follows [Bruns and Conca 2001; 2003]. However, for better compatibility with the definition of the ideals J_S we have exchanged the roles of the left and right tableau.

If one starts from bitableaux, the correspondence is constructed from the algorithm *KRS-step* [Bruns and Conca 2003, 4.2] (based on *deletion* [Bruns and Conca, 4.1]). Let $\Sigma = (a_{ij}|b_{ij})$ be a nonempty standard bitableau. The output of KRS-step is a triple (Σ', ℓ, r) consisting of a standard bitableau Σ' and a pair of integers (ℓ, r) constructed as follows.

(a) One chooses the largest entry r in the *right* tableau of Σ; suppose that $\{(i_1, j_1), \ldots, (i_u, j_u)\}$, $i_1 < \cdots < i_u$, is the set of indices (i, j) such that $r = b_{ij}$. (Note that $j_1 \geq \cdots \geq j_u$.)

(b) Then the boxes at the *pivot position* $(p, q) = (i_u, j_u)$ in the right and the left tableau are removed.

(c) The entry $r = b_{pq}$ of the removed box in the right tableau is the third component of the output, and a_{pq} is stored in s, an auxiliary memory cell.

(d) The first and the second component of the output are determined by a "push out" procedure on the *left* tableau as follows:

(i) if $p = 1$, then $\ell = s$ is the second component of the output, and the first is the standard bitableau Σ' that has now been created;

(ii) otherwise s is moved one row up and pushes out the left most entry a_{p-1k} such that $a_{p-1k} \le s$ whereas a_{p-1k} is stored in s.

(iii) one replaces p by $p-1$ and goes to step (i).

It is now possible to define KRS recursively: One sets $\mathrm{KRS}([\,|\,]) = 1$, and $\mathrm{KRS}(\Sigma) = \mathrm{KRS}(\Sigma')x_{\ell r}$ for $\Sigma \neq [\,|\,]$.

There is an inverse to deletion, called insertion that can be easily constructed by inverting all steps in deletion. Together they prove the main theorem on KRS:

Theorem 3.1. *The map* KRS *is a bijection between the set of standard bitableaux on* $\{1, \dots, m\} \times \{1, \dots, n\}$ *and the monomials of* $K[X]$.

For insertion one must order the factors of a monomial in a way that respects the monotonicity properties of KRS-step: let $x_{r_1\ell_1} \cdots x_{r_k\ell_k} = \mathrm{KRS}(\Sigma)$ with the factors ordered as in the definition of KRS; then

$$r_i \le r_{i+1} \quad \text{and} \quad r_i = r_{i+1} \Longrightarrow \ell_i \ge \ell_{i+1}. \tag{$*$}$$

See [Bruns and Conca 2003, p. 37] (with r and ℓ exchanged). Property $(*)$ allows us to take care of a superstandard subtableau, but some additional bookkeeping is necessary. To this end we extend the output of KRS-step by a further component ρ, the *row mark* that we will now define. (Here "row" refers to the tableau, not to a minor.)

Let $S = s_1, \dots, s_v$ a nonincreasing sequence as above, and suppose that Σ contains a superstandard bitableau of shape S. Then we can distinguish boxes in the left tableau that belong to the superstandard bitableau from those that do not belong to it, namely the box at position (i, j) belongs to the superstandard subtableau if and only if $a_{ij} = j$ and $j \le s_i$. We supplement step (d) above by

(iv) if $a_{ij} = j$ and $j \le s_i$, but (i, j) is the pivot position or $a'_{ij} > a_{ij}$, then $\rho = i$ is the fourth component of the output of KRS-step. Otherwise we set $\rho = 0$.

Let us first make sure that rule (iv) makes sense by showing that there can be at most one row i with $a_{ij} = j$ and $a'_{ij} > a_{ij}$. This is clear if (i, j) is the pivot position since all remaining positions remain unchanged. In the other case, if $a_{ij} = j$ and $a'_{ij} > a_{ij}$, then $i = \max\{k : a_{kj}\} = j$. In fact, if the box at position (i, j) is hit by the push out sequence in KRS-step(d) and $a_{ij} = j$, then the entry j is pushed out into the next upper row and replaces $a_{i-1j} = j$ by j.

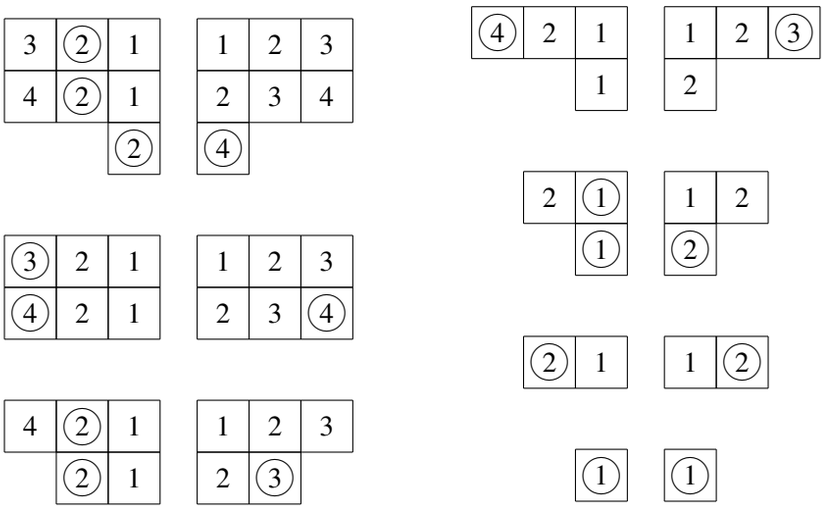

Figure 2. The KRS algorithm.

The triples (ℓ, r, ρ) form the columns of a three row array $\mathrm{krs}(\Sigma)$ that we build by listing the triples (ℓ, r, ρ) from right to left as follows:

$$\mathrm{krs}(\Sigma) = \mathrm{krs}(\Sigma') \begin{pmatrix} \ell \\ r \\ \rho \end{pmatrix}.$$

We give an example in Figure 2 with $S = 3, 2$: the circles in the right tableau mark the pivot position, those in the left mark the chains of "pushouts".

The three row array produced by the example of Figure 2 is

$$\mathrm{krs}(\Sigma) = \begin{pmatrix} 1 & 2 & 1 & 4 & 2 & 3 & 2 \\ 1 & 2 & 2 & 3 & 3 & 4 & 4 \\ 1 & 1 & 2 & 0 & 2 & 1 & 0 \end{pmatrix},$$

and

$$\mathrm{KRS}(\Sigma) = x_{11}x_{22}x_{12}x_{23}x_{43}x_{34}x_{24}.$$

Let us extract the subarrays with row marks 1 and 2:

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 2 & 2 \end{pmatrix}.$$

The product of the corresponding monomials

$$x_{11}x_{22}x_{34} \quad \text{and} \quad x_{12}x_{23}$$

is the KRS image of a superstandard bitableau of shape $(3, 2)$ (though it is not the KRS image of the superstandard subtableau contained in Σ). What we have observed in this special case is always true, as we will be stated in Lemma 3.2 below.

Let

$$\operatorname{diag}[a_1 \dots a_t \mid b_1 \dots b_t] = \prod_{i=1}^{t} x_{a_i b_i}$$

be the product of the indeterminates in the main diagonal of $[a_1 \dots a_t \mid b_1 \dots b_t]$. If $\Delta = \delta_1 \cdots \delta_w$ is an arbitrary bitableau, then we set

$$\operatorname{diag}(\Delta) = \prod_{i=1}^{w} \operatorname{diag}(\delta_i).$$

It is easy to see that the map diag is not injective on standard bitableaux (let alone all bitableaux), in contrast to KRS. (Otherwise KRS would be completely superfluous in the study of determinantal ideals.) However, if Σ is a superstandard bitableau, then

$$\operatorname{diag}(\Sigma) = \operatorname{KRS}(\Sigma) \tag{3-1}$$

since the whole push out sequence in KRS-step(d) always replaces the entry of a box by itself. (One can also use the symmetry of KRS correspondence; see [Bruns and Conca 2003, Remark 4.7(a)].)

Lemma 3.2. *Let Σ be a standard bitableau containing a superstandard bitableau of shape S. Then there exists a superstandard bitableau* T *of shape S such that* diag(T) *divides* KRS(Σ).

Proof. Suppose T is a (not necessarily standard) bitableau whose row tableau is superstandard of shape S. Then $\operatorname{diag}(\mathrm{T}) = \operatorname{diag}(\mathrm{T}')$ where T' is standard of shape S. This is easy to see and left to the reader. Therefore it is enough to prove the lemma without the requirement that T is standard. (Equation (3-1) would allow us to replace diag(T) by KRS(T), but this is irrelevant.)

Let $\Sigma = (a_{ij} \mid b_{ij})$ and choose an index k such that row k of Σ occurs in the superstandard subtableau. Let $s = \max\{j : a_{kj} = j\}$. As in the example we extract the subarray A from $\operatorname{krs}(\Sigma)$ with row mark k. We claim that the corresponding monomial is the diagonal of an s-minor $[1 \dots s \mid c_1 \dots c_s]$. This claim amounts to the following conditions for the subarray A:

(1) The entries of the first row are $1, \dots, s$ in ascending order;

(2) the entries $c_1, \dots, c_s$ of the second row are strictly increasing.

First of all we note that the row mark is k if a box at position (k, z) with $a_{kz} = z$ changes its content in KRS-step: either (k, z) is the pivot position

or in $\Sigma' = (a'_{ij} \mid b'_{ij})$ one has $a'_{kz} > z$. This change happens exactly once for $j = 1, \dots, s$. Therefore the entries of the first row of A are indeed $1, \dots, s$.

But $1, \dots, s$ are also produced in the right order. If $a_{kz} = z$, then $a_{kw} = w$ for $w = 1, \dots, z-1$, and so these boxes have not yet changed content. Moreover the component r of the output of KS-step is exactly z, and $1, \dots, z-1$ will be produced later. This proves (1).

The entries $c_1, \dots, c_s$ in the second row are automatically weakly increasing by the first inequality in $(*)$, and an equality of two entries would contradict (1) because of the second inequality in $(*)$. In other words, (1) implies (2). □

It is now time to introduce a *diagonal monomial* (or *term*) *order* $\prec$ on the polynomial ring $K[X]$. This is a term order on the polynomial ring under which the initial monomial of each minor is the product of the elements in the main diagonal:

$$\operatorname{in}_\prec[a_1 \dots a_t \mid b_1 \dots b_t] = \operatorname{diag}[a_1 \dots a_t \mid b_1 \dots b_t].$$

Diagonal monomial orders are the standard choice in the study of determinantal ideals from the Gröbner basis viewpoint. See [Bruns and Conca 2003] for a survey that also contains a brief introduction to general Gröbner bases and initial ideals.

Theorem 3.3. *Let $S = s_1 \dots s_u$ be a nonincreasing sequence. Then the following hold*:

(1) *the row superstandard bitableaux of shape S form a Gröbner basis of J_S.*

(2) *In particular*, $\operatorname{in}_\prec(J_S) = \operatorname{KRS}(J_S)$.

(3) *Furthermore*, $\operatorname{in}_\prec(J_S) = \prod \operatorname{in}_\prec(J_{s_i})$.

(4) *And lastly* $\operatorname{in}_\prec(J_S) = \bigcap_{i=1}^u \operatorname{in}_\prec(J_{t_i}^{e_i}) = \bigcap_{i=1}^u \operatorname{in}_\prec(J_{t_i})^{e_i}$ *where the sequences $\{t_1, \dots, t_u\}$ and $e_1, \dots, e_u$ are defined as in Corollary 2.3.*

Proof. Claims (1) and (2) result immediately from Lemma 3.2 and [Bruns and Conca 2003, Lemma 5.2].

Since $\prod \operatorname{in}_\prec(J_{s_i}) \subset \operatorname{in}_\prec(J_S)$ for obvious reasons, it is enough to observe the converse for (3). But this follows again from Lemma 3.2 since $\operatorname{in}_\prec(\mathrm{T})$ is contained in $\prod \operatorname{in}_\prec(J_{s_i})$.

In the terminology of [Bruns and Conca 2001; 2003], claim (2), applied to the sequence $t_i, \dots, t_i$ (e_i repetitions), says that the ideal $J_{t_i}^{e_i}$ are in-KRS, and for in-KRS ideals the formation of initial ideals commutes with intersection; see [Bruns and Conca 2003, Lemma 5.2]. So it remains to use Corollary 2.3. □

4. Sagbi deformation

Sagbi bases are the *subalgebra analog of Gröbner bases for ideals*. They have been introduced by Robbiano and Sweedler [1990]. Conca, Herzog and Valla [1996] showed how to use Sagbi bases and Sagbi deformation (also called toric deformation) in the study of homological properties of subalgebras of polynomials rings and, in particular, to Rees algebras.

In this section we will use Sagbi deformations of Rees algebras to study the ideals J_S defined in the previous sections. By definition, these ideals are products of powers of the ideas $J_1, \ldots, J_m$ (we do not assume that $n \geq m$; if $m > n$ then all results in this section hold with $J_{n+1} = 0, \ldots, J_m = 0$.)

Before we turn to our class of ideals we study the Sagbi approach via Rees algebras in general. Let $A = K[x_1, \ldots, x_r]$ the polynomial ring in r indeterminates, endowed with a monomial order $\prec$. For every K-vector subspace V of A we may consider the vector space $\mathrm{in}_\prec(V)$ generated by the monomials $\mathrm{in}_\prec(f)$ as $f \neq 0$ varies in V. If V is an ideal of A, then $\mathrm{in}_\prec(V)$ will be an ideal of A, and if V is a K-subalgebra of A, then $\mathrm{in}_\prec(V)$ will be a K-subalgebra of A as well. If V is an ideal, then a subset G of V is a Gröbner basis if $\mathrm{in}_\prec(V)$ is generated (as an ideal) by $\{\mathrm{in}_\prec(f) : f \in G\}$. Similarly, if V is an algebra, then a subset G of V is a *Sagbi basis* if $\mathrm{in}_\prec(V)$ is generated (as a K-algebra) by $\{\mathrm{in}_\prec(f) : f \in G\}$. A variation of the Buchberger criterion allows us to detect whether a given set G of polynomials is a Sagbi basis. One has to replace the so called S-pairs with the binomial relations defining the toric ring $K[\mathrm{in}_\prec(f) : f \in G]$. We refer the reader to [Conca et al. 1996] for further details.

Let now $I_1, \ldots, I_v$ homogeneous ideals of A. We want to express the condition

$$\mathrm{in}_\prec(I_1^{a_1} \cdots I_v^{a_v}) = \mathrm{in}_\prec(I_1)^{a_1} \cdots \mathrm{in}_\prec(I_v)^{a_v} \quad \text{for all } (a_1, \ldots, a_v) \in \mathbb{N}^v \tag{4-1}$$

in terms of Sagbi deformations. Let

$$\mathcal{R}(I_1, \ldots, I_v) = \bigoplus_{a \in \mathbb{N}^v} I_1^{a_1} \cdots I_v^{a_v}$$

be the (multi-)Rees ring $\mathcal{R}(I_1, \ldots, I_v)$ associated to the family $I_1, \ldots, I_v$. In order to describe it as a subalgebra of a polynomial ring, we take new variables $y_1, \ldots, y_v$. Then we can identify $\mathcal{R}(I_1, \ldots, I_v)$ with the subalgebra

$$A[I_1 y_1, \ldots, I_v y_v] \subset A[y] = A[y_1, \ldots, y_v].$$

By construction, $\mathcal{R}(I_1, \ldots, I_v)$ has a $\mathbb{Z} \oplus \mathbb{Z}^v$-graded structure induced by the assignment $\deg(x_i) = e_0$ for all i and $\deg(y_j) = e_j$ for all j where $e_0, e_1, \ldots, e_v$ denotes the canonical basis of $\mathbb{Z} \oplus \mathbb{Z}^v$.

We extend $\prec$ to a monomial order on $K[x, y]$. It is indeed irrelevant which extension is chosen because the polynomials we will consider are "monomials" in the y's and so we denote the extension by $\prec$ as well.

Then

$$\operatorname{in}_\prec(\mathcal{R}(I_1, \ldots, I_v)) = \bigoplus_{a \in \mathbb{N}^v} \operatorname{in}_\prec(I_1^{a_1} \cdots I_v^{a_v}),$$

and hence (4-1) holds if and only if

$$\operatorname{in}_\prec(\mathcal{R}(I_1, \ldots, I_v)) = \mathcal{R}(\operatorname{in}_\prec(I_1), \ldots, \operatorname{in}_\prec(I_v)). \tag{4-2}$$

Condition (4-2) can be expressed in terms of Sagbi basis.

For every i let $F_{i1}, \ldots, F_{ic_i}$ a Gröbner basis of I_i with respect to $\prec$. As a K-algebra, the Rees ring $\mathcal{R}(I_1, \ldots, I_v)$ is generated by two sets of polynomials:

(1) $X = \{x_1, \ldots, x_r\}$, and

(2) $\mathcal{F} = \{F_{ij} y_i : i = 1, \ldots, v \text{ and } j = 1, \ldots, c_i\}$.

Condition (4-2) is equivalent to the statement

$$X \cup \mathcal{F} \text{ is a Sagbi basis with respect to } \prec. \tag{4-3}$$

To test whether condition (4-3) holds we can use the Sagbi variant of the Buchberger criterion [Conca et al. 1996]. Set

$$M_{ij} = \operatorname{in}_\prec(F_{ij}).$$

and consider two A-algebra maps from the polynomial ring

$$P = A[p_{ij} : i = 1, \ldots, v, \ \ j = 1, \ldots, c_i]$$

to $A[y]$ defined as follows:

$$\Phi(p_{ij}) = M_{ij} y_i \quad \text{and} \quad \Psi(p_{ij}) = F_{ij} y_i.$$

By construction

$$\operatorname{Im} \Phi = \mathcal{R}(\operatorname{in}_\prec(I_1), \ldots, \operatorname{in}_\prec(I_v)) \quad \text{and} \quad \operatorname{Im} \Psi = \mathcal{R}(I_1, \ldots, I_v).$$

The kernel of Φ is a toric ideal, i.e., a prime ideal generated by binomials since $\mathcal{R}(\operatorname{in}_\prec(I_1), \ldots, \operatorname{in}_\prec(I_v))$ is a K-algebra generated by monomials. These binomials replace the S-pairs in the Buchberger criterion for Gröbner bases. Roughly speaking, the following criterion says that every such binomial relation of the initial monomials can be "lifted" to a relation of the elements of G themselves.

Lemma 4.1 (Sagbi version of the Buchberger criterion). *Let G be a set of binomials generating* $\operatorname{Ker}\Phi$. *Suppose that for every $g \in G$ such that $\Psi(g) \neq 0$ one has*

$$\Psi(g) = \sum \lambda_{a,b} X^a \mathcal{F}^b,$$

where $\lambda_{a,b} \in K^*$, and $X^a\mathcal{F}^b$ is a monomial in the set $X \cup \mathcal{F}$ such that

$$\operatorname{in}_\prec(X^a\mathcal{F}^b) \preceq \operatorname{in}_\prec(\Psi(g)) \quad \textit{for all } a, b.$$

Then $X \cup \mathcal{F}$ is a Sagbi basis.

Remark 4.2. If g has total degree 1 in the p_{ij}'s, then the condition required in Lemma 4.1 is automatically satisfied because $F_{i1}, \ldots, F_{ic_i}$ is a Gröbner basis of the ideal I_i. So we have only to worry about the $g \in G$ of degree > 1 in the p_{ij}'s.

Assume now that each ideal I_i is generated in a single degree, say d_i. Then $\mathcal{R}(I_1, \ldots, I_v)$ can be given the structure of a standard $\mathbb{Z} \oplus \mathbb{Z}^v$-graded K-algebra by assigning the degree $e_j - d_j e_0$ to y_j, $j = 1, \ldots, v$, and e_0 to the variables x_i. On P we define the grading by $\deg(x_i) = e_0$ and $\deg(p_{ij}) = e_i$. Then the maps Φ and Ψ are $\mathbb{Z} \oplus \mathbb{Z}^v$-graded.

The following theorem relates a ring theoretic property of the Rees algebra to the free resolutions of the ideals involved:

Theorem 4.3 (Blum). *If each I_i is generated in a single degree and $\mathcal{R}(I_1, \ldots, I_v)$ is a Koszul algebra* (*for example, it is defined by a Gröbner basis of quadrics*) *then $I_1^{a_1} \cdots I_v^{a_v}$ has a linear resolution for all $a_1, \ldots, a_v \in \mathbb{N}$.*

This was proved by Blum [2001, Corollary 3.6] for $v = 1$, but the proof generalizes immediately to the multigraded setting.

Now we return to the family of determinantal ideals we are interested in. Let $R = K[X]$ where $X = (x_{ij})$ is an $m \times n$-matrix of indeterminates as introduced in Section 2. For the ideals of minors considered in this article, the equality (4-1) is part of Theorem 3.3, but it will be proved independently by the Sagbi approach. Recall that, by definition, for $t = 1, \ldots, m$ we denote by J_t is the ideal generated by the t-minors of the first t rows of X. For a nonincreasing sequence $S = s_1, \ldots, s_v$ the ideal $J_S = J_{s_1} \cdots J_{s_v}$ can be written as a product of powers of the ideals J_t, and in this section it is more convenient to use the latter representation. To simplify notation we omit the row indices in a superstandard tableau by setting

$$[a_1 \ldots a_s] = [1 \ldots s \mid a_1 \ldots a_s].$$

We know from [Sturmfels 1990] that the minors $[a_1 \ldots a_s]$ are a Gröbner basis of J_s with respect to a diagonal monomial order. For the application of Lemma 4.1 below we set

(1) $X = \{x_{ij} : 1 \le i \le m \text{ and } 1 \le j \le n\}$,

(2) $\mathcal{F} = \{[a_1, \dots, a_s]y_s : 1 \le s \le m \text{ and } 1 \le a_1 < \cdots < a_s \le n\}$.

Let

$$\mathcal{A} = \{[a_1 \dots a_s] : \ 1 \le s \le m \text{ and } 1 \le a_1 < \cdots < a_s \le n\}.$$

The set $\mathcal{A}$ inherits the partial order from the set of all minors that has been introduced for the straightening law (see Section 2). The set of all minors is a distributive lattice with respect to this order, and $\mathcal{A}$ is a sublattice: suppose that $r \le s$; to wit,

$$\begin{aligned}[a_1 \dots a_s] \wedge [b_1 \dots b_r] \\ &= [\min(a_1, b_1), \min(a_2, b_2), \dots, \min(a_r, b_r), a_{r+1}, \dots, a_s], \\ [a_1 \dots a_s] \vee [b_1 \dots b_r] \\ &= [\max(a_1, b_1), \max(a_2, b_2), \dots, \max(a_r, b_r)].\end{aligned}$$

For $a = [a_1 \dots a_s] \in \mathcal{A}$ we set

$$m_a = \text{in}([a_1 \dots a_s]) = \text{diag}[a_1 \dots a_s].$$

For each $a \in \mathcal{A}$ we introduce an indeterminate p_a and consider the R-algebra map

$$\Phi : R[p_a : a \in \mathcal{A}] \to R[y_1, \dots, y_n], \quad \Phi(p_a) = m_a y_s.$$

Proposition 4.4. $\text{Ker}\,\Phi$ *is generated by*

(1) *the Hibi relations*

$$\underline{p_a p_b} - p_{a\wedge b} p_{a\vee b}$$

with $a, b \in \mathcal{A}$ *incomparable, and*

(2) *the relations of degree* 1 *in the* p*'s — more precisely, relations of the form*

$$\underline{x_{ij} p_a} - x_{ik} p_b$$

with $a = [a_1 \dots a_i, \dots, a_s]$, $a_{i-1} < j \le a_i$ *and* $b = a \setminus \{a_i\} \cup \{j\}$.

These polynomials form a Gröbner basis of $\text{Ker}\,\Phi$ *with respect to every monomial order in which the underlined terms are initial.*

Proof. It is enough to prove that the given elements are a Gröbner basis of $\text{Ker}\,\Phi$. The argument is quite standard (see, for example, [Sturmfels 1996, Chapter 14] for similar statements) and so we just sketch it. First note that a monomial order selecting the underlined monomials is given by taking the reverse lexicographic order associated to a total order on the p_a's that refines the partial order $\mathcal{A}$.

To prove the assertion we choose an arbitrary monomial in the image of Φ, say

$$wy_{s_1}\cdots y_{s_e}, \quad s_1 \geq \cdots \geq s_e, \ w \text{ a monomial in the } x_{ij}\text{'s},$$

and check that the preimage $\Phi^{-1}(wy_{s_1}\cdots y_{s_e})$ contains exactly one monomial of the form

$$up_{a_1}\cdots p_{a_e}$$

with $|a_i| = s_i$ for $i = 1, \ldots, e$ and a monomial u in the x_{ij}'s such that

(i) $a_1 \leq a_2 \leq \cdots \leq a_e$ in the poset $\mathcal{A}$;

(ii) for every x_{ij} dividing u and for every k, $1 \leq k \leq e$, one has either $j \geq a_{k,i}$ or $j \leq a_{k,i-1}$ where $a_k = \{a_{k,1}, \ldots, a_{k,s_k}\}$ and, by convention, $a_{k,0} = 0$.

To check the claim one observes that a_1 is determined uniquely as the minimum of the $b \in \mathcal{A}$ such that $|b| = s_1$ and $m_b|w$, then a_2 is the minimum of the $b \in \mathcal{A}$ such that $|b| = s_2$ and $m_{a_1}m_b|w$ and so on. □

Remark 4.5. For every finite lattice L one may consider the ring

$$K[L] = K[x : x \in L]/(xy - (x \wedge y)(x \vee b) : x, y \in L).$$

Hibi [1987] proved in that $K[L]$ is a domain if and only if L is distributive and in that case $K[L]$ turns out to be (isomorphic to) a normal semigroup ring. When L is a distributive lattice $K[L]$ is called the Hibi ring of L. That is why the elements $p_a p_b - p_{a\wedge b}p_{a\vee b}$ in Proposition 4.4 are called Hibi relations. In our setting the Hibi ring associated to $\mathcal{A}$ coincides with the multi-graded coordinate ring of flag variety associated to the sequence $1, 2, \ldots, m$ and also with the special fiber $\mathcal{R}/(x_{ij})\mathcal{R}$ of the multi-Rees algebra $\mathcal{R}(J_1, \ldots, J_m)$.

Example 4.6. For $m = n = 4$ the generators of $\operatorname{Ker}\Phi$ are

$$\begin{array}{lll}
x_{1,3}p_4 - x_{1,4}p_3, & x_{1,2}p_4 - x_{1,4}p_2, & x_{1,1}p_4 - x_{1,4}p_1,\\
x_{1,2}p_3 - x_{1,3}p_2, & x_{1,1}p_3 - x_{1,3}p_1, & x_{1,1}p_2 - x_{1,2}p_1,\\
x_{2,3}p_{24} - x_{2,4}p_{23}, & x_{2,3}p_{14} - x_{2,4}p_{13}, & x_{2,2}p_{14} - x_{2,4}p_{12},\\
x_{1,2}p_{34} - x_{1,3}p_{24}, & x_{1,1}p_{34} - x_{1,3}p_{14}, & x_{1,1}p_{24} - x_{1,2}p_{14},\\
x_{2,2}p_{13} - x_{2,3}p_{12}, & x_{1,1}p_{23} - x_{1,2}p_{13}, & x_{3,3}p_{124} - x_{3,4}p_{123},\\
x_{2,2}p_{134} - x_{2,3}p_{124}, & x_{1,1}p_{234} - x_{1,2}p_{134}, & p_{34}p_2 - p_{24}p_3,\\
p_{34}p_1 - p_{14}p_3, & p_{24}p_1 - p_{14}p_2, & p_{23}p_1 - p_{13}p_2,\\
p_{14}p_{23} - p_{13}p_{24}, & p_{234}p_1 - p_{134}p_2, & p_{234}p_{14} - p_{134}p_{24},\\
p_{234}p_{13} - p_{134}p_{23}, & p_{234}p_{12} - p_{124}p_{23}, & p_{134}p_{12} - p_{124}p_{13}.
\end{array}$$

Now we have collected all arguments for our main result.

Theorem 4.7. (1) *The set $X \cup \mathcal{F}$ is a Sagbi basis of the multi-Rees algebra $\mathcal{R}(J_1, \dots, J_m)$.*

(2) *For all $a_1, \dots, a_m \in \mathbb{N}$ we have*

$$\operatorname{in}_\prec(J_1^{a_1} \cdots J_m^{a_m}) = \operatorname{in}_\prec(J_1)^{a_1} \cdots \operatorname{in}_\prec(J_m)^{a_m},$$

and $J_1^{a_1} \cdots J_m^{a_m}$ has a linear resolution.

(3) *$\mathcal{R}(J_1, \dots, J_m)$ is a normal and Koszul domain.*

Proof. (1) The first statement follows from Proposition 4.4, Lemma 4.1 and Remark 4.2, provided we can "lift" the Hibi relations. For incomparable $a, b \in \mathcal{A}$ consider the nonstandard product $[a][b]$. In its standard representation we have only standard monomials with the same shape. A standard monomial with superstandard row tableau can be reconstructed from its initial (diagonal) term and the only standard monomial with superstandard row with initial term equal to that of $[a][b]$ is $[a \wedge b][a \vee b]$. It follows that $[a \wedge b][a \vee b]$ appears in the standard representation of $[a][b]$ and all the other standard monomials have leading term strictly smaller than that of $[a][b]$. This shows that the Hibi relations lifts.

(2) The equation $\operatorname{in}_\prec(J_1^{a_1} \cdots J_m^{a_m}) = \operatorname{in}_\prec(J_1)^{a_1} \cdots \operatorname{in}_\prec(J_m)^{a_m}$ has already been stated in Theorem 3.3, but it follows again from the equivalence of (4-1) and (4-3).

Note that Theorem 3.3 conversely implies the liftability of the Hibi relations since it shows that $X \cup \mathcal{F}$ is a Sagbi basis.

The algebra $\mathcal{R}(\operatorname{in}_\prec(J_1), \dots, \operatorname{in}_\prec(J_m))$ is Koszul since it is defined by a Gröbner basis of quadrics as stated in Proposition 4.4. But

$$\operatorname{in}_\prec(\mathcal{R}(J_1, \dots, J_m)) = \mathcal{R}(\operatorname{in}_\prec(J_1), \dots, \operatorname{in}_\prec(J_m)),$$

and the Koszulness of $\mathcal{R}(J_1, \dots, J_m)$ is a consequence of the preservation of Koszulness under Sagbi deformation [Bruns and Conca 2003, 3.14]. This proves part of (3) and Theorem 4.3 implies that the ideals $J_1^{a_1} \cdots J_m^{a_m}$ have a linear resolution.

(3) Only the normality of the multi-Rees algebra is still open. To this end one can apply the preservation of normality under Sagbi deformation [Bruns and Conca 2003, 3.12] and apply [Sturmfels 1996, Proposition 13.15] which implies that $\operatorname{in}_\prec(\mathcal{R}(J_1, \dots, J_m))$ is normal since its defining ideal has a square-free initial ideal. □

5. Equivariant R-modules

In this section we make the assumption that $m \geq n$. This will simplify the conclusion of main result of the section, which has a less pleasing analogue when $m < n$.

Let $T^m \subset \mathrm{GL}_m(K)$ denote the diagonal torus, and set $G := T^m \times \mathrm{GL}_n(K)$. Then G acts on R as in Section 1. In this section we consider the Grothendieck group of finitely generated G-equivariant R-modules with a rational G-action, denoted $K_G^0(R)$.

Since R is a polynomial ring, the group $K_G^0(R)$ can be identified with the representation ring of G. Hence $K_G^0(R)$ is generated by the free equivariant modules $R \otimes V$, as V ranges over all finite dimensional rational G modules. The group $K_G^0(R)$ inherits a product from the tensor product of G-modules. The product of the classes of two general equivariant R-modules can be expressed in terms of their Tor-modules, a fact we will not need here.

Using the multigrading of Section 2, an equivariant R-module M is at once seen to be a multigraded module. We write its Hilbert series as

$$\mathrm{Hilb}(M) = \sum_{\boldsymbol{a}\oplus\boldsymbol{b}\in\mathbb{Z}^m\oplus\mathbb{Z}^n} \dim_K(M_{\boldsymbol{a}\oplus\boldsymbol{b}})u^{\boldsymbol{a}}v^{\boldsymbol{b}} \in \mathbb{Z}[\![u_1^{\pm 1},\ldots,u_m^{\pm 1},v_1^{\pm 1},\ldots,v_n^{\pm 1}]\!]^{\mathfrak{S}_n}.$$

Here the group $\mathfrak{S}_n$ is permuting the v variables, and the $\mathrm{GL}_n(K)$-invariance of M forces $\mathrm{Hilb}(M)$ to be invariant under this action. The Hilbert series $\mathrm{Hilb}(M)$ can alternately be described as the character of the G-module M. There is a Laurent polynomial $K(M;u,v)$ such that

$$\mathrm{Hilb}(M) = \frac{K(M;u,v)}{\prod_{i=1}^m\prod_{j=1}^n(1-u_iv_j)},$$

and hence we identify the class of a module M in $K_G^0(R)$ with $K(M;u,v)$ [Miller and Sturmfels 2005, Theorem 8.20]. This makes the identification of $K_G^0(R)$ with the representation ring of G explicit:

$$K_G^0(R) = \mathbb{Z}[u_1^{\pm 1},\ldots,u_m^{\pm 1},v_1^{\pm 1},\ldots,v_n^{\pm 1}]^{\mathfrak{S}_n}, \quad M \mapsto K(M;u,v).$$

The superstandard bitableaux of shape S span a representation of G [Bruns and Vetter 1988, Theorem 11.5(a)]. It follows that the ideals J_S are G-invariant, and hence the quotient ring R/J_S defines an element of $K_G^0(R)$. This stands in contrast to an ideal generated by standard bitableaux with a fixed left tableau, which is not necessarily a G-invariant ideal (see [Bruns and Vetter 1988, Remark 11.12]).

Proposition 5.1. *The classes of the modules R/J_S, as S ranges over shapes S with part sizes at most $n-1$, freely generate $K_G^0(R)$ as a module over*

$$\mathbb{Z}[u_1^{\pm 1}, \dots, u_m^{\pm 1}, (v_1 \cdots v_n)^{\pm 1}].$$

Multiplication by $(v_1 \cdots v_n)^{\pm 1}$ corresponds to tensoring with the determinantal character of $\mathrm{GL}_n(K)$ or its dual, and multiplication by a u variable corresponds to tensoring with a character of T.

Proof. It is sufficient to show that the polynomials $K(J_S; u, v)$, as S ranges over all shapes, generate

$$\mathbb{Z}[u_1^{\pm 1}, \dots, u_m^{\pm 1}, v_1^{\pm 1}, \dots, v_n^{\pm 1}]^{\mathfrak{S}_n}$$

as a module over $\mathbb{Z}[u_1^{\pm 1}, \dots, u_m^{\pm 1}, (v_1 \cdots v_n)^{-1}]$. This is because all rational representations of $\mathrm{GL}_n(K)$ are obtained by tensoring polynomial representations with a power of the determinantal representation, and

$$K(R/J_S; u, v) = 1 - K(J_S; u, v).$$

For any shape S whose part sizes are at most n, let $\sigma_S(v)$ denote the *Schur polynomial* in variables $v_1, \dots, v_n$. That is, $\sigma_S(v)$ will be the generating function in v for the content of tableaux of shape S with strictly increasing rows, weakly increasing columns and entries in $\{1, \dots, n\}$.

The ideal J_S is generated by an irreducible representation of G whose character is $u_1^{s'_1} \cdots u_m^{s'_\ell} \cdot \sigma_S(v)$, where $S' = s'_1, \dots, s'_\ell$ denotes the transpose of S. This is the shape whose j-th part is $s'_j = \#\{s_i : i \geq j\}$. It follows that

$$\mathrm{Hilb}(J_S) = u_1^{s'_1} \cdots u_m^{s'_\ell} \cdot \sigma_S(v) + \cdots,$$

where the ellipsis denotes a $\mathbb{Z}[v]$-linear combination of Schur polynomials of degree larger than $\sum_i s_i$. Multiplying by $\prod_{i=1}^m \prod_{j=1}^n (1 - u_i v_j)$, this proves that $K(J_S; u, v)$ takes the same form. We conclude the linear independence of the classes, since the Schur polynomials are linearly independent.

To finish the proof, we must show that every Schur polynomial can be written as a finite $\mathbb{Z}[u^{\pm 1}]$-linear combination of these classes. The difficulty with this lays in demanding the finiteness of the expression. We will show, first, that the Schur polynomials appearing in $K(J_S; u, v)$ never get too long, and second, when $S = n, \dots, n$ (ℓ-factors) that $K(J_S; u; v) = (u_1 \cdots u_n)^\ell \sigma_S(v)$.

We will use the fact that passing to an initial ideal does not alter K-classes: $K(J_S; u, v) = K(\mathrm{in}(J_S); u, v)$ [Miller and Sturmfels 2005, Proposition 8.28]. Although $\mathrm{in}(J_S)$ is no longer a G-equivariant ideal, we can compute its K-polynomial in the Grothendieck group of multigraded modules. To understand $K(\mathrm{in}(J_S); u, v)$ we resolve the quotient $R/\mathrm{in}(J_S)$ by its highly nonminimal Taylor resolution [Miller and Sturmfels 2005, Chapter 6]. Write

$$\mathrm{in}(J_S) = \langle m_1, \dots, m_r \rangle,$$

where the m_i are the leading terms of the superstandard bitableaux of shape S in the diagonal term order. Given a subset $I \subseteq \{1, \ldots, r\}$, set $m_I = \operatorname{lcm}\{m_i : i \in I\}$. If the degree of m_I is $(\boldsymbol{a}_I, \boldsymbol{b}_I) \in \mathbb{N}^m \oplus \mathbb{N}^n$, then the i-th piece of the Taylor resolution of $\operatorname{in}(J_S)$ is $\bigoplus_{I:|I|=i} R(-\boldsymbol{a}_I, -\boldsymbol{b}_I)$. It is a fact that this can be endowed with a differential yielding a resolution of $R/\operatorname{in}(J_S)$.

We claim that all Schur polynomials that appear with a nonzero coefficient in $K(J_S; u, v)$ have length at most s_1'. Suppose that this were not true. Writing $K(J_S; u, v)$ in the standard basis of monomials of $\mathbb{Z}[u, v]$ this implies that the variable v_1 appears with exponent greater than s_1'. However, appealing to the fact that the Taylor resolution can be used to compute $K(R/\operatorname{in}(J_S); u, v)$, this means that there is some monomial m_I whose associated degree $(\boldsymbol{a}_I, \boldsymbol{b}_I)$ has $(\boldsymbol{b}_I)_1 > s_1'$. However, the least common multiple of all the m_i is of the form

$$x_{11}^{s_1'} \cdot (\text{a monomial in } x_{ij} \text{ with } j \neq 1),$$

which is a contradiction.

It follows that $K(J_S; u, v)$ can be written as a finite $\mathbb{Z}[u]$-linear combination of Schur polynomials whose shape is contained in a $s_1' \times n$ box. Suppose that $S = n, \ldots, n$. Then the ideal J_S is principal, generated by a power of a maximal minor of X. That $K(J_S; u, v) = (u_1 \cdots u_n)^{s_1'}(v_1 \cdots v_n)^{s_1'}$ is immediate. By induction, we may write $\sigma_S(v)$ as a linear $\mathbb{Z}[u^{\pm 1}]$-linear combination of the classes of ideals generated by superstandard tableaux. □

Example 5.2. Take $n = m = 3$ and $S = 2, 1$. The least common multiple of the initial monomials of the superstandard bitableaux of shape S is $x_{11}^2 x_{12}^2 x_{22} x_{13} x_{23}$. Using Macaulay2, we have,

$$K(J_S; u, v) = \sigma_{2,1}(v)u_1^2u_2 - \sigma_{2,2}(v)u_1^3u_2 - \sigma_{3,1}(v)(u_1^3u_2 + u_1^2u_2^2) + \sigma_{3,2}(v)(u_1^4u_2 + u_1^3u_2^2) - \sigma_{3,3}(v)u_1^4u_2^2.$$

Observe that each shape appearing has at most two parts.

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