Unipotent and Nakayama automorphisms of quantum nilpotent algebras

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Automorphisms of algebras R from a very large axiomatic class of quantum nilpotent algebras are studied using techniques from noncommutative unique factorization domains and quantum cluster algebras. First, the Nakayama automorphism of R (associated to its structure as a twisted Calabi–Yau algebra) is determined and shown to be given by conjugation by a normal element, namely, the product of the homogeneous prime elements of R (there are finitely many up to associates). Second, in the case when R is connected graded, the unipotent automorphisms of R are classified up to minor exceptions. This theorem is a far reaching extension of the classification results previously used to settle the Andruskiewitsch–Dumas and Launois–Lenagan conjectures. The result on unipotent automorphisms has a wide range of applications to the determination of the full automorphisms groups of the connected graded algebras in the family. This is illustrated by a uniform treatment of the automorphism groups of the generic algebras of quantum matrices of both rectangular and square shape.

1. Introduction

This paper is devoted to a study of automorphisms of *quantum nilpotent algebras*, a large, axiomatically defined class of algebras. The algebras in this class are known under the name Cauchon–Goodearl–Letzter extensions and consist of iterated skew polynomial rings satisfying certain common properties for algebras appearing in the area of quantum groups. The class contains the quantized coordinate rings of the Schubert cells for all simple algebraic groups, multiparameter quantized coordinate rings of many algebraic varieties, quantized Weyl algebras,

and related algebras. The quantized coordinate rings of all double Bruhat cells are localizations of special algebras in the class.

Extending the results of [Alev and Chamarie 1992; Launois and Lenagan 2007; Yakimov 2013; 2014b], we prove that all of these algebras are relatively rigid in terms of symmetry, in the sense that they have far fewer automorphisms than their classical counterparts. This allows strong control, even exact descriptions in many cases, of their automorphism groups. We pursue this theme in two directions. First, results of Liu, Wang, and Wu [Liu et al. 2014] imply that any quantum nilpotent algebra R is a twisted Calabi–Yau algebra. In particular, R thus has a special associated automorphism, its *Nakayama automorphism*, which controls twists appearing in the cohomology of R. At the same time, all algebras R in the class that we consider are equivariant noncommutative unique factorization domains [Launois et al. 2006] in the sense of [Chatters 1984]. We develop a formula for the Nakayama automorphism ν of R, and show that ν is given by commutation with a special normal element. Specifically, if $u_1, \ldots, u_n$ is a complete list of the homogeneous prime elements of R up to scalar multiples, then $a(u_1 \cdots u_n) = (u_1 \cdots u_n)\nu(a)$ for all $a \in R$. (Here homogeneity is with respect to the grading of R arising from an associated torus action.) It was an open problem to understand what is the role of the special element of the equivariant UFD R that equals the product of all (finitely many up to associates) homogeneous prime elements of R. The first main result in the paper answers this: conjugation by this special element is the Nakayama automorphism of R.

In a second direction, we obtain very general rigidity results for the connected graded algebras R in the abovementioned axiomatic class. This is done by combining the quantum cluster algebra structures that we constructed in [Goodearl and Yakimov 2012; 2013] with the rigidity of quantum tori theorem of [Yakimov 2014b]. The quantum clusters of R constructed in [Goodearl and Yakimov 2012; 2013] provide a huge supply of embeddings $\mathcal{A} \subseteq R \subset \mathcal{T}$ where $\mathcal{A}$ is a quantum affine space algebra and $\mathcal{T}$ is the corresponding quantum torus. This allows for strong control of the *unipotent* automorphisms of R relative to a nonnegative grading on R, those being automorphisms ψ such that for any homogeneous element $x \in R$ of degree d, the difference $\psi(x) - x$ is supported in degrees greater than d. Such a ψ induces [Yakimov 2013] a continuous automorphism of the completion of any quantum torus $\mathcal{T}$ as above, to which a general rigidity theorem proved in [Yakimov 2014b] applies. We combine this rigidity with the large supply of quantum clusters in [Goodearl and Yakimov 2012; 2013] and a general theorem for separation of variables from the first of these two papers.

With this combination of methods and the noncommutative UFD property of R, we show here that the unipotent automorphisms of a quantum nilpotent algebra R have a very restricted form, which is a very general improvement of

the earlier results in that direction [Yakimov 2013; 2014b] that were used in proving the Andruskiewitsch–Dumas and Launois–Lenagan conjectures. Our theorem essentially classifies the unipotent automorphisms of all connected graded algebras in the class, up to the presence of certain types of torsion in the scalars involved in the algebras. In a variety of cases the full automorphism group $\mathrm{Aut}(R)$ can be completely determined as an application of this result. We illustrate this by presenting, among other examples, a new route to the determination of the automorphism groups of generic quantum matrix algebras [Launois and Lenagan 2007; Yakimov 2013] of both rectangular and square shape, in particular giving a second proof of the conjecture in [Launois and Lenagan 2007].

In a recent paper, Ceken, Palmieri, Wang and Zhang [Ceken et al. 2015] classified the automorphism groups of certain PI algebras using discriminants. Their methods apply to quantum affine spaces at roots of unity but not to general quantum matrix algebras at roots of unity. It is an interesting problem whether the methods of quantum cluster algebras and rigidity of quantum tori can be applied in conjunction with the methods of [Ceken et al. 2015] to treat the automorphism groups of the specializations of all algebras in this paper to roots of unity.

We finish the introduction by describing the class of quantum nilpotent algebras that we address. These algebras are iterated skew polynomial extensions

$$R := \mathbb{K}[x_1][x_2; \sigma_2, \delta_2] \cdots [x_N; \sigma_N, \delta_N] \tag{1.1}$$

over a base field $\mathbb{K}$, equipped with rational actions by tori $\mathcal{H}$ of automorphisms which cover the σ_k in a suitably generic fashion, and such that the skew derivations δ_k are locally nilpotent. They have been baptized *CGL extensions* in [Launois et al. 2006]; see Definition 2.3 for the precise details. We consider the class of CGL extensions to be the best current definition of quantum nilpotent algebras from a ring theoretic perspective. All important CGL extensions that we are aware of are *symmetric* in the sense that they possess CGL extension presentations with the generators in both forward and reverse orders, that is, both (1.1) and

$$R = \mathbb{K}[x_N][x_{N-1}; \sigma_{N-1}^*, \delta_{N-1}^*] \cdots [x_1; \sigma_1^*, \delta_1^*].$$

The results outlined above apply to the class of symmetric CGL extensions satisfying a mild additional assumption on the scalars that appear.

Throughout, fix a base field $\mathbb{K}$. All automorphisms are assumed to be $\mathbb{K}$-algebra automorphisms, and all skew derivations are assumed to be $\mathbb{K}$-linear. We also assume that in all Ore extensions (skew polynomial rings) $B[x; \sigma, \delta]$, the map σ is an automorphism. Recall that $B[x; \sigma, \delta]$ denotes a ring generated by a unital subring B and an element x satisfying $xs = \sigma(s)x + \delta(s)$ for all $s \in S$, where σ is an automorphism of B and δ is a (left) σ-derivation of B.

We will denote $[j, k] := \{n \in \mathbb{Z} \mid j \le n \le k\}$ for $j, k \in \mathbb{Z}$. In particular, $[j, k] = \varnothing$ if $j \nleq k$.

2. Symmetric CGL extensions

In this section, we give some background on $\mathcal{H}$-UFDs and CGL extensions, including some known results, and then establish a few additional results that will be needed in later sections.

2A. $\mathcal{H}$-UFDs. Recall that a *prime element* of a domain R is any nonzero normal element $p \in R$ (*normality* meaning that $Rp = pR$) such that Rp is a completely prime ideal, that is, R/Rp is a domain. Assume that in addition R is a $\mathbb{K}$-algebra and $\mathcal{H}$ a group acting on R by $\mathbb{K}$-algebra automorphisms. An $\mathcal{H}$-*prime ideal* of R is any proper $\mathcal{H}$-stable ideal P of R such that $(IJ \subseteq P \implies I \subseteq P$ or $J \subseteq P)$ for all $\mathcal{H}$-stable ideals I and J of R. In general, $\mathcal{H}$-prime ideals need not be prime, but they are prime in the case of CGL extensions [Brown and Goodearl 2002, II.2.9].

One says that R is an $\mathcal{H}$-*UFD* if each nonzero $\mathcal{H}$-prime ideal of R contains a prime $\mathcal{H}$-eigenvector. This is an equivariant version of Chatters' notion [Chatters 1984] of noncommutative unique factorization domain given in [Launois et al. 2006, Definition 2.7].

The following fact is an equivariant version of results of Chatters and Jordan [Chatters 1984, Proposition 2.1; Chatters and Jordan 1986, p. 24]; see [Goodearl and Yakimov 2012, Proposition 2.2] and [Yakimov 2014a, Proposition 6.18 (ii)].

Proposition 2.1. *Let R be a noetherian $\mathcal{H}$-UFD. Every normal $\mathcal{H}$-eigenvector in R is either a unit or a product of prime $\mathcal{H}$-eigenvectors. The factors are unique up to reordering and taking associates.*

We shall also need the following equivariant version of [Chatters and Jordan 1986, Lemma 2.1]. A nonzero ring R equipped with an action of a group $\mathcal{H}$ is said to be $\mathcal{H}$-*simple* provided the only $\mathcal{H}$-stable ideals of R are 0 and R.

Lemma 2.2. *Let R be a noetherian $\mathcal{H}$-UFD and $E(R)$ the multiplicative subset of R generated by the prime $\mathcal{H}$-eigenvectors of R. All nonzero $\mathcal{H}$-stable ideals of R meet $E(R)$, and so the localization $R[E(R)^{-1}]$ is $\mathcal{H}$-simple.*

Proof. The second conclusion is immediate from the first. To see the first, let I be a nonzero $\mathcal{H}$-stable ideal of R. Since R is noetherian, $P_1 P_2 \cdots P_m \subset I$ for some prime ideals P_j minimal over I. For each j, the intersection of the $\mathcal{H}$-orbit of P_j is an $\mathcal{H}$-prime ideal Q_j of R such that $I \subseteq Q_j \subseteq P_j$. Each Q_j contains a prime $\mathcal{H}$-eigenvector q_j, and the product $q_1 q_2 \cdots q_m$ lies in I. Thus, $I \cap E(R) \neq \varnothing$, as desired. (Alternatively, suppose that $I \cap E(R) = \varnothing$, enlarge I to an $\mathcal{H}$-stable ideal P maximal with respect to being disjoint from $E(R)$,

check that P is $\mathcal{H}$-prime, and obtain a prime $\mathcal{H}$-eigenvector in P, yielding a contradiction.) □

2B. *CGL extensions.* Throughout the paper, we focus on iterated Ore extensions of the form

$$R := \mathbb{K}[x_1][x_2; \sigma_2, \delta_2] \cdots [x_N; \sigma_N, \delta_N]. \qquad (2.1)$$

We refer to such an algebra as an *iterated Ore extension over* $\mathbb{K}$, to emphasize that the initial step equals the base field $\mathbb{K}$, and we call the integer N the *length* of the extension. For $k \in [0, N]$, we let R_k denote the subalgebra of R generated by $x_1, \ldots, x_k$. In particular, $R_0 = \mathbb{K}$ and $R_N = R$. Each R_k is an iterated Ore extension over $\mathbb{K}$, of length k.

Definition 2.3. An iterated Ore extension (2.1) is called a *CGL extension* [Launois et al. 2006, Definition 3.1] if it is equipped with a rational action of a $\mathbb{K}$-torus $\mathcal{H}$ by $\mathbb{K}$-algebra automorphisms satisfying the following conditions:

(i) The elements $x_1, \ldots, x_N$ are $\mathcal{H}$-eigenvectors.

(ii) For every $k \in [2, N]$, δ_k is a locally nilpotent σ_k-derivation of the algebra R_{k-1}.

(iii) For every $k \in [1, N]$, there exists $h_k \in \mathcal{H}$ such that $\sigma_k = (h_k \cdot)|_{R_{k-1}}$ and the h_k-eigenvalue of x_k, to be denoted by λ_k, is not a root of unity.

Conditions (i) and (iii) imply that

$$\sigma_k(x_j) = \lambda_{kj} x_j \text{ for some } \lambda_{kj} \in \mathbb{K}^* \quad \text{for all } 1 \le j < k \le N. \qquad (2.2)$$

We then set $\lambda_{kk} := 1$ and $\lambda_{jk} := \lambda_{kj}^{-1}$ for $j < k$. This gives rise to a multiplicatively skew-symmetric matrix $\boldsymbol{\lambda} := (\lambda_{kj}) \in M_N(\mathbb{K}^*)$.

The CGL extension R is called *torsion-free* if the subgroup $\langle \lambda_{kj} \mid k, j \in [1, N] \rangle$ of $\mathbb{K}^*$ is torsion-free. Define the *rank* of R by

$$\text{rank}(R) := \big| \{ k \in [1, N] \mid \delta_k = 0 \} \big| \in \mathbb{Z}_{>0} \qquad (2.3)$$

(compare [Goodearl and Yakimov 2012, (4.3)]).

Denote the character group of the torus $\mathcal{H}$ by $X(\mathcal{H})$ and express this group additively. The action of $\mathcal{H}$ on R gives rise to an $X(\mathcal{H})$-grading of R. The $\mathcal{H}$-eigenvectors are precisely the nonzero homogeneous elements with respect to this grading. We denote the $\mathcal{H}$-eigenvalue of a nonzero homogeneous element $u \in R$ by χ_u. In other words, $\chi_u = X(\mathcal{H})\text{-deg}(u)$ in terms of the $X(\mathcal{H})$-grading.

Proposition 2.4 [Launois et al. 2006, Proposition 3.2]. *Every CGL extension is an* $\mathcal{H}$-*UFD, with* $\mathcal{H}$ *as in the definition.*

The sets of homogeneous prime elements in the subalgebras R_k of a CGL extension (2.1) were characterized in [Goodearl and Yakimov 2012]. The statement of the result involves the standard predecessor and successor functions, $p = p_\eta$ and $s = s_\eta$, of a function $\eta : [1, N] \to \mathbb{Z}$, defined as follows:

$$\begin{aligned} p(k) &= \max\{j < k \mid \eta(j) = \eta(k)\}, \\ s(k) &= \min\{j > k \mid \eta(j) = \eta(k)\}, \end{aligned} \tag{2.4}$$

where $\max \varnothing = -\infty$ and $\min \varnothing = +\infty$. Define corresponding order functions $O_\pm : [1, N] \to \mathbb{N}$ by

$$\begin{aligned} O_-(k) &:= \max\{m \in \mathbb{N} \mid p^m(k) \neq -\infty\}, \\ O_+(k) &:= \max\{m \in \mathbb{N} \mid s^m(k) \neq +\infty\}. \end{aligned} \tag{2.5}$$

Theorem 2.5 [Goodearl and Yakimov 2012, Theorem 4.3, Corollary 4.8]. *Let R be a CGL extension of length N as in (2.1). There exist a function $\eta : [1, N] \to \mathbb{Z}$ whose range has cardinality $\mathrm{rank}(R)$ and elements*

$$c_k \in R_{k-1} \quad \textit{for all } k \in [2, N] \textit{ with } p(k) \neq -\infty$$

such that the elements $y_1, \ldots, y_N \in R$, recursively defined by

$$y_k := \begin{cases} y_{p(k)} x_k - c_k & \textit{if } p(k) \neq -\infty, \\ x_k & \textit{if } p(k) = -\infty, \end{cases} \tag{2.6}$$

are homogeneous and have the property that for every $k \in [1, N]$,

$$\{y_j \mid j \in [1, k], \ s(j) > k\} \tag{2.7}$$

is a list of the homogeneous prime elements of R_k up to scalar multiples.

The elements $y_1, \ldots, y_N \in R$ with these properties are unique. The function η satisfying the above conditions is not unique, but the partition of $[1, N]$ into a disjoint union of the level sets of η is uniquely determined by R, as are the predecessor and successor functions p and s. The function p has the property that $p(k) = -\infty$ if and only if $\delta_k = 0$.

Furthermore, the elements y_k of R satisfy

$$y_k x_j = \alpha_{jk}^{-1} x_j y_k, \quad \textit{for all } k, j \in [1, N], s(k) = +\infty, \tag{2.8}$$

where

$$\alpha_{jk} := \prod_{m=0}^{O_-(k)} \lambda_{j, p^m(k)} \quad \textit{for all } j, k \in [1, N].$$

The uniqueness of the level sets of η was not stated in Theorem 4.3 of [Goodearl and Yakimov 2012], but it follows at once from Theorem 4.2 of the

same work. This uniqueness immediately implies the uniqueness of p and s. In the setting of the theorem, the rank of R is also given by

$$\mathrm{rank}(R) = \big|\{j \in [1, N] \mid s(j) = +\infty\}\big| \tag{2.9}$$

(compare [Goodearl and Yakimov 2012, (4.3)]).

Example 2.6. Let $R := \mathcal{O}_q(M_{t,n}(\mathbb{K}))$ be the standard quantized coordinate ring of $t \times n$ matrices over $\mathbb{K}$, with $q \in \mathbb{K}^*$, generators X_{ij} for $i \in [1, t]$, $j \in [1, n]$, and relations

$$X_{ij} X_{im} = q X_{im} X_{ij} \qquad\qquad X_{ij} X_{lj} = q X_{lj} X_{ij}$$

$$X_{im} X_{lj} = X_{lj} X_{im} \qquad X_{ij} X_{lm} - X_{lm} X_{ij} = (q - q^{-1}) X_{im} X_{lj}$$

for $i < l$ and $j < m$. It is well known that R has an iterated Ore extension presentation with variables X_{ij} listed in lexicographic order, that is,

$$R = \mathbb{K}[x_1][x_2; \sigma_2, \delta_2] \cdots [x_N; \sigma_N, \delta_N], \quad N := tn,$$
$$x_{(i-1)n+j} := X_{ij} \quad \text{for all } i \in [1, t], \, j \in [1, n]. \tag{2.10}$$

Now assume that q is not a root of unity. There is a rational action of the torus $\mathcal{H} := (\mathbb{K}^*)^{t+n}$ on R by $\mathbb{K}$-algebra automorphisms such that

$$(\alpha_1, \ldots, \alpha_{t+n}) \cdot X_{ij} = \alpha_i \alpha_{t+j} X_{ij} \text{ for all } (\alpha_1, \ldots, \alpha_{t+n}) \in \mathcal{H}, \, i \in [1, t], \, j \in [1, n],$$

and it is well known that R equipped with this action is a CGL extension. Moreover, $\mathrm{rank}(R) = t + n - 1$, because for $k \in [1, N]$ we have $\delta_k = 0$ if and only if either $k \in [1, n]$ or $k = (i-1)n + 1$ with $i \in [2, t]$.

The function η from Theorem 2.5 can be chosen so that

$$\eta((i-1)n + j) = j - i \quad \text{for all } i \in [1, t], \, j \in [1, n].$$

The element $y_{(i-1)n+j}$ is the largest solid quantum minor with lower right corner in row i, column j, that is,

$$y_{(i-1)n+j} = \big[[i - \min(i, j) + 1, i] \mid [j - \min(i, j) + 1, j]\big],$$
$$\text{for all } i \in [1, t], \, j \in [1, n],$$

and the list of homogeneous prime elements of R, up to scalar multiples, given by Theorem 2.5 is

$$y_n, y_{2n}, \ldots, y_{(t-1)n}, y_{(t-1)n+1}, y_{(t-1)n+2}, \ldots, y_{tn}. \tag{2.11}$$

Proposition 2.7. *Let R be a CGL extension of length N as in* (2.1). *The following are equivalent for an integer $i \in [1, N]$:*

(a) *The integer i satisfies $\eta^{-1}(\eta(i)) = \{i\}$ for the function η from Theorem 2.5.*

(b) *The element x_i is prime in R.*

(c) *The element x_i satisfies $x_i x_j = \lambda_{ij} x_j x_i$ for all $j \in [1, N]$.*

We will denote by $P_x(R)$ the set of integers $i \in [1, N]$ satisfying the conditions (a)–(c).

Proof. Denote by A, B, and C the sets of integers i occurring in parts (a), (b) and (c). We will prove that $A \subseteq B \subseteq A \subseteq C \subseteq A$. The inclusions $A \subseteq B$ and $A \subseteq C$ follow at once from Theorem 2.5 and (2.8). Moreover, $B \subseteq A$ because of (2.6) and (2.7), since if x_i is prime it must be a scalar multiple of y_i.

Let $i \in C$. Then $x_i x_j = \sigma_i(x_j) x_i$ for all $j < i$, whence $\delta_i = 0$ and so $p(i) = -\infty$. Thus, $\eta^{-1}(\eta(i)) \subseteq [i, N]$. Assume that $\eta^{-1}(\eta(i)) \neq \{i\}$, which implies $s(i) \neq +\infty$. Set $j := s(i) \in \eta^{-1}(\eta(i))$. Then $p(j) = i \neq -\infty$ and so $\delta_j(x_i) \neq 0$ by [Goodearl and Yakimov 2012, Proposition 4.7(b)], which contradicts the equality $x_j x_i = \lambda_{ij}^{-1} x_i x_j = \sigma_j(x_i) x_j$. Therefore $\eta^{-1}(\eta(i)) = \{i\}$ and $i \in A$. □

One can show that the conditions in Proposition 2.7 (a)–(c) are equivalent to x_i being a normal element of R.

Recall that *quantum tori* and *quantum affine space algebras* over $\mathbb{K}$ are defined by

$$\mathcal{T}_{\boldsymbol{p}} = \mathcal{O}_{\boldsymbol{p}}((\mathbb{K}^*)^N) := \mathbb{K}\langle Y_1^{\pm 1}, \ldots, Y_N^{\pm 1} \mid Y_k Y_j = p_{kj} Y_j Y_k, \ \text{for all } k, j \in [1, N]\rangle,$$

$$\mathcal{A}_{\boldsymbol{p}} = \mathcal{O}_{\boldsymbol{p}}(\mathbb{K}^N) := \mathbb{K}\langle Y_1, \ldots, Y_N \mid Y_k Y_j = p_{kj} Y_j Y_k, \ \text{for all } k, j \in [1, N]\rangle,$$

for any multiplicatively skew-symmetric matrix $\boldsymbol{p} = (p_{ij}) \in M_N(\mathbb{K}^*)$.

Proposition 2.8 [Goodearl and Yakimov 2012, Theorem 4.6]. *For any CGL extension R of length N, the elements $y_1, \ldots, y_N$ generate a quantum affine space algebra $\mathcal{A}$ inside R. The corresponding quantum torus $\mathcal{T}$ is naturally embedded in* $\mathrm{Fract}(R)$ *and we have the inclusions*

$$\mathcal{A} \subseteq R \subset \mathcal{T}.$$

The algebras $\mathcal{A}$ and $\mathcal{T}$ in Proposition 2.8 are isomorphic to $\mathcal{A}_{\boldsymbol{q}}$ and $\mathcal{T}_{\boldsymbol{q}}$, respectively, where by [Goodearl and Yakimov 2012, (4.17)] the entries of the matrix $\boldsymbol{q} = (q_{ij})$ are given by

$$q_{kj} = \prod_{m=0}^{O_-(k)} \prod_{l=0}^{O_-(j)} \lambda_{p^m(k), p^l(j)}, \quad \text{for all } k, j \in [1, N]. \tag{2.12}$$

Definition 2.9. Let $\boldsymbol{p} = (p_{ij}) \in M_N(\mathbb{K}^*)$ be a multiplicatively skew-symmetric matrix. Define the (skew-symmetric) multiplicative bicharacter $\Omega_{\boldsymbol{p}} : \mathbb{Z}^N \times \mathbb{Z}^N \to \mathbb{K}^*$ by

$$\Omega_{\boldsymbol{p}}(e_i, e_j) = p_{ij} \quad \text{for all } i, j \in [1, N],$$

where $e_1, \ldots, e_N$ denotes the standard basis of $\mathbb{Z}^N$. The *radical* of $\Omega_{\boldsymbol{p}}$ is the subgroup

$$\operatorname{rad} \Omega_{\boldsymbol{p}} := \{ f \in \mathbb{Z}^N \mid \Omega_{\boldsymbol{p}}(f, g) = 1 \quad \text{for all } g \in \mathbb{Z}^N \}$$

of $\mathbb{Z}^N$. We say that the bicharacter $\Omega_{\boldsymbol{p}}$ is *saturated* if $\mathbb{Z}^N / \operatorname{rad} \Omega_{\boldsymbol{p}}$ is torsion-free, that is,

$$nf \in \operatorname{rad} \Omega_{\boldsymbol{p}} \implies f \in \operatorname{rad} \Omega_{\boldsymbol{p}} \quad \text{for all } n \in \mathbb{Z}_{>0}, \, f \in \mathbb{Z}^N.$$

Carrying the terminology forward, we say that the quantum torus $\mathcal{T}_{\boldsymbol{p}}$ is *saturated* provided $\Omega_{\boldsymbol{p}}$ is saturated.

Finally, we apply this terminology to a CGL extension R via its associated matrix $\boldsymbol{\lambda}$, and say that R is *saturated* provided the bicharacter $\Omega_{\boldsymbol{\lambda}}$ is saturated.

For example, any torsion-free CGL extension R is saturated, because all values of $\Omega_{\boldsymbol{\lambda}}$ lie in the torsion-free group $\langle \lambda_{kj} \mid k, j \in [1, N] \rangle$ in that case.

Lemma 2.10. *Let R be a CGL extension of length N as in (2.1), and let $\mathcal{T}$ be the quantum torus in Proposition 2.8. Then R is saturated if and only if $\mathcal{T}$ is saturated.*

Proof. In view of (2.12), $\Omega_{\boldsymbol{q}}(e_k, e_j) = q_{kj} = \Omega_{\boldsymbol{\lambda}}(\bar{e}_k, \bar{e}_j)$ for all $k, j \in [1, N]$, where

$$\bar{e}_i := e_i + e_{p(i)} + \cdots + e_{p^{O_-(i)}} \quad \text{for all } i \in [1, N].$$

Since $\bar{e}_1, \ldots, \bar{e}_N$ is a basis for $\mathbb{Z}^N$, it follows that $\Omega_{\boldsymbol{\lambda}}$ is saturated if and only if $\Omega_{\boldsymbol{q}}$ is saturated. $\qquad\square$

Continue to let R be a CGL extension of length N as in (2.1). Denote by $\mathcal{N}(R)$ the unital subalgebra of R generated by its homogeneous prime elements $y_k, k \in [1, N], s(k) = +\infty$. By [Goodearl and Yakimov 2012, Proposition 2.6], $\mathcal{N}(R)$ coincides with the unital subalgebra of R generated by all normal elements of R. As in Lemma 2.2, denote by $E(R)$ the multiplicative subset of R generated by the homogeneous prime elements of R. In the present situation, $E(R)$ is also generated by the set $\mathbb{K}^* \sqcup \{ y_k \mid k \in [1, N], s(k) = +\infty \}$. It is an Ore set in R and $\mathcal{N}(R)$ since it is generated by elements which are normal in both algebras. Note that $\mathcal{N}(R)[E(R)^{-1}] \subseteq R[E(R)^{-1}] \subseteq \mathcal{T}$, where $\mathcal{T}$ is the torus of Proposition 2.8.

Proposition 2.11. *The center of the quantum torus $\mathcal{T}$ in Proposition 2.8 coincides with the center of $R[E(R)^{-1}]$ and is contained in $\mathcal{N}(R)[E(R)^{-1}]$, i.e.,*

$$Z(\mathcal{T}) = Z\big(R[E(R)^{-1}]\big) = \{ z \in \mathcal{N}(R)[E(R)^{-1}] \mid zx = xz, \text{ for all } x \in R \}. \quad (2.13)$$

Proof. It is clear that $Z\big(R[E(R)^{-1}]\big) \subseteq Z(\mathcal{T})$, because these centers consist of the elements in $R[E(R)^{-1}]$ and $\mathcal{T}$ that commute with all elements of R, and that

the set on the right hand side of (2.13) is contained in $Z(R[E(R)^{-1}])$. Hence, it suffices to show that $Z(\mathcal{T}) \subseteq \mathcal{N}(R)[E(R)^{-1}]$.

Recall that the center of any quantum torus equals the linear span of the central Laurent monomials in its generators. If m is a central Laurent monomial in the generators $y_1^{\pm 1}, \ldots, y_N^{\pm 1}$ of $\mathcal{T}$, then m is an $\mathcal{H}$-eigenvector and

$$I := \{r \in R \mid mr \in R\}$$

is a nonzero $\mathcal{H}$-stable ideal of R. By Lemma 2.2, there exists $c \in I \cap E(R)$, and $m = ac^{-1}$ for some $a \in R$. Since m centralizes R and c normalizes it, the element $a = mc$ is normal in R. Hence, $a \in \mathcal{N}(R)$, and we conclude that $m \in \mathcal{N}(R)[E(R)^{-1}]$. Therefore $Z(\mathcal{T}) \subseteq \mathcal{N}(R)[E(R)^{-1}]$, as required. $\qquad \square$

Definition 2.12. We will say that an automorphism ψ of a CGL extension R as in (2.1) is *diagonal* provided $x_1, \ldots, x_N$ are eigenvectors for ψ. Set

$$\mathrm{DAut}(R) := \{\text{diagonal automorphisms of } R\},$$

a subgroup of $\mathrm{Aut}(R)$.

In particular, the group $\{(h \cdot) \mid h \in \mathcal{H}\}$ is contained in $\mathrm{DAut}(R)$. It was shown in [Goodearl and Yakimov 2012, Theorems 5.3, 5.5] that $\mathrm{DAut}(R)$ is naturally isomorphic to a $\mathbb{K}$-torus of rank equal to $\mathrm{rank}(R)$, exhibited as a closed connected subgroup of the torus $(\mathbb{K}^*)^N$. This allows us to think of $\mathrm{DAut}(R)$ as a torus, and to replace $\mathcal{H}$ by $\mathrm{DAut}(R)$ if desired. A description of this torus, as a specific subgroup of $(\mathbb{K}^*)^N$, was established in [Goodearl and Yakimov 2012, Theorem 5.5]. Finally, it follows from Corollary 5.4 of the same work that for any nonzero normal element $u \in R$, there exists $\psi \in \mathrm{DAut}(R)$ such that $ua = \psi(a)u$ for all $a \in R$.

2C. *Symmetric CGL extensions.* For a CGL extension R as in (2.1) and $j, k \in [1, N]$, denote by $R_{[j,k]}$ the unital subalgebra of R generated by $\{x_i \mid j \le i \le k\}$. So, $R_{[j,k]} = \mathbb{K}$ if $j \not\le k$.

Definition 2.13. We call a CGL extension R of length N as in Definition 2.3 *symmetric* if the following two conditions hold:

(i) For all $1 \le j < k \le N$,

$$\delta_k(x_j) \in R_{[j+1,k-1]}.$$

(ii) For all $j \in [1, N]$, there exists $h_j^* \in \mathcal{H}$ such that

$$h_j^* \cdot x_k = \lambda_{kj}^{-1} x_k = \lambda_{jk} x_k \quad \text{for all } k \in [j+1, N]$$

and $h_j^* \cdot x_j = \lambda_j^* x_j$ for some $\lambda_j^* \in \mathbb{K}^*$ which is not a root of unity.

For example, all quantum Schubert cell algebras $U^+[w]$ are symmetric CGL extensions, cf. Example 3.10 below.

Given a symmetric CGL extension R as in Definition 2.13, set

$$\sigma_j^* := (h_j^* \cdot) \in \text{Aut}(R) \quad \text{for all } j \in [1, N-1].$$

Then for all $j \in [1, N-1]$, the inner σ_j^*-derivation on R given by

$$a \mapsto x_j a - \sigma_j^*(a) x_j$$

restricts to a σ_j^*-derivation δ_j^* of $R_{[j+1,N]}$. It is given by

$$\delta_j^*(x_k) := x_j x_k - \lambda_{jk} x_k x_j = -\lambda_{jk} \delta_k(x_j) \quad \text{for all } k \in [j+1, N].$$

For all $1 \le j < k \le N$, σ_k and δ_k preserve $R_{[j,k-1]}$ and σ_j^* and δ_j^* preserve $R_{[j+1,k]}$. This gives rise to the skew polynomial extensions

$$R_{[j,k]} = R_{[j,k-1]}[x_k; \sigma_k, \delta_k] \quad \text{and} \quad R_{[j,k]} = R_{[j+1,k]}[x_j; \sigma_j^*, \delta_j^*]. \tag{2.14}$$

In particular, it follows that R has an iterated Ore extension presentation with the variables x_k in descending order:

$$R = \mathbb{K}[x_N][x_{N-1}; \sigma_{N-1}^*, \delta_{N-1}^*] \cdots [x_1; \sigma_1^*, \delta_1^*]. \tag{2.15}$$

This is the reason for the name "symmetric".

Denote the following subset of the symmetric group S_N:

$$\Xi_N := \{\tau \in S_N \mid \tau(k) = \max \tau([1, k-1]) + 1 \text{ or}$$
$$\tau(k) = \min \tau([1, k-1]) - 1, \text{ for all } k \in [2, N]\}. \tag{2.16}$$

In other words, Ξ_N consists of those $\tau \in S_N$ such that $\tau([1, k])$ is an interval for all $k \in [2, N]$. For each $\tau \in \Xi_N$, we have the iterated Ore extension presentation

$$R = \mathbb{K}[x_{\tau(1)}][x_{\tau(2)}; \sigma_{\tau(2)}'', \delta_{\tau(2)}''] \cdots [x_{\tau(N)}; \sigma_{\tau(N)}'', \delta_{\tau(N)}''], \tag{2.17}$$

where

$$\sigma_{\tau(k)}'' := \sigma_{\tau(k)} \quad \text{and} \quad \delta_{\tau(k)}'' := \delta_{\tau(k)},$$

if $\tau(k) = \max \tau([1, k-1]) + 1$, while

$$\sigma_{\tau(k)}'' := \sigma_{\tau(k)}^* \quad \text{and} \quad \delta_{\tau(k)}'' := \delta_{\tau(k)}^*,$$

if $\tau(k) = \min \tau([1, k-1]) - 1$.

Proposition 2.14 [Goodearl and Yakimov 2012, Remark 6.5]. *For every symmetric CGL extension R of length N and any $\tau \in \Xi_N$, the iterated Ore extension presentation* (2.17) *of R is a CGL extension presentation for the same choice of $\mathbb{K}$-torus $\mathcal{H}$, and the associated elements $h_{\tau(1)}'', \ldots, h_{\tau(N)}'' \in \mathcal{H}$ required by*

Definition 2.3(iii) are given by $h''_{\tau(k)} = h_{\tau(k)}$ *if* $\tau(k) = \max \tau([1, k-1]) + 1$ *and* $h''_{\tau(k)} = h^*_{\tau(k)}$ *if* $\tau(k) = \min \tau([1, k-1]) - 1$.

It follows from Proposition 2.14 that in the given situation,

$$\sigma''_{\tau(k)}(x_{\tau(j)}) = \lambda_{\tau(k),\tau(j)} x_{\tau(j)},$$

for $1 \le j < k \le N$. Hence, the $\boldsymbol{\lambda}$-matrix for the presentation (2.17) is the matrix

$$\boldsymbol{\lambda}_\tau := (\lambda_{\tau(k),\tau(j)}). \tag{2.18}$$

If R is a symmetric CGL extension of length N and $\tau \in \Xi_N$, we write $y_{\tau,1}, \ldots, y_{\tau,N}$ for the y-elements obtained from applying Theorem 2.5 to the CGL extension presentation (2.17). Proposition 2.8 then shows that $y_{\tau,1}, \ldots, y_{\tau,N}$ generate a quantum affine space algebra $\mathcal{A}_\tau$ inside R, the corresponding quantum torus $\mathcal{T}_\tau$ is naturally embedded in $\mathrm{Fract}(R)$, and we have the inclusions

$$\mathcal{A}_\tau \subseteq R \subset \mathcal{T}_\tau.$$

Proposition 2.15. *If R is a saturated symmetric CGL extension of length N, then the quantum tori $\mathcal{T}_\tau$ are saturated, for all $\tau \in \Xi_N$.*

Proof. Let $\tau \in \Xi_N$, and recall (2.18). It follows that

$$\Omega_{\boldsymbol{\lambda}_\tau}(f, g) = \Omega_{\boldsymbol{\lambda}}(\tau \cdot f, \tau \cdot g) \quad \text{for all } f, g \in \mathbb{Z}^N,$$

where we identify τ with the corresponding permutation matrix in $GL_N(\mathbb{Z})$ and write elements of $\mathbb{Z}^N$ as column vectors. Since $\Omega_{\boldsymbol{\lambda}}$ is saturated by hypothesis, it follows immediately that $\Omega_{\boldsymbol{\lambda}_\tau}$ is saturated. Applying Lemma 2.10 to the presentation (2.17), we conclude that $\mathcal{T}_\tau$ is saturated. $\qquad\square$

3. Nakayama automorphisms of iterated Ore extensions

Every iterated Ore extension R over $\mathbb{K}$ is a twisted Calabi–Yau algebra (see Definition 3.1 and Corollary 3.3), and as such has an associated Nakayama automorphism, which is unique in this case because the inner automorphisms of R are trivial. Our main aim is to determine this automorphism ν when R is a symmetric CGL extension. In that case, we show that ν is the restriction to R of an inner automorphism $u^{-1}(-)u$ of $\mathrm{Fract}(R)$, where $u = u_1 \cdots u_n$ for a list $u_1, \ldots, u_n$ of the homogeneous prime elements of R up to scalar multiples. On the way, we formalize a technique of Liu, Wang and Wu [2014] and use it to give a formula for ν in a more general symmetric situation, where we show that each standard generator of R is an eigenvector for ν and determine the eigenvalues.

Recall that the *right twist* of a bimodule M over a ring R by an automorphism ν of R is the R-bimodule M^ν based on the left R-module M and with right R-module multiplication $*$ given by $m * r = m\nu(r)$ for $m \in M, r \in R$.

Definition 3.1. A $\mathbb{K}$-algebra R is ν-*twisted Calabi–Yau of dimension d*, where ν is an automorphism of R and $d \in \mathbb{Z}_{\geq 0}$, provided

(i) R is *homologically smooth*, meaning that as a module over $R^e := R \otimes_{\mathbb{K}} R^{\mathrm{op}}$, it has a finitely generated projective resolution of finite length;

(ii) As R^e-modules, $\mathrm{Ext}^i_{R^e}(R, R^e) \cong \begin{cases} 0 & \text{if } i \neq d, \\ R^\nu & \text{if } i = d. \end{cases}$

When these conditions hold, ν is called the *Nakayama automorphism* of R. It is unique up to an inner automorphism. The case of a Calabi–Yau algebra in the sense of [Ginzburg 2007] is recovered when ν is inner.

Theorem 3.2 [Liu et al. 2014, Theorem 3.3]. *Let B be a ν_0-twisted Calabi–Yau algebra of dimension d, and let $R := B[x; \sigma, \delta]$ be an Ore extension of B. Then R is a ν-twisted Calabi–Yau algebra of dimension $d + 1$, where ν satisfies the following conditions*:

(a) $\nu|_B = \sigma^{-1}\nu_0$.

(b) $\nu(x) = ux + b$ *for some unit $u \in B$ and some $b \in B$.*

Corollary 3.3. *Every iterated Ore extension of length N over $\mathbb{K}$ is a twisted Calabi–Yau algebra of dimension N.*

Note that the only units in an iterated Ore extension R over $\mathbb{K}$ are scalars, and so the only inner automorphism of R is the identity. Hence, the Nakayama automorphism of R is unique.

Liu, Wang and Wu [2014] gave several examples for which the Nakayama automorphism can be completely pinned down by Theorem 3.2. These examples are iterated Ore extensions which can be rewritten as iterated Ore extensions with the original variables in reverse order. We present a general result of this form in the following subsection, and apply it to symmetric CGL extensions in Section 3B.

3A. *Nakayama automorphisms of reversible iterated Ore extensions.*

Definition 3.4. Let R be an iterated Ore extension of length N as in (2.1). We shall say that R (or, more precisely, the presentation (2.1)) is *diagonalized* if there are scalars $\lambda_{kj} \in \mathbb{K}^*$ such that $\sigma_k(x_j) = \lambda_{kj}x_j$ for all $1 \leq j < k \leq N$. When R is diagonalized, we extend the λ_{kj} to a multiplicatively skew-symmetric matrix just as in the CGL case.

A diagonalized iterated Ore extension R is called *reversible* provided there is a second iterated Ore extension presentation

$$R = \mathbb{K}[x_N][x_{N-1}; \sigma^*_{N-1}, \delta^*_{N-1}] \cdots [x_1; \sigma^*_1, \delta^*_1], \tag{3.1}$$

such that $\sigma^*_j(x_k) = \lambda_{jk}x_k$ for all $1 \leq j < k \leq N$.

Every symmetric CGL extension is a reversible diagonalized iterated Ore extension, by virtue of the presentation (2.15).

For any iterated Ore extension R as in (2.1), we define the subalgebras $R_{[j,k]}$ of R just as in Section 2C.

Lemma 3.5. *Let R be a diagonalized iterated Ore extension of length N as in Definition 3.4. Then R is reversible if and only if*

$$\delta_k(x_j) \in R_{[j+1,k-1]} \quad \textit{for all } 1 \le j < k \le N. \tag{3.2}$$

Proof. Assume first that R is reversible, and let $1 \le j < k \le N$. From the structure of the iterated Ore extensions (2.1) and (3.1), we see that

$$\delta_k(x_j) \in R_{[1,k-1]} \quad \text{and} \quad \delta_j^*(x_k) \in R_{[j+1,N]}.$$

Since R is diagonalized, we also have

$$\delta_j^*(x_k) = x_j x_k - \lambda_{jk} x_k x_j = -\lambda_{jk}\big(x_k x_j - \sigma_k(x_j)x_k\big) = -\lambda_{jk}\delta_k(x_j),$$

and thus $\delta_k(x_j) \in R_{[j+1,N]}$. Since $R_{[1,k-1]}$ and $R_{[j+1,N]}$ are iterated Ore extensions with PBW bases $\{x_1^\bullet \cdots x_{k-1}^\bullet\}$ and $\{x_{j+1}^\bullet \cdots x_N^\bullet\}$, respectively, it follows that $\delta_k(x_j) \in R_{[j+1,k-1]}$, verifying (3.2).

Conversely, assume that (3.2) holds. We establish the following by a downward induction on $l \in [1, N]$:

(a) The monomials

$$x_l^{a_l} \cdots x_N^{a_N} \quad \text{for all } a_l, \ldots, a_N \in \mathbb{Z}_{\ge 0} \tag{3.3}$$

form a basis of $R_{[l,N]}$.

(b) $R_{[l,N]} = R_{[l+1,N]}[x_l; \sigma_l^*, \delta_l^*]$ for some automorphism σ_l^* and σ_l^*-derivation δ_l^* of $R_{[l+1,N]}$, such that $\sigma_l^*(x_k) = \lambda_{lk} x_k$ for all $k \in [l+1, N]$.

When $l = N$, both (a) and (b) are clear, since $R_{[N,N]} = \mathbb{K}[x_N]$ and $R_{[N+1,N]} = \mathbb{K}$.

Now let $l \in [1, N-1]$ and assume that (a) and (b) hold for $R_{[l+1,N]}$. For $k \in [l+1, N]$, it follows from (3.2) that

$$x_k x_l - \lambda_{kl} x_l x_k = \delta_k(x_l) \in R_{[l+1,k-1]} \subset R_{[l+1,N]}.$$

Consequently, we see that

$$R_{[l+1,N]} + x_l R_{[l+1,N]} = R_{[l+1,N]} + R_{[l+1,N]} x_l. \tag{3.4}$$

In particular, (3.4) implies that $\sum_{a=0}^\infty x_l^a R_{[l+1,N]}$ is a subalgebra of R. In view of our induction hypothesis, it follows that the monomials (3.3) span $R_{[l,N]}$. Consequently, they form a basis, since they are part of the standard PBW basis for R. This establishes (a) for $R_{[l,N]}$.

Given the above bases for $R_{[l,N]}$ and $R_{[l+1,N]}$, we see that $R_{[l,N]}$ is a free right $R_{[l+1,N]}$-module with basis $(1, x_l, x_l^2, \dots)$. Via (3.4) and an easy induction on degree, we confirm that $R_{[l,N]}$ is also a free left $R_{[l+1,N]}$-module with the same basis. A final application of (3.4) then yields $R_{[l,N]} = R_{[l+1,N]}[x_l; \sigma_l^*, \delta_l^*]$ for some automorphism σ_l^* and σ_l^*-derivation δ_l^* of $R_{[l+1,N]}$. For $k \in [l+1, N]$, we have

$$x_l x_k - \lambda_{lk} x_k x_l = -\lambda_{lk}\big(x_k x_l - \sigma_k(x_l)x_k\big)$$
$$= -\lambda_{lk}\delta_k(x_l) \in R_{[l+1,k-1]} \subset R_{[l+1,N]},$$

from which it follows that $\sigma_l^*(x_k) = \lambda_{lk} x_k$. Thus, (b) holds for $R_{[l,N]}$.

Therefore, the induction works. Combining statements (b) for $l = N, \dots, 1$, we conclude that R is reversible. $\qquad\square$

Theorem 3.6. *Let $R = \mathbb{K}[x_1][x_2; \sigma_2, \delta_2] \cdots [x_N; \sigma_N, \delta_N]$ be a reversible, diagonalized iterated Ore extension over $\mathbb{K}$, let ν be the Nakayama automorphism of R, and let $(\lambda_{jk}) \in M_N(\mathbb{K}^*)$ be the multiplicatively antisymmetric matrix such that $\sigma_k(x_j) = \lambda_{kj} x_j$ for all $1 \le j < k \le N$. Then*

$$\nu(x_k) = \left(\prod_{j=1}^{N} \lambda_{kj}\right) x_k \quad \text{for all } k \in [1, N]. \tag{3.5}$$

Proof. In case $N = 1$, the algebra R is a polynomial ring $\mathbb{K}[x_1]$. Then R is Calabi–Yau (e.g., as in [Farinati 2005, Example 13]), that is, ν is the identity. Thus, the theorem holds in this case.

Now let $N \ge 2$, and assume the theorem holds for all reversible, diagonalized iterated Ore extensions of length less than N. It is clear from the original and the reversed iterated Ore extension presentations of R that R_{N-1} and $R_{[2,N]}$ are diagonalized iterated Ore extensions, and it follows from Lemma 3.5 that R_{N-1} and $R_{[2,N]}$ are reversible. If ν_0 denotes the Nakayama automorphism of R_{N-1}, then the inductive statement together with Theorem 3.2 gives us

$$\nu(x_k) = \sigma_N^{-1}\nu_0(x_k) = \lambda_{Nk}^{-1}\left(\prod_{j=1}^{N-1} \lambda_{kj}\right) x_k = \left(\prod_{j=1}^{N} \lambda_{kj}\right) x_k,$$

$$\text{for all } k \in [1, N-1]. \tag{3.6}$$

Similarly, if ν_1 denotes the Nakayama automorphism of $R_{[2,N]}$, we obtain

$$\nu(x_k) = (\sigma_1^*)^{-1}\nu_1(x_k) = \lambda_{1k}^{-1}\left(\prod_{j=2}^{N} \lambda_{kj}\right) x_k = \left(\prod_{j=1}^{N} \lambda_{kj}\right) x_k \quad \text{for all } k \in [2, N].$$

$$\tag{3.7}$$

The formulas (3.6) and (3.7) together yield (3.5), establishing the induction step. $\qquad\square$

In particular, Theorem 3.6 immediately determines the Nakayama automorphisms of the multiparameter quantum affine spaces $\mathcal{O}_q(\mathbb{K}^N)$, as in [Liu et al. 2014, Proposition 4.1], and those of the Weyl algebras $A_n(\mathbb{K})$ [loc.cit., Remark 4.2].

Examples 3.7. Let $R := \mathcal{O}_q(M_{t,n}(\mathbb{K}))$ as in Example 2.6, with no restriction on $q \in \mathbb{K}^*$. It is clear that the iterated Ore extension presentation (2.10) of R is diagonalized. Since R also has an iterated Ore extension presentation with the X_{ij} in reverse lexicographic order, one easily checks that R is thus reversible.

The scalars $\lambda_{(i-1)n+j,\,(l-1)n+m}$ from (2.2) are equal to 1 except in the following cases:

$$\lambda_{(i-1)n+j,\,(i-1)n+m} = q^{-1} \quad (m < j), \qquad \lambda_{(i-1)n+j,\,(i-1)n+m} = q \quad (m > j),$$
$$\lambda_{(i-1)n+j,\,(l-1)n+j} = q^{-1} \quad (l < i), \qquad \lambda_{(i-1)n+j,\,(l-1)n+j} = q \quad (l > i).$$

In view of Theorem 3.6, we thus find that the Nakayama automorphism ν of R is given by the rule

$$\nu(X_{ij}) = q^{t+n-2i-2j+2} X_{ij},$$

for $i \in [1, t]$, $j \in [1, n]$.

Let us consider the multiparameter version of R only in the $n \times n$ case. This is the $\mathbb{K}$-algebra $R' := \mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(\mathbb{K}))$, where $\lambda \in \mathbb{K} \setminus \{0, 1\}$ and $\boldsymbol{p}$ is a multiplicatively skew-symmetric $n \times n$ matrix over $\mathbb{K}^*$, with generators X_{ij} for $i, j \in [1, n]$ and relations

$$X_{lm}X_{ij} = \begin{cases} p_{li}p_{jm}X_{ij}X_{lm} + (\lambda - 1)p_{li}X_{im}X_{lj} & (l > i, \ m > j), \\ \lambda p_{li}p_{jm}X_{ij}X_{lm} & (l > i, \ m \le j), \\ p_{jm}X_{ij}X_{lm} & (l = i, \ m > j). \end{cases}$$

Iterated Ore extension presentations of R' are well known, and as above, we see that R' is diagonalized and reversible. It follows from Theorem 3.6 that

$$\nu(X_{ij}) = \left(\prod_{l=1}^{n} p_{il}^n\right)\left(\prod_{m=1}^{n} p_{mj}^n\right)\lambda^{n(i-j-1)+i+j-1} X_{ij} \quad \text{for all } i, j \in [1, n].$$

3B. *Nakayama automorphisms of symmetric CGL extensions.* As noted above, any symmetric CGL extension is reversible and diagonalized, so Theorem 3.6 provides a formula for its Nakayama automorphism. We prove that in this case, the Nakayama automorphism arises from the action of a normal element, as follows.

Theorem 3.8. *Let R be a symmetric CGL extension of length N as in Definition 2.13 and ν its Nakayama automorphism. Let $u_1, \dots, u_n$ be a complete, irredundant list of the homogeneous prime elements of R up to scalar multiples,*

and set $u = u_1 \cdots u_n$. *Then* ν *satisfies* (*and is determined by*) *the following condition*:

$$au = u\nu(a) \quad \text{for all } a \in R. \tag{3.8}$$

Proof. Replacing the u_i by scalar multiples of these elements has no effect on (3.8). Thus, we may assume that, in the notation of Theorem 2.5,

$$\{u_1, \ldots, u_n\} = \{y_l \mid l \in [1, N], s(l) = +\infty\}.$$

Hence, (2.8) implies that

$$x_k u = \beta_k u x_k \text{ with } \beta_k := \prod_{\substack{l \in [1,N] \\ s(l) = +\infty}} \alpha_{kl} \quad \text{for all } k \in [1, N]. \tag{3.9}$$

As l runs through the elements of $[1, N]$ with $s(l) = +\infty$ and m runs from 0 to $O_-(l)$, the numbers $p^m(l)$ run through the elements of $[1, N]$ exactly once each. Hence,

$$\beta_k = \prod_{\substack{l \in [1,N] \\ s(l) = +\infty}} \prod_{m=0}^{O_-(l)} \lambda_{k, p^m(l)} = \prod_{j=1}^{N} \lambda_{kj}. \tag{3.10}$$

In view of Theorem 3.6, we obtain from (3.9) and (3.10) that $x_k u = u\nu(x_k)$ for all $k \in [1, N]$. The relation (3.8) follows. $\square$

Example 3.9. Return to $R := \mathcal{O}_q(M_{t,n}(\mathbb{K}))$ as in Examples 2.6, 3.7, and assume that q is not a root of unity. Recall the list of homogeneous prime elements of R from (2.11). The product of these $t + n - 1$ quantum minors gives the element u that determines the Nakayama automorphism of R as in Theorem 3.8.

Example 3.10. Let $\mathfrak{g}$ be a simple Lie algebra with set of simple roots Π, Weyl group W, and root lattice Q, and set $Q^+ := \mathbb{Z}_{\geq 0}\Pi$. For each $\alpha \in \Pi$, denote by $s_\alpha \in W$ and ϖ_α the corresponding reflection and fundamental weight. Denote by $\langle . , . \rangle$ the W-invariant, symmetric, bilinear form on $\mathbb{Q}\Pi$, normalized by $\langle \alpha, \alpha \rangle = 2$ for short roots α. Let $\mathcal{U}_q(\mathfrak{g})$ be the quantized universal enveloping algebra of $\mathfrak{g}$ over an arbitrary base field $\mathbb{K}$ for a deformation parameter $q \in \mathbb{K}^*$ which is not a root of unity. We will use the notation of [Jantzen 1996]. In particular, we will denote the standard generators of $\mathcal{U}_q(\mathfrak{g})$ by E_α, $K_\alpha^{\pm 1}$, F_α, $\alpha \in \Pi$. The subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by $\{E_\alpha \mid \alpha \in \Pi\}$ will be denoted by $\mathcal{U}_q^+(\mathfrak{g})$. It is naturally Q^+-graded with $\deg E_\alpha = \alpha$ for $\alpha \in \Pi$. For each $w \in W$, De Concini–Kac–Procesi and Lusztig defined a graded subalgebra $\mathcal{U}^+[w]$ of $\mathcal{U}_q^+(\mathfrak{g})$, given by [loc.cit., Sections 8.21–8.22]. It is well known that $\mathcal{U}^+[w]$ is a symmetric CGL extension for the torus $\mathcal{H} := (\mathbb{K}^*)^{|\Pi|}$ and the action

$$t \cdot x := \left(\prod_{\alpha \in \Pi} t_\alpha^{\langle \alpha, \gamma \rangle} \right) x \quad \text{for all } t = (t_\alpha)_{\alpha \in \Pi} \in (\mathbb{K}^*)^{|\Pi|}, x \in \mathcal{U}_q^+(\mathfrak{g})_\gamma, \gamma \in Q^+.$$

Here and below, for a Q^+-graded algebra R we denote by R_γ the homogeneous component of R of degree $\gamma \in Q^+$. (Note that the $\mathcal{H}$-eigenvectors in $\mathcal{U}^+[w]$ are precisely the homogeneous elements with respect to the Q^+-grading.)

The algebra $\mathcal{U}^+[w]$ is a deformation of the universal enveloping algebra $\mathcal{U}(\mathfrak{n}_+ \cap w(\mathfrak{n}_-))$ where $\mathfrak{n}_\pm$ are the nilradicals of a pair of opposite Borel subalgebras. For each $w \in W$, Joseph [1995, Section 10.3.1] defined a Q^+-graded algebra S_w^- in terms of a localization of the related quantum group algebra $R_q[G]$. The grading is given by [Yakimov 2014a, (3.22)]; here we will omit the trivial second component. An explicit Q^+-graded isomorphism $\varphi_w^- : S_w^- \to \mathcal{U}^+[w]$ was constructed in [loc.cit., Theorem 2.6]. Denote the support of w

$$\mathcal{S}(w) := \{\alpha \in \Pi \mid s_\alpha \leq w\} \subseteq \Pi,$$

where $\leq$ denotes the Bruhat order on W.

For a subset $I \subseteq \Pi$, define the subset of dominant integral weights

$$P_I^+ := \mathbb{Z}_{\geq 0}\{\varpi_\alpha \mid \alpha \in I\}.$$

Denote

$$\rho_I := \sum_{\alpha \in I} \varpi_\alpha,$$

which also equals the half-sum of positive roots of the standard Levi subalgebra of $\mathfrak{g}$ corresponding to I.

For each $\lambda \in P_{\mathcal{S}(w)}^+$, there is a nonzero normal element $d_{w,\lambda}^- \in (S_w^-)_{(1-w)\lambda}$ given by [loc.cit., (3.29)]. It commutes with the elements of S_w^- by

$$d_{w,\lambda}^- s = q^{\langle (w+1)\lambda, \gamma \rangle} s d_{w,\lambda}^- \quad \text{for all } s \in (S_w^-)_\gamma, \gamma \in Q^+.$$

We have

$$d_{w,\lambda_1}^- d_{w,\lambda_2}^- = q^{\langle \lambda_1, (1-w)\lambda_2 \rangle} d_{w,\lambda_1+\lambda_2}^-, \quad \text{for all } \lambda_1, \lambda_2 \in P_{\mathcal{S}(w)}^+;$$

see [Yakimov 2014a, (3.31)]. By Theorem 6.1(ii) of the same work,

$$\{d_{w,\varpi_\alpha}^- \mid \alpha \in \mathcal{S}(w)\}$$

is a list of the homogeneous prime elements of S_-^w. Therefore, up to a nonzero scalar multiple the product of the homogeneous prime elements of $\mathcal{U}^+[w]$ is $\varphi_w^-(d_{w,\rho_{\mathcal{S}(w)}}^-)$. Theorem 3.8 implies that the Nakayama automorphism of $\mathcal{U}^+[w]$ is given by

$$\nu(a) = \varphi_w^-(d_{w,\rho_{\mathcal{S}(w)}}^-)^{-1} a \varphi_w^-(d_{w,\rho_{\mathcal{S}(w)}}^-), \quad \text{for all } a \in \mathcal{U}^+[w].$$

Furthermore, the above facts imply that it is also given by

$$\nu(a) = q^{-\langle (w+1)\rho_{\mathcal{S}(w)}, \gamma \rangle} a, \quad \text{for all } a \in \mathcal{U}^+[w]_\gamma, \gamma \in Q^+.$$

This is a more explicit form than a previous formula for the Nakayama automorphism of $\mathcal{U}^+[w]$ obtained by Liu and Wu [2014].

4. Unipotent automorphisms

In this section, we prove a theorem stating that the unipotent automorphisms (see Definition 4.2) of a symmetric CGL extension have a very restricted form. The theorem improves the results in [Yakimov 2013; 2014b]. It is sufficient to classify the full groups of unipotent automorphisms of concrete CGL extensions apart from examples which have a nontrivial quantum torus factor in a suitable sense. This is illustrated by giving a second proof of the Launois–Lenagan conjecture [Launois and Lenagan 2007] on automorphisms of square quantum matrix algebras, and by determining the automorphism groups of several other generic quantized coordinate rings.

4A. *Algebra decompositions of symmetric CGL extensions.* Next, we define a unique decomposition of every symmetric CGL extension into a crossed product of a symmetric CGL extension by a free abelian monoid which has the property that the first term cannot be further so decomposed.

Let R be a symmetric CGL extension of length N as in (2.1). Recall from Section 2 that $P_x(R) \subseteq [1, N]$ consists of those indices i for which x_i is a prime element of R. They satisfy

$$x_i x_k = \lambda_{ik} x_k x_i \quad \text{for all } k \in [1, N]. \tag{4.1}$$

For all $1 \leq j < k \leq N$, the element

$$Q_{kj} := x_k x_j - \lambda_{kj} x_j x_k = \delta_k(x_j) \in R_{[j+1,k-1]}$$

is uniquely a linear combination of monomials $x_{j+1}^{m_{j+1}} \cdots x_{k-1}^{m_{k-1}}$. Of course, $Q_{kj} = 0$ if k or j is in $P_x(R)$.

Denote by $F_x(R)$ the set of those $i \in P_x(R)$ such that x_i does not appear in Q_{kj} (more precisely, no monomial which appears with a nonzero coefficient in Q_{kj} contains a positive power of x_i) for any $k, j \in [1, N] \backslash P_x(R)$, $j < k$. Let $C_x(R) := [1, N] \backslash F_x(R)$. The idea for the notation is that $F_x(R)$ indexes the set of x s which will be factored out and $C_x(R)$ indexes the set of essential x s which generate the core of R. Denote the subalgebras

$$\mathcal{C}(R) := \mathbb{K}\langle x_k \mid k \in C_x(R) \rangle \quad \text{and} \quad \mathcal{A}(R) := \mathbb{K}\langle x_i \mid i \in F_x(R) \rangle.$$

We observe that R is a split extension of either of these subalgebras by a corresponding ideal:

$$R = \mathcal{C}(R) \oplus \langle x_i \mid i \in F_x(R) \rangle = \mathcal{A}(R) \oplus \langle x_k \mid k \in C_x(R) \rangle.$$

Let $C_x(R) = \{k_1 < k_2 < \ldots < k_t\}$. The algebra $\mathcal{C}(R)$ is a symmetric CGL extension of the form

$$\mathcal{C}(R) = \mathbb{K}[x_{k_1}][x_{k_2}; \sigma'_{k_2}, \delta'_{k_2}] \ldots [x_{k_t}; \sigma'_{k_t}, \delta'_{k_t}],$$

where the automorphisms $\sigma'_\bullet$, the skew derivations $\delta'_\bullet$, and the torus action $\mathcal{H}$ are obtained by restricting those for the CGL extension R. The elements $h_\bullet$ and $h_\bullet^*$ entering in the definition of a symmetric CGL extension are not changed in going from R to $\mathcal{C}(R)$; we just use a subset of those. The CGL extension $\mathcal{C}(R)$ will be called the *core* of R. The algebra $\mathcal{A}(R)$ is a quantum affine space algebra with commutation relations

$$x_{i_1} x_{i_2} = \lambda_{i_1 i_2} x_{i_2} x_{i_1} \quad \text{for all } i_1, i_2 \in F_x(R). \tag{4.2}$$

It is a symmetric CGL extension with the restriction of the action of $\mathcal{H}$, but this will not play any role below.

Finally, we can express R as a crossed product

$$R = \mathcal{C}(R) * M, \tag{4.3}$$

where M is a free abelian monoid on $|F_x(R)|$ generators. The actions of these generators on $\mathcal{C}(R)$ are given by the automorphisms formed from the commutation relations

$$x_i x_k = \lambda_{ik} x_k x_i \quad \text{for all } i \in F_x(R), k \in C_x(R), \tag{4.4}$$

and products of the images of the elements of M are twisted by a 2-cocycle $M \times M \to \mathbb{K}^*$. Both (4.2) and (4.4) are specializations of (4.1). An alternative description of R is as an iterated Ore extension over $\mathcal{C}(R)$ of the form

$$R = \mathcal{C}(R)[x_{l_1}; \sigma'_{l_1}][x_{l_2}; \sigma'_{l_2}] \cdots [x_{l_s}; \sigma'_{l_s}],$$

where $F_x(R) = \{l_1 < l_2 < \cdots < l_s\}$.

Examples 4.1. Any multiparameter quantum affine space algebra $R = \mathcal{O}_p(\mathbb{K}^N)$ is a CGL extension with all $\delta_k = 0$. In this case, $F_x(R) = P_x(R) = [1, N]$, so $C_x(R) = \varnothing$ and $\mathcal{C}(R) = \mathbb{K}$.

At the other extreme, many generic quantized algebras are CGL extensions for which $F_x(R) = \varnothing$ and so $\mathcal{C}(R) = R$. This holds, for instance, when q is not a root of unity and $R = \mathcal{U}_q^+(\mathfrak{g})$ with $\mathfrak{g} \neq \mathfrak{sl}_2$ (Proposition 4.6) or $R = \mathcal{O}_q(M_{t,n}(\mathbb{K}))$ with $n, t \geq 2$ (Proposition 4.9).

For an intermediate situation, consider

$$R := \mathcal{O}_q(M_3(\mathbb{K}))/\langle X_{21}, X_{31}, X_{32}\rangle,$$

a quantized coordinate ring of the monoid of upper triangular 3×3 matrices, with q not a root of unity. This is a CGL extension with variables $x_1, \ldots, x_6$

equal to the cosets of the generators X_{11}, X_{12}, X_{13}, X_{22}, X_{23}, X_{33}. Here

$$P_x(R) = \{1, 3, 4, 6\} \quad \text{and} \quad F_x(R) = \{1, 6\},$$

whence $\mathcal{C}(R) = \mathbb{K}\langle x_2, x_3, x_4, x_5\rangle \cong \mathcal{O}_q(M_2(\mathbb{K}))$ and $\mathcal{A}(R) = \mathbb{K}[x_1, x_6]$ is a commutative polynomial ring.

4B. *Main theorem on unipotent automorphisms.* Recall that a *connected graded algebra* is a nonnegatively graded algebra $R = \bigoplus_{n=0}^{\infty} R^n$ such that $R^0 = \mathbb{K}$. For such an algebra, set $R^{\geq m} := \bigoplus_{n=m}^{\infty} R^n$ for all $m \in \mathbb{Z}_{\geq 0}$. We have used the notation R^n for homogeneous components to avoid conflict with the notation R_k for partial iterated Ore extensions (Section 2B). The algebra R is called *locally finite* if all of its homogeneous components R^d are finite dimensional over $\mathbb{K}$.

Suppose R is a CGL extension as in Theorem 2.5. Every group homomorphism

$$\pi : X(\mathcal{H}) \to \mathbb{Z}$$

gives rise to an algebra $\mathbb{Z}$-grading on R, such that $u \in R^{\pi(\chi_u)}$ for all $\mathcal{H}$-eigenvectors u in R. This makes the algebra R connected graded if and only if $\pi(\chi_{x_j}) > 0$ for all $j \in [1, N]$. A homomorphism with this property exists if and only if the subsemigroup generated by $\chi_{x_1}, \ldots, \chi_{x_N}$ in $X(\mathcal{H})$ does not contain 0.

Definition 4.2. We call an automorphism ψ of a connected graded algebra R *unipotent* if

$$\psi(x) - x \in R^{\geq m+1} \quad \text{for all } x \in R^m, m \in \mathbb{Z}_{\geq 0}.$$

It is obvious that those automorphisms form a subgroup of $\mathrm{Aut}(R)$, which will be denoted by $\mathrm{UAut}(R)$.

Theorem 4.3. *Let R be a symmetric saturated CGL extension which is a connected graded algebra via a homomorphism $\pi : X(\mathcal{H}) \to \mathbb{Z}$. Then the restriction of every unipotent automorphism of R to the core $\mathcal{C}(R)$ is the identity.*

In other words, every unipotent automorphism ψ of R satisfies

$$\psi(x_k) = x_k, \qquad\qquad \text{for all } k \in C_x(R), \qquad (4.5)$$

$$\psi(x_i) = x_i + a_i, \qquad\qquad \text{for all } i \in F_x(R), \qquad (4.6)$$

where for every $i \in F_x(R)$, a_i is a normal element of R lying in $R^{\geq \deg x_i + 1}$ such that $a_i x_i^{-1}$ is a central element of $R[E(R)^{-1}]$.

The proof of Theorem 4.3 is given in Section 4D.

The restriction of a unipotent automorphism to $\mathcal{A}(R)$ can have a very general form as illustrated by the next two remarks.

Remark 4.4. Consider the quantum affine space algebra

$$R = \mathcal{O}_q(\mathbb{K}^3) := \mathbb{K}\langle x_1, x_2, x_3 \mid x_i x_j = q x_j x_i \text{ for all } i < j \rangle,$$

for a nonroot of unity $q \in \mathbb{K}^*$, which is a symmetric CGL extension with respect to the natural action of $(\mathbb{K}^*)^3$. In this case, $\mathcal{A}(R) = R$ and $\mathcal{C}(R) = \mathbb{K}$. All the generators x_i are prime, thus $P_x(R) = \{1, 2, 3\}$. Introduce the grading such that x_1, x_2, x_3 all have degree 1. The unipotent automorphisms of this algebra are determined [Alev and Chamarie 1992, Théorème 1.4.6] by

$$\psi(x_1) = x_1, \quad \psi(x_2) = x_2 + \xi x_1 x_3, \quad \psi(x_3) = x_3, \qquad \text{for some } \xi \in \mathbb{K}.$$

In particular, in this case the normal element $a_2 = \xi x_1 x_3$ is generally nonzero. At the same time, the normal elements a_1 and a_3 vanish.

Remark 4.5. It is easy to see that the polynomial algebra $R = \mathbb{K}[x_1, \ldots, x_N]$ is a symmetric CGL extension with the standard action of $(\mathbb{K}^*)^N$. In this case, again we have $\mathcal{A}(R) = R$. Currently, little is known for the very large group of unipotent automorphisms of the polynomial algebras in at least 3 variables.

In Section 4C we show how one can explicitly describe the full automorphism groups of many symmetric saturated CGL extensions R with small factors $\mathcal{A}(R)$ using Theorem 4.3 together with graded methods. These "essentially noncommutative" CGL extensions are very rigid; typically, all automorphisms are graded with respect to the grading of Theorem 4.3, and often there are few or no graded automorphisms beyond the diagonal ones. These types of CGL extensions are very common in the theory of quantum groups. We illustrate this by giving a second proof of the Launois–Lenagan conjecture [Launois and Lenagan 2007] that states that

$$\text{Aut}(\mathcal{O}_q(M_n(\mathbb{K})) \cong \mathbb{Z}_2 \ltimes (\mathbb{K}^*)^{2n-1},$$

for all $n > 1$, base fields $\mathbb{K}$, and nonroots of unity $q \in \mathbb{K}^*$. Here, the nontrivial element of $\mathbb{Z}_2$ acts by the transpose automorphism $(X_{lm} \mapsto X_{ml})$ and the torus acts by rescaling the X_{lm}. This conjecture was proved for $n = 2$ in [Alev and Chamarie 1992], for $n = 3$ in [Launois and Lenagan 2013] and for all n in [Yakimov 2013]. We reexamine this in Section 4C, reprove it in a new way, and give a very general approach to such relationships based on Theorem 4.3.

For a simple Lie algebra $\mathfrak{g}$, the algebra $\mathcal{U}_q^+(\mathfrak{g})$ is the subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by all positive Chevalley generators E_α, $\alpha \in \Pi$, recall the setting of Example 3.10. The Andruskiewitsch–Dumas conjecture [Andruskiewitsch and Dumas 2008] predicted an explicit description of the full automorphism group of $\mathcal{U}_q^+(g)$. This conjecture was proved in [Yakimov 2014b] in full generality. The

key part of the conjecture was to show that

$$\mathrm{UAut}(\mathcal{U}_q^+(\mathfrak{g})) = \{\mathrm{id}\}, \tag{4.7}$$

for the $\mathbb{Z}_{\geq 0}$-grading given by $\deg E_\alpha = 1$, $\alpha \in \Pi$. The next proposition establishes that $\mathcal{C}(\mathcal{U}_q^+(\mathfrak{g})) = \mathcal{U}_q^+(\mathfrak{g})$ for all simple Lie algebras $\mathfrak{g} \neq \mathfrak{sl}_2$, and thus (4.7) also follows from Theorem 4.3. Since the pieces of the proof in [Yakimov 2014b] were embedded in the different steps of the proof of Theorem 4.3, this does not give an independent second proof of the Andruskiewitsch–Dumas conjecture. However, it illustrates the broad range of applications of Theorem 4.3 which cover the previous conjectures on automorphism groups in this area.

Proposition 4.6. *For all finite dimensional simple Lie algebras* $\mathfrak{g} \neq \mathfrak{sl}_2$, *base fields* $\mathbb{K}$ *and nonroots of unity* $q \in \mathbb{K}^*$,

$$F_x(\mathcal{U}_q^+(\mathfrak{g})) = \varnothing, \quad i.e., \mathcal{C}(\mathcal{U}_q^+(\mathfrak{g})) = \mathcal{U}_q^+(\mathfrak{g}).$$

Proof. In the setting of Example 3.10, the algebra $\mathcal{U}_q^+(\mathfrak{g})$ coincides with the algebra $\mathcal{U}^+[w_0]$ for the longest element w_0 of the Weyl group of $\mathfrak{g}$. Fix a reduced decomposition $w_0 = s_{\alpha_1} \ldots s_{\alpha_N}$ for $\alpha_1, \ldots, \alpha_N \in \Pi$. Define the roots

$$\beta_1 := \alpha_1, \beta_2 := s_{\alpha_1}(\alpha_1), \ldots, \beta_N := s_{\alpha_1} \ldots s_{\alpha_{N-1}}(\alpha_N)$$

and Lusztig's root vectors

$$E_{\beta_1} := E_{\alpha_1}, E_{\beta_2} := T_{\alpha_1}(E_{\alpha_1}), \ldots, E_{\beta_N} := T_{\alpha_1} \ldots T_{\alpha_{N-1}}(E_{\alpha_N})$$

in terms of Lusztig's braid group action [Jantzen 1996, Section 8.14] on $\mathcal{U}_q(\mathfrak{g})$. The algebra $\mathcal{U}_q^+(\mathfrak{g})$ has a torsion-free CGL extension presentation of the form

$$\mathcal{U}_q^+(\mathfrak{g}) = \mathbb{K}[E_{\beta_1}][E_{\beta_2}; \sigma_2, \delta_2] \ldots [E_{\beta_N}; \sigma_N, \delta_N],$$

for some automorphisms $\sigma_\bullet$ and skew derivations $\delta_\bullet$, the exact form of which will not play a role in the present proof. Since $\beta_1, \ldots, \beta_N$ is a list of all positive roots of $\mathfrak{g}$, for each $\alpha \in \Pi$ there exists $k(\alpha) \in [1, N]$ such that

$$\beta_{k(\alpha)} = \alpha.$$

By [loc.cit., Proposition 8.20],

$$E_{\beta_{k(\alpha)}} = E_\alpha \quad \text{for all } \alpha \in \Pi. \tag{4.8}$$

Given $\alpha \in \Pi$, choose $\alpha' \in \Pi$ which is connected to α in the Dynkin graph of $\mathfrak{g}$. (This is the only place we use that $\mathfrak{g} \neq \mathfrak{sl}_2$.) The Serre relations imply that $E_\alpha E_{\alpha'} \neq \xi E_{\alpha'} E_\alpha$ for all $\xi \in \mathbb{K}$. By (4.8),

$$k(\alpha) \notin P_x(\mathcal{U}_q^+(\mathfrak{g})) \quad \text{for all } \alpha \in \Pi.$$

Thus, $E_\alpha \in \mathcal{C}(\mathcal{U}_q^+(\mathfrak{g}))$ for all $\alpha \in \Pi$. Since $\mathcal{U}_q^+(\mathfrak{g})$ is generated by $\{E_\alpha \mid \alpha \in \Pi\}$, we obtain that

$$\mathcal{C}(\mathcal{U}_q^+(\mathfrak{g})) = \mathcal{U}_q^+(\mathfrak{g}).$$

The decomposition equation (4.3) then implies that $F_x(\mathcal{U}_q^+(\mathfrak{g}))$ is empty.   □

4C. *Full automorphism groups.* There is a large class of quantum nilpotent algebras R for which Theorem 4.3 applies and $\mathcal{C}(R) = R$. For such R, the only unipotent automorphism is the identity. This lack of unipotent automorphisms often combines with other relations to imply that all automorphisms of R are homogeneous with respect to the grading from Theorem 4.3. We flesh out this statement and analyze several examples in this subsection.

Definition 4.7. Let ψ be an automorphism of a connected graded algebra R. The *degree zero component* of ψ is the linear map $\psi_0 : R \to R$ such that

$$\psi_0(x) \text{ is the degree } d \text{ component of } x \quad \text{for all } x \in R^d, d \in \mathbb{Z}_{\geq 0}.$$

The automorphism ψ is said to be *graded* (or *homogeneous of degree zero*) if $\psi = \psi_0$, that is, $\psi(R^d) = R^d$ for all $d \in \mathbb{Z}_{\geq 0}$.

Lemma 4.8. *Let R be a locally finite connected graded algebra, ψ an automorphism of R, and ψ_0 the degree zero component of ψ. Assume that $\psi(R^d) \subseteq R^{\geq d}$, for all $d \in \mathbb{Z}_{\geq 0}$. Then ψ_0 is a graded automorphism of R, and the automorphism $\psi_0^{-1}\psi$ is unipotent.*

Proof. It follows immediately from the hypotheses that ψ_0 is an algebra endomorphism of R. We show that it is an automorphism by proving that ψ_0 maps R^d isomorphically onto R^d, for all $d \in \mathbb{Z}_{\geq 0}$. It suffices to show that $\psi_0(R^d) = R^d$, since R^d is finite dimensional.

Obviously $\psi_0(R^0) = R^0$. Now assume, for some $d \in \mathbb{Z}_{>0}$, that $\psi_0(R^j) = R^j$ for all $j \in [0, d-1]$. Our hypotheses imply that $R^{\geq d} \subseteq \psi^{-1}(R^{\geq d})$, and we next show that this is an equality. If $x \in R \backslash R^{\geq d}$, then $x = y + z$ with y nonzero, $y \in R^j$, and $z \in R^{\geq j+1}$, for some $j \in [0, d-1]$. The assumption $\psi_0(R^j) = R^j$ implies $R^j \cap \ker \psi_0 = 0$, so $\psi_0(y) \neq 0$. Since $\psi(x) - \psi_0(y) \in R^{\geq j+1}$, it follows that $\psi(x) \notin R^{\geq d}$. This shows that, indeed, $R^{\geq d} = \psi^{-1}(R^{\geq d})$, whence $\psi(R^{\geq d}) = R^{\geq d}$. Consequently, any $v \in R^d$ can be expressed as $v = \psi(u)$ for some $u \in R^{\geq d}$, and thus $v = \psi_0(u_d)$ where u_d is the degree d component of u. This verifies $\psi_0(R^d) = R^d$ and establishes the required inductive step.

Therefore ϕ_0 is an automorphism of R. It is clear that $\psi_0^{-1}\psi$ is unipotent. □

The condition on ψ in Lemma 4.8 is often satisfied in quantum algebras. In particular, Launois and Lenagan established it [2007, Proposition 4.2] when R is a locally finite connected graded domain, generated in degree 1 by elements

$x_1, \ldots, x_n$ such that for all $i \in [1, n]$, there exist $x_i' \in R$ with $x_i x_i' = q_i x_i' x_i$ for some $q_i \in \mathbb{K}^*$, $q_i \neq 1$. If, in addition, R is a symmetric saturated CGL extension such that $\mathcal{C}(R) = R$ and R is connected graded via a homomorphism $\pi : X(\mathcal{H}) \to \mathbb{Z}$, we can conclude from Theorem 4.3 that all automorphisms of R are graded. We illustrate this by giving a second proof of the descriptions of $\mathrm{Aut}(\mathcal{O}_q(M_{t,n}(\mathbb{K})))$ in [Launois and Lenagan 2007, Theorem 4.9, Corollary 4.11] and [Yakimov 2013, Theorem 3.2].

Proposition 4.9. *For all integers $n, t \geq 2$, base fields $\mathbb{K}$, and nonroots of unity $q \in \mathbb{K}^*$,*

$$F_x\big(\mathcal{O}_q(M_{t,n}(\mathbb{K}))\big) = \varnothing, \quad i.e., \mathcal{C}\big(\mathcal{O}_q(M_{t,n}(\mathbb{K}))\big) = \mathcal{O}_q(M_{t,n}(\mathbb{K})). \qquad (4.9)$$

Consequently,

$$\mathrm{UAut}(\mathcal{O}_q(M_{t,n}(\mathbb{K}))) = \{\mathrm{id}\},$$

for the grading of $\mathcal{O}_q(M_{t,n}(\mathbb{K}))$ with $\deg X_{lm} = 1$ for all $l, m \in [1, n]$.

Proof. Recall the CGL extension presentation of $R = \mathcal{O}_q(M_{t,n}(\mathbb{K}))$ from (2.10) and the function η from Example 2.6. We have already noted that R is a symmetric CGL extension. The scalars λ_{kl} are all equal to powers of q. Thus, R is a torsion-free CGL extension, and in particular it is saturated.

The only level sets of η of cardinality 1 are $\eta^{-1}(n-1)$ and $\eta^{-1}(1-t)$, that is, the only generators of R that are prime are X_{1n} and X_{t1}. Thus, $P_x(R) = \{n, (t-1)n+1\}$. The identities

$$X_{1,n-1}X_{2n} - X_{2n}X_{1,n-1} = (q - q^{-1})X_{1n}X_{2,n-1},$$

$$X_{t-1,1}X_{t2} - X_{t2}X_{t-1,1} = (q - q^{-1})X_{t-1,2}X_{t1}$$

imply (4.9). The final conclusion of the proposition now follows from Theorem 4.3. $\qquad \square$

Theorem 4.10 (Launois–Lenagan, Yakimov). *For all integers $n, t \geq 2$, base fields $\mathbb{K}$, and nonroots of unity $q \in \mathbb{K}^*$,*

$$\mathrm{Aut}(\mathcal{O}_q(M_{t,n}(\mathbb{K})) = \begin{cases} \mathrm{DAut}(\mathcal{O}_q(M_{t,n}(\mathbb{K})) \cong (\mathbb{K}^*)^{t+n-1} & \textit{if } n \neq t, \\ \mathrm{DAut}(\mathcal{O}_q(M_{t,n}(\mathbb{K})) \cdot \{\mathrm{id}, \tau\} \cong (\mathbb{K}^*)^{t+n-1} \rtimes \mathbb{Z}_2 & \textit{if } n = t, \end{cases}$$

where τ is the transpose automorphism of $\mathcal{O}_q(M_{n,n}(\mathbb{K}))$ given by $\tau(X_{ij}) = X_{ji}$, for all $i, j \in [1, n]$.

Remark. In the cases where t or n is 1, $\mathcal{O}_q(M_{t,n}(\mathbb{K}))$ is a quantum affine space algebra. In these cases a description of the automorphism groups was found much earlier in [Alev and Chamarie 1992] using direct arguments.

Proof. Let $R = \mathcal{O}_q(M_{t,n}(\mathbb{K}))$ as in Examples 2.6, 3.7, with $n, t \geq 2$. This algebra is a locally finite connected graded domain in which all generators X_{ij} have degree 1. By [Launois and Lenagan 2007, Corollary 4.3], all automorphisms ψ of R satisfy $\psi(R^d) \subseteq R^{\geq d}$, for all $d \in \mathbb{Z}_{\geq 0}$. Thus, by Lemma 4.8, Theorem 4.3, and Proposition 4.9, all automorphisms of R are graded.

It remains to show that any graded automorphism ψ of R has the stated form. We first look at the induced automorphism $\overline{\psi}$ on the abelianization $\overline{R} := R/[R, R]$. Note that the cosets in $\overline{R}$ of the generators X_{ij} satisfy

$$\overline{X}_{ij}\overline{X}_{lm} = 0 \text{ if } \begin{cases} (i = l, j \neq m), & \text{or} \\ (i \neq l, j = m), & \text{or} \\ (i < l, j > m), \end{cases}$$

and that the $\overline{X}_{ij}^2$ together with the products $\overline{X}_{ij}\overline{X}_{lm}$ for $i < l$ and $j < m$ form a basis for $\overline{R}^2$. It is easily checked that the degree 1 part of the annihilator of $\overline{X}_{1n}$ has dimension $tn - 1$, as does that of $\overline{X}_{t1}$, while no degree 1 elements of $\overline{R}$ other than scalar multiples of $\overline{X}_{1n}$ or $\overline{X}_{t1}$ have this property. Thus, $\overline{\psi}(\overline{X}_{1n})$ must be a scalar multiple of either $\overline{X}_{1n}$ or $\overline{X}_{t1}$, and similarly for $\overline{\psi}(\overline{X}_{t1})$. It follows that in R, we have $\psi(X_{1n}), \psi(X_{t1}) \in \mathbb{K}^* X_{1n} \cup \mathbb{K}^* X_{t1}$.

Now define

$$C_s(x) := \{y \in R^1 \mid xy = q^s yx\} \quad \text{for all } s \in \mathbb{Z}, x \in R^1,$$

and observe that $\psi(C_s(x)) = C_s(\psi(x))$. Since $C_1(X_{1n})$ and $C_1(X_{t1})$ have dimensions $t - 1$ and $n - 1$, respectively, we conclude that

$$\psi(X_{1n}) \in \mathbb{K}^* X_{1n} \quad \text{and} \quad \psi(X_{t1}) \in \mathbb{K}^* X_{t1} \tag{4.10}$$

if $t \neq n$. If $t = n$ and $\psi(X_{1n}) \in \mathbb{K}^* X_{t1}$, $\psi(X_{t1}) \in \mathbb{K}^* X_{1n}$, then the composition $\psi\tau$ will have the property (4.10). Thus, it remains to show that every graded automorphism ψ of R that satisfies (4.10) is a diagonal automorphism. It follows from (4.10) that ψ preserves the space

$$V := R^1 \cap C_{-1}(X_{1n}) = \mathbb{K}X_{11} + \cdots + \mathbb{K}X_{1,n-1}.$$

For $j \in [1, n - 1]$, the elements $v \in V$ for which

$$\dim_{\mathbb{K}}(V \cap C_1(v)) = n - j - 1 \quad \text{and} \quad \dim_{\mathbb{K}}(V \cap C_{-1}(v)) = j - 1$$

are just the nonzero scalar multiples of X_{1j}. Hence, $\psi(X_{1j}) \in \mathbb{K}^* X_{1j}$ for all $j \in [1, n]$. Similarly, $\psi(X_{i1}) \in \mathbb{K}^* X_{i1}$ for all $i \in [1, t]$.

Finally, for $i \in [2, t]$ and $j \in [2, n]$, the elements of $C_1(X_{i1}) \cap C_1(X_{1j})$ are exactly the nonzero scalar multiples of X_{ij}. We conclude that $\psi(X_{ij}) \in \mathbb{K}^* X_{ij}$ for all $i \in [1, t]$, $j \in [1, n]$, showing that ψ is a diagonal automorphism of R. $\square$

We give two additional examples which can be established in similar fashion, leaving details to the reader.

Example 4.11. First, let $R := \mathcal{O}_q(\mathfrak{sp}\, k^{2n})$ be the quantized coordinate ring of $2n$-dimensional symplectic space, with generators $x_1, \ldots, x_{2n}$ and relations as in [Musson 1993, Section 1.1]. (This presentation gives a symmetric CGL extension presentation, whereas the original presentation in [Reshetikhin et al. 1989, Definition 14] is not symmetric.) Then:

For all integers $n > 0$, base fields $\mathbb{K}$, and nonroots of unity $q \in \mathbb{K}^*$,

$$\mathrm{Aut}(\mathcal{O}_q(\mathfrak{sp}\, k^{2n})) = \mathrm{DAut}(\mathcal{O}_q(\mathfrak{sp}\, k^{2n}) \cong (\mathbb{K}^*)^{n+1}.$$

Now let $R := \mathcal{O}_q(\mathfrak{o}\, k^m)$ be the quantized coordinate ring of m-dimensional euclidean space, with generators $x_1, \ldots, x_m$ and relations as in [Musson 1993, Sections 2.1 and 2.2]. Then:

For all integers $n > 0$, base fields $\mathbb{K}$, and nonroots of unity $q \in \mathbb{K}^*$,

$$\mathrm{Aut}(\mathcal{O}_q(\mathfrak{o}\, k^{2n})) = \mathrm{DAut}(\mathcal{O}_q(\mathfrak{o}\, k^{2n})) \cdot \langle \tau \rangle \cong (\mathbb{K}^*)^{n+1} \rtimes \mathbb{Z}_2,$$

$$\mathrm{Aut}(\mathcal{O}_q(\mathfrak{o}\, k^{2n+1})) = \mathrm{DAut}(\mathcal{O}_q(\mathfrak{o}\, k^{2n+1})) \cong (\mathbb{K}^*)^{n+1},$$

where τ is the automorphism of $\mathcal{O}_q(\mathfrak{o}\, k^{2n})$ that interchanges x_n, x_{n+1} and fixes x_i for all $i \neq n, n+1$.

4D. *Proof of Theorem 4.3.* The proof of Theorem 4.3 is based on the rigidity of quantum tori result of [Yakimov 2014b]. This proof is carried out in six steps via Lemmas 4.12–4.17 below. Some parts of it are similar to the proof of the Andruskiewitsch–Dumas conjecture in [loc.cit., Theorem 1.3], other parts are different. Throughout the proof we use the general facts for CGL extensions established in [Goodearl and Yakimov 2012; 2013].

Note that the $\mathbb{Z}_{\geq 0}$-grading on the algebra R in Theorem 4.3 extends to a $\mathbb{Z}$-grading on $R[E(R)^{-1}]$, since $E(R)$ is generated by homogeneous elements.

Lemma 4.12. *In the setting of Theorem 4.3, for every $k \in [1, N]$ there exists $z_k \in Z(R[E(R)^{-1}])^{\geq 1}$ such that*

$$\psi(x_k) = (1 + z_k)x_k.$$

Note. It follows from Proposition 2.11 that the elements z_k satisfy

$$z_k \in \mathcal{N}(R)[E(R)^{-1}] \quad \text{for all } k \in [1, N]. \tag{4.11}$$

Proof. Fix $k \in [1, N]$. There exists an element τ of the subset Ξ_N of the symmetric group S_N defined in (2.16) such that $\tau(1) = k$. For example, one can choose

$$\tau = [k, k+1, \ldots, n, k-1, k-2, \ldots, 1]$$

in the one-line notation for permutations. For the corresponding sequence of prime elements, we have $y_{\tau,1} = x_k$. The corresponding embeddings $\mathcal{A}_\tau \subseteq R \subset \mathcal{T}_\tau$ are $X(\mathcal{H})$-graded. We use the homomorphism $\pi : X(\mathcal{H}) \to \mathbb{Z}$ to obtain a $\mathbb{Z}_{\geq 0}$-grading on $\mathcal{A}_\tau$ and a $\mathbb{Z}$-grading on $\mathcal{T}_\tau$ for which all generators $y_{\tau,1}, \ldots, y_{\tau,N}$ have positive degree. The embeddings $\mathcal{A}_\tau \subseteq R \subset \mathcal{T}_\tau$ become $\mathbb{Z}$-graded. It follows from Proposition 2.15 that $\mathcal{T}_\tau$ is a saturated quantum torus since R is a saturated CGL extension.

Applying the rigidity of quantum tori result in [Yakimov 2014b, Theorem 1.2] and the conversion result [Yakimov 2013, Proposition 3.3], we obtain that

$$\psi(y_{\tau,k}) = (1 + c_k)y_{\tau,k} \text{ for some } c_k \in Z(\mathcal{T}_\tau)^{\geq 1} \quad \text{for all } k \in [1, N].$$

By Proposition 2.11, $Z(\mathcal{T}_\tau) = Z(R[E(R)^{-1}])$. Using that $y_{\tau,1} = x_k$ and setting $z_k := c_1$ leads to the desired result. $\qquad\square$

From now on, all characters will be computed with respect to the torus $\mathrm{DAut}(R)$, recall Definition 2.12 and the discussion after it. For an algebra A, we denote by A^* its group of units.

Lemma 4.13. *In the setting of Theorem 4.3, the elements* $z_k \in Z(R[E(R)^{-1}])$, $k \in [1, N]$, *from Lemma 4.12 define a group homomorphism*

$$X(\mathrm{DAut}(R)) \to \mathrm{Fract}(Z(R[E(R)^{-1}]))^*$$

such that

$$\chi_{x_k} \mapsto 1 + z_k, \tag{4.12}$$

for $k \in [1, N]$.

Consequently, if u is any homogeneous element of R and

$$\chi_u = j_1 \chi_{x_1} + \cdots + j_N \chi_{x_N},$$

for some $j_1, \ldots, j_N \in \mathbb{Z}$, *then*

$$\psi(u) = (1 + z_1)^{j_1} \cdots (1 + z_N)^{j_N} u. \tag{4.13}$$

Proof. It follows from [Goodearl and Yakimov 2012, Theorem 5.5] that the character lattice $X(\mathrm{DAut}(R))$ is generated by $\chi_{x_1}, \ldots, \chi_{x_N}$. For $l \in [1, N]$, denote by $X(\mathrm{DAut}(R))_l$ the subgroup of $X(\mathrm{DAut}(R))$ generated by $\chi_{x_1}, \ldots, \chi_{x_l}$.

We show by induction on l that there exists a group homomorphism

$$X(\mathrm{DAut}(R))_l \to \mathrm{Fract}(Z(R[E(R)^{-1}]))^*$$

satisfying (4.12) for $k \in [1, l]$. The statement is obvious for $l = 1$. Assume its validity for $l - 1$, where $l \geq 2$. If $\delta_l = 0$, then by [Goodearl and Yakimov 2012,

Theorem 5.5],

$$X(\mathrm{DAut}(R))_l = X(\mathrm{DAut}(R))_{l-1} \oplus \mathbb{Z}\chi_{x_l},$$

and the statement follows trivially. Now consider the case $\delta_l \neq 0$. Choose $j < l$ such that $\delta_l(x_j) \neq 0$, in other words, $Q_{lj} \neq 0$. Choose a monomial $x_{j+1}^{m_{j+1}} \ldots x_{l-1}^{m_{l-1}}$ which appears with a nonzero coefficient in Q_{lj}, and observe that

$$\chi_{x_l} = -\chi_{x_j} + m_{j+1}\chi_{x_{j+1}} + \cdots + m_{l-1}\chi_{x_{l-1}}.$$

The inductive step thus amounts to proving that

$$(1 + z_l) = (1 + z_j)^{-1}(1 + z_{j+1})^{m_{j+1}} \ldots (1 + z_{l-1})^{m_{l-1}}. \qquad (4.14)$$

The inductive hypothesis, the fact that $z_1, \ldots, z_l$ are central, and that all monomials appearing with nonzero coefficients in Q_{lj} have the same $X(\mathrm{DAut}(R))$-degrees give

$$\psi(Q_{lj}) = (1 + z_{j+1})^{m_{j+1}} \ldots (1 + z_{l-1})^{m_{l-1}} Q_{lj}.$$

Applying ψ to the identity $Q_{lj} = x_l x_j - \lambda_{lj} x_j x_l$ and again using that $z_1, \ldots, z_l$ are central leads to

$$(1 + z_{j+1})^{m_{j+1}} \ldots (1 + z_{l-1})^{m_{l-1}} Q_{lj} = (1 + z_l)(1 + z_j)(x_l x_j - \lambda_{lj} x_j x_l)$$
$$= (1 + z_l)(1 + z_j) Q_{lj}.$$

This implies (4.14) because $Q_{lj} \neq 0$, and completes the induction, establishing the first part of the lemma.

The last statement of the lemma follows from the first part of the lemma, the centrality of the z_k, and the fact that all monomials $x_1^{m_1} \cdots x_N^{m_N}$ appearing with nonzero coefficients in u have the same $X(\mathrm{DAut}(R))$-degree as u. $\qquad \square$

Lemma 4.14. *Any symmetric CGL extension R of length N is a free left $\mathcal{N}(R)$-module in which $\mathcal{N}(R)x_k$ is a direct summand, for all $k \in [1, N] \backslash P_x(R)$.*

If $ux_k \in R$ for some $u \in \mathcal{N}(R)[E(R)^{-1}]$ and $k \in [1, N] \backslash P_x(R)$, then $u \in \mathcal{N}(R)$.

Proof. Theorem 4.11 in [Goodearl and Yakimov 2012] proves that R is a free left module over $\mathcal{N}(R)$ and constructs an explicit basis of it. For $k \in [1, N] \backslash P_x(R)$, the element x_k becomes one of the basis elements, because $|\eta^{-1}(\eta(k))| > 1$. This proves the first part of the lemma.

For the second part, write $r := ux_k$ and $u = e^{-1}y$ for some $e \in E(R)$ and $y \in \mathcal{N}(R)$. Then $er = yx_k \in \mathcal{N}(R)x_k$. It follows from the first part of the lemma that $r \in \mathcal{N}(R)x_k$, and therefore $u \in \mathcal{N}(R)$. $\qquad \square$

Lemma 4.15. *In the setting of Theorem 4.3, the elements z_k from Lemma 4.12 satisfy*

$$z_k \in Z(R)^{\geq 1} \quad \text{for all } k \in [1, N] \backslash P_x(R).$$

Proof. By (4.11), $z_k \in \mathcal{N}(R)[E(R)^{-1}]$. Furthermore, $z_k x_k = \psi(x_k) - x_k \in R$. We apply the second part of Lemma 4.14 to $u := z_k$ to obtain $z_k \in \mathcal{N}(R)$ and so

$$z_k \in R \cap Z(R[E(R)^{-1}])^{\geq 1} = Z(R)^{\geq 1},$$

for all $k \in [1, N] \backslash P_x(R)$. $\square$

Lemma 4.16. *In the setting of Theorem 4.3, the elements z_k from Lemma 4.12 satisfy*

$$z_k = 0 \quad \text{for all } k \in [1, N] \backslash P_x(R).$$

Proof. Let $k \in [1, N] \backslash P_x(R)$ and denote

$$\eta^{-1}(\eta(k)) = \{k_1 < \cdots < k_m\}.$$

By Theorem 2.5, y_{k_m} is a homogeneous prime element of R and

$$\chi_{y_m} = \chi_{x_{k_1}} + \cdots + \chi_{x_{k_m}}.$$

Applying (4.13) with $u = y_{k_m}$ gives

$$\psi(y_{k_m}) = (1 + z_{k_1}) \ldots (1 + z_{k_m}) y_{k_m}.$$

From Lemma 4.15, $z_{k_1}, \ldots, z_{k_m} \in Z(R)$. So, $\psi(Ry_{k_m}) \subseteq Ry_{k_m}$. At the same time, Ry_{k_m} is a height one prime ideal of R, and so $\psi(Ry_{k_m})$ is a height one prime ideal. Therefore $\psi(Ry_{k_m}) = Ry_{k_m}$, which implies that $(1 + z_{k_1}) \ldots (1 + z_{k_m})$ is a unit of R. The group of units of a CGL extension is reduced to scalars, thus

$$(1 + z_{k_1}) \ldots (1 + z_{k_m}) \in \mathbb{K}^*.$$

Since $z_{k_1}, \ldots, z_{k_m} \in R^{\geq 1}$, this is only possible if $z_{k_1} = \cdots = z_{k_m} = 0$. Therefore $z_k = 0$. $\square$

Lemma 4.17. *In the setting of Theorem 4.3, the elements $z_k \in Z(R[E(R)^{-1}]$ from Lemma 4.12 satisfy*

$$z_k = 0 \quad \text{for all } k \in C_x(R).$$

Proof. The statement was proved for $k \in [1, N] \backslash P_x(R)$ in Lemma 4.16. Now let $k \in C_x(R) \cap P_x(R)$. There exist $j, l \in [1, N] \backslash P_x(R)$ such that $j < k < l$ and there is a monomial $x_{j+1}^{m+1} \ldots x_{l-1}^{m_{l-1}}$ with $m_k > 0$ that appears with a nonzero coefficient in Q_{lj}. Applying ψ to the identity $Q_{lj} = x_l x_j - \lambda_{lj} x_j x_l$ and using Lemmas 4.13 and 4.16 gives

$$(1 + z_{j+1})^{m_{j+1}} \ldots (1 + z_{l-1})^{m_{l-1}} = (1 + z_l)(1 + z_j) = 1.$$

Since $z_{j+1}, \ldots, z_{l-1} \in Z(R[E(R)^{-1}])^{\geq 1}$ and $R[E(R)^{-1}]$ is a graded domain, $z_t = 0$ for all $t \in [j+1, l-1]$ such that $m_t > 0$. Thus $z_k = 0$, because $m_k > 0$. $\square$

Proof of Theorem 4.3. This follows from Lemmas 4.12 and 4.17, setting $a_i = z_i x_i$ for $i \in F_x(R)$, recalling that x_i is normal in R for all $i \in F_x(R)$. $\square$

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References

[Alev and Chamarie 1992] J. Alev and M. Chamarie, "Dérivations et automorphismes de quelques algèbres quantiques", *Comm. Algebra* **20**:6 (1992), 1787–1802.

[Andruskiewitsch and Dumas 2008] N. Andruskiewitsch and F. Dumas, "On the automorphisms of $U_q^+(\mathfrak{g})$", pp. 107–133 in *Quantum groups*, edited by B. Enriquez, IRMA Lect. Math. Theor. Phys. **12**, Eur. Math. Soc., Zürich, 2008.

[Brown and Goodearl 2002] K. A. Brown and K. R. Goodearl, *Lectures on algebraic quantum groups*, Birkhäuser, Basel, 2002.

[Ceken et al. 2015] S. Ceken, J. H. Palmieri, Y.-H. Wang, and J. J. Zhang, "The discriminant controls automorphism groups of noncommutative algebras", *Adv. Math.* **269** (2015), 551–584.

[Chatters 1984] A. W. Chatters, "Noncommutative unique factorization domains", *Math. Proc. Cambridge Philos. Soc.* **95**:1 (1984), 49–54.

[Chatters and Jordan 1986] A. W. Chatters and D. A. Jordan, "Noncommutative unique factorisation rings", *J. London Math. Soc.* (2) **33**:1 (1986), 22–32.

[Farinati 2005] M. Farinati, "Hochschild duality, localization, and smash products", *J. Algebra* **284**:1 (2005), 415–434.

[Ginzburg 2007] V. Ginzburg, "Calabi–Yau algebras", preprint, 2007. arXiv 0612139v3

[Goodearl and Yakimov 2012] K. R. Goodearl and M. T. Yakimov, "From quantum Ore extensions to quantum tori via noncommutative UFDs", preprint, 2012. *Advances in Math.* (in press). arXiv 1208.6267

[Goodearl and Yakimov 2013] K. R. Goodearl and M. T. Yakimov, "Quantum cluster algebra structures on quantum nilpotent algebras", preprint, 2013. *Memoirs Amer. Math. Soc.* (in press). arXiv 1309.7869

[Jantzen 1996] J. C. Jantzen, *Lectures on quantum groups*, Graduate Studies in Mathematics **6**, Amer. Math. Soc., Providence, RI, 1996.

[Joseph 1995] A. Joseph, *Quantum groups and their primitive ideals*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **29**, Springer, Berlin, 1995.

[Launois et al. 2006] S. Launois, T. H. Lenagan, and L. Rigal, "Quantum unique factorisation domains", *J. London Math. Soc.* (2) **74**:2 (2006), 321–340.

[Launois and Lenagan 2007] S. Launois and T. H. Lenagan, "Primitive ideals and automorphisms of quantum matrices", *Algebr. Represent. Theory* **10**:4 (2007), 339–365.

[Launois and Lenagan 2013] S. Launois and T. H. Lenagan, "Automorphisms of quantum matrices", *Glasg. Math. J.* **55**:A (2013), 89–100.

[Liu and Wu 2014] L.-Y. Liu and Q.-S. Wu, "Twisted Calabi–Yau property of right coideal subalgebras of quantized enveloping algebras", *J. Algebra* **399** (2014), 1073–1085.

[Liu et al. 2014] L.-Y. Liu, S.-Q. Wang, and Q.-S. Wu, "Twisted Calabi–Yau property of Ore extensions", *J. Noncomm. Geom.* **8**:2 (2014), 587–609.

[Musson 1993] I. M. Musson, "Ring-theoretic properties of the coordinate rings of quantum symplectic and Euclidean space", pp. 248–258 in *Ring theory* (Granville, OH, 1992), edited by S. K. Jain and S. T. Rizvi, World Sci. Publ., River Edge, NJ, 1993.

[Reshetikhin et al. 1989] N. Y. Reshetikhin, L. A. Takhtadzhyan, and L. D. Faddeev, "Quantization of Lie groups and Lie algebras", *Algebra i Analiz* **1**:1 (1989), 178–206. In Russian; translated in *Leningrad Math. J.* **1**:1 (1990), 193–225.

[Yakimov 2013] M. Yakimov, "The Launois–Lenagan conjecture", *J. Algebra* **392** (2013), 1–9.

[Yakimov 2014a] M. Yakimov, *On the spectra of quantum groups*, Mem. Amer. Math. Soc. **1078**, Amer. Math. Soc., Providence, RI, 2014.

[Yakimov 2014b] M. Yakimov, "Rigidity of quantum tori and the Andruskiewitsch–Dumas conjecture", *Selecta Math. (N.S.)* **20**:2 (2014), 421–464.

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