

# Formal fibers of prime ideals in polynomial rings

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Let  $(R, \mathfrak{m})$  be a Noetherian local domain of dimension  $n$  that is essentially finitely generated over a field and let  $\hat{R}$  be the  $\mathfrak{m}$ -adic completion of  $R$ . Matsumura has shown that  $n - 1$  is the maximal height possible for prime ideals  $P$  of  $\hat{R}$  such that  $P \cap R = (0)$ . In this article we prove that  $\text{ht } P = n - 1$ , for every prime ideal  $P$  of  $\hat{R}$  that is maximal with respect to  $P \cap R = (0)$ . We also present a related result concerning generic formal fibers of certain extensions of mixed polynomial-power series rings.

## 1. Introduction

Let  $(R, \mathfrak{m})$  be a Noetherian local domain and let  $\hat{R}$  be the  $\mathfrak{m}$ -adic completion of  $R$ . The *generic formal fiber ring* of  $R$  is the localization  $(R \setminus (0))^{-1} \hat{R}$  of  $\hat{R}$  with respect to the multiplicatively closed set of nonzero elements of  $R$ . Let  $\text{Gff}(R)$  denote the generic formal fiber ring of  $R$ . If  $R$  is essentially finitely generated over a field and  $\dim R = n$ , then  $\dim(\text{Gff}(R)) = n - 1$  by the result of Matsumura [1988, Theorem 2] mentioned in the abstract. In this article we show every maximal ideal of  $\text{Gff}(R)$  has height  $n - 1$ ; equivalently,  $\text{ht } P = n - 1$ , for every prime ideal  $P$  of  $\hat{R}$  that is maximal with respect to  $P \cap R = (0)$ , a sharpening of Matsumura's result.

In earlier articles we encounter formal fibers and generic fibers; these concepts are related to our study of prime spectral maps among “mixed polynomial-power series rings” over a field; see [Heinzer et al. 2006a; 2006b; 2007]. For  $P$  in  $\text{Spec } R$ , the *formal fiber* over  $P$  is the affine scheme  $\text{Spec}((R \setminus P)^{-1}(\hat{R}/P\hat{R}))$ , or equivalently  $\text{Spec}((R_P/PR_P) \otimes_R \hat{R})$ . Let  $\text{Gff}(R/P)$  denote the generic formal fiber ring of  $R/P$ . Since  $\hat{R}/P\hat{R}$  is the completion of  $R/P$ , the formal fiber over  $P$  is  $\text{Spec}(\text{Gff}(R/P))$ .

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The authors are grateful for the hospitality, cooperation and support of Michigan State, Nebraska, Purdue, and MSRI in Berkeley, where we worked on this research.

We thank the referee for helpful suggestions about this paper.

MSC2010: 13B35, 13J10, 13A15.

Keywords: formal fibers, generic formal fiber ring.

Let  $n$  be a positive integer, let  $X = \{x_1, \dots, x_n\}$  be a set of  $n$  variables over a field  $k$ , and let  $A := k[x_1, \dots, x_n]_{(x_1, \dots, x_n)} = k[X]_{(X)}$  denote the localized polynomial ring in these  $n$  variables over  $k$ . Then the completion of  $A$  is  $k[\widehat{X}] = k[[X]]$ .

With this notation, we have:

**Theorem 1.1** [Heinzer et al. 2006a, Theorem 1.1.1]. *Let  $A = k[X]_{(X)}$  be the localized polynomial ring as defined above. Every maximal ideal of the generic formal fiber ring  $\text{Gff}(A)$  has height  $n - 1$ . Equivalently, if  $Q$  is an ideal of  $\widehat{A}$  maximal with respect to  $Q \cap A = (0)$ , then  $Q$  is a prime ideal of height  $n - 1$ .*

We were inspired to revisit and generalize Theorem 1.1 by Youngsu Kim. His interest in formal fibers and the material in [Heinzer et al. 2006a] inspired us to consider the second question below.

**Questions 1.2.** For  $n \in \mathbb{N}$ , let  $x_1, \dots, x_n$  be indeterminates over a field  $k$ , and let  $R = k[x_1, \dots, x_n]_{(x_1, \dots, x_n)}$  denote the localized polynomial ring with maximal ideal  $\mathfrak{m} = (x_1, \dots, x_n)R$ . Let  $\widehat{R}$  be the  $\mathfrak{m}$ -adic completion of  $R$ .

- (1) For  $P \in \text{Spec } R$ , what is the dimension of the generic formal fiber ring  $\text{Gff}(R/P)$ ?
- (2) What heights are possible for maximal ideals of the ring  $\text{Gff}(R/P)$ ?

In connection with Questions 1.2(1), for  $P \in \text{Spec } R$ , the ring  $R/P$  is essentially finitely generated over a field, and a result of Matsumura [1988, Corollary, p. 263] states that  $\dim(\text{Gff}(R/P)) = n - 1 - \text{ht } P$ .

Sharpening Matsumura's result and Theorem 1.1, we prove Theorem 1.3; see also Theorem 3.5. Thus the answer to Question 1.2(2) is that the height of every maximal ideal of  $\text{Gff}(R/P)$  is  $n - 1 - \text{ht } P$ .

**Theorem 1.3.** *Let  $S$  be a local domain essentially finitely generated over a field; thus  $S = k[s_1, \dots, s_r]_{\mathfrak{p}}$ , where  $k$  is a field,  $r \in \mathbb{N}$ , the elements  $s_i$  are in  $S$  and  $\mathfrak{p}$  is a prime ideal of the finitely generated  $k$ -algebra  $k[s_1, \dots, s_r]$ . Let  $\mathfrak{n} := \mathfrak{p}S$  and let  $\widehat{S}$  denote the  $\mathfrak{n}$ -adic completion of  $S$ . Then every maximal ideal of  $\text{Gff}(S)$  has height  $\dim S - 1$ . Equivalently, if  $Q \in \text{Spec } \widehat{S}$  is maximal with respect to  $Q \cap S = (0)$ , then  $\text{ht } Q = \dim S - 1$ .*

In Theorem 4.2 of Section 4, we prove that all maximal ideals in the generic formal fiber of certain extensions of a mixed polynomial-power series ring have the same height.

## 2. Background and preliminaries

We begin with historical remarks concerning dimensions and heights of maximal ideals of generic formal fiber rings for Noetherian local domains:

**Remarks 2.1.** 1. Let  $(R, \mathfrak{m})$  be an  $n$ -dimensional Noetherian local domain. Matsumura [1988] remarks that as the ring  $R$  gets closer to its  $\mathfrak{m}$ -adic completion  $\widehat{R}$ , it is natural to think that the dimension of the generic formal fiber ring  $\text{Gff}(R)$  gets smaller. Matsumura describes examples where  $\dim(\text{Gff}(R))$  has one of the three values  $n-1$ ,  $n-2$  or  $0$ , and speculates [loc.cit., p. 261] as to whether these are the only possible values for  $\dim(\text{Gff}(R))$ .

2. Matsumura's question in item 1 is answered by Rotthaus [1991]; she establishes the following result: For every positive integer  $n$  and every integer  $t$  between  $0$  and  $n-1$ , there exists an excellent regular local ring  $R$  such that  $\dim R = n$  and such that the generic formal fiber ring of  $R$  has dimension  $t$ .

3. For  $(R, \mathfrak{m})$  an  $n$ -dimensional universally catenary Noetherian local domain, Loepp and Rotthaus [2004] compare the dimension of the generic formal fiber ring of  $R$  with that of the localized polynomial ring  $R[x]_{(\mathfrak{m}, x)}$ . It is shown in [Matsumura 1988] that the dimension of the generic formal fiber ring  $\text{Gff}(R[x]_{(\mathfrak{m}, x)})$  is either  $n$  or  $n-1$ . Loepp and Rotthaus [2004, Theorem 2] prove that  $\dim(\text{Gff}(R[x]_{(\mathfrak{m}, x)})) = n$  implies that  $\dim(\text{Gff } R) = n-1$ . They show by example that in general the converse is not true, and they give sufficient conditions for the converse to hold.

4. Let  $(T, M)$  be a complete Noetherian local domain that contains a field of characteristic zero. Assume that  $T/M$  has cardinality at least the cardinality of the real numbers. Loepp [1997; 1998] adapts techniques developed by Heitmann [1993] and proves, among other things, for every prime ideal  $p$  of  $T$  with  $p \neq M$ , there exists an excellent regular local ring  $R$  that has completion  $T$  and has generic formal fiber ring  $\text{Gff}(R) = T_p$ . By varying the height of  $p$ , Loepp obtains examples where the dimension of the generic formal fiber ring is any integer  $t$  with  $0 \leq t < \dim T$ . Loepp shows for these examples that there exists a unique prime  $q$  of  $T$  with  $q \cap R = P$  and  $q = PT$ , for each nonzero prime  $P$  of  $R$ .

5. If  $R$  is an  $n$ -dimensional countable Noetherian local domain, Heinzer, Rotthaus and Sally [Heinzer et al. 1993, Proposition 4.10, p. 36] show that:

- (a) The generic formal fiber ring  $\text{Gff}(R)$  is a Jacobson ring in the sense that each prime ideal of  $\text{Gff}(R)$  is an intersection of maximal ideals of  $\text{Gff}(R)$ .
- (b)  $\dim(\widehat{R}/P) = 1$  for each prime ideal  $P \in \text{Spec } \widehat{R}$  that is maximal with respect to  $P \cap R = (0)$ .
- (c) If  $\widehat{R}$  is equidimensional, then  $\text{ht } P = n-1$  for each prime ideal  $P \in \text{Spec } \widehat{R}$  that is maximal with respect to  $P \cap R = (0)$ .
- (d) If  $Q \in \text{Spec } \widehat{R}$  with  $\text{ht } Q \geq 1$ , then there exists a prime ideal  $P \subset Q$  such that  $P \cap R = (0)$  and  $\text{ht}(Q/P) = 1$ .

It follows from this result that all ideals maximal in the generic formal fiber ring of the ring  $A$  of Theorem 1.1 have the same height, if the field  $k$  is countable.

6. The work of Matsumura [1988], referenced in item 1 above, does not address the question of whether all ideals maximal in the generic formal fiber rings have the same height for the rings he studies. In general, for an excellent regular local ring  $R$ , it can happen that  $\text{Gff}(R)$  contains maximal ideals of different heights; see [Rotthaus 1991, Corollary 3.2].

7. Charters and Loepp [2004, Theorem 3.1] extend Rotthaus' result of the previous item: Let  $(T, M)$  be a complete Noetherian local ring and let  $G$  be a nonempty subset of  $\text{Spec } T$  such that the number of maximal elements of  $G$  is finite. They prove there exists a Noetherian local domain  $R$  whose completion is  $T$  and whose generic formal fiber is exactly  $G$  if  $G$  satisfies the following conditions:

- (a)  $M \notin G$  and  $G$  contains the associated primes of  $T$ .
- (b) If  $P \subset Q$  are in  $\text{Spec } T$  and  $Q \in G$ , then  $P \in G$ .
- (c) Every  $Q \in G$  meets the prime subring of  $T$  in  $(0)$ .

Charters and Loepp [2004, Theorem 4.1] also show that, if  $T$  contains the ring of integers and if, in addition to conditions (a)–(c),

- (d)  $T$  is equidimensional and
- (e)  $T_P$  is a regular local ring for each maximal element  $P$  of  $G$ ,

then there exists an excellent local domain  $R$  whose completion is  $T$  and whose generic formal fiber is exactly  $G$ . Since the maximal elements of the set  $G$  may be chosen to have different heights, this result provides many examples where the generic formal fiber ring contains maximal ideals of different heights.

We make the following observations concerning injective local maps of Noetherian local rings:

**Discussion 2.2.** Let  $\phi : (R, \mathfrak{m}) \hookrightarrow (S, \mathfrak{n})$  be an injective local map of the Noetherian local ring  $(R, \mathfrak{m})$  into a Noetherian local ring  $(S, \mathfrak{n})$ . Let

$$\widehat{R} = \varprojlim_n R/\mathfrak{m}^n \quad \text{and} \quad \widehat{S} = \varprojlim_n S/\mathfrak{n}^n$$

denote the  $\mathfrak{m}$ -adic completion of  $R$  and the  $\mathfrak{n}$ -adic completion of  $S$ . For each  $n \in \mathbb{N}$ , we have  $\mathfrak{m}^n \subseteq \mathfrak{n}^n \cap R$ . Hence there exists a map

$$\phi_n : R/\mathfrak{m}^n \rightarrow R/(\mathfrak{n}^n \cap R) \hookrightarrow S/\mathfrak{n}^n, \quad \text{for each } n \in \mathbb{N}.$$

The family of maps  $\{\phi_n\}_{n \in \mathbb{N}}$  determines a unique map  $\widehat{\phi} : \widehat{R} \rightarrow \widehat{S}$ .

Since  $\mathfrak{m}^n \subseteq \mathfrak{n}^n \cap R$ , the  $\mathfrak{m}$ -adic topology on  $R$  is the subspace topology from  $S$  if and only if for each positive integer  $n$  there exists a positive integer  $s_n$  such

that  $\mathfrak{n}^{s_n} \cap R \subseteq \mathfrak{m}^n$ . Since  $R/\mathfrak{m}^n$  is Artinian, the descending chain of ideals  $\{\mathfrak{m}^n + (\mathfrak{n}^s \cap R)\}_{s \in \mathbb{N}}$  stabilizes. The ideal  $\mathfrak{m}^n$  is closed in the  $\mathfrak{m}$ -adic topology, and it is closed in the subspace topology if and only if

$$\bigcap_{s \in \mathbb{N}} (\mathfrak{m}^n + (\mathfrak{n}^s \cap R)) = \mathfrak{m}^n.$$

Hence  $\mathfrak{m}^n$  is closed in the subspace topology if and only if there exists a positive integer  $s_n$  such that  $\mathfrak{n}^{s_n} \cap R \subseteq \mathfrak{m}^n$ . Thus the subspace topology from  $S$  is the same as the  $\mathfrak{m}$ -adic topology on  $R$  if and only if  $\hat{\phi}$  is injective.

### 3. Gff( $R$ ) and Gff( $S$ ) for $S$ an extension domain of $R$

**Definition 3.1.** Let  $R$  and  $S$  be integral domains with  $R$  a subring of  $S$  and let  $\varphi : R \hookrightarrow S$  denote the inclusion map of  $R$  into  $S$ . The integral domain  $S$  is a *trivial generic fiber extension* of  $R$ , or a *TGF extension* of  $R$ , if every nonzero prime ideal of  $S$  has nonzero intersection with  $R$ . In this case, we also say that  $\varphi$  is a *trivial generic fiber extension* or *TGF extension*.

Theorem 3.2 is useful in considering properties of generic formal fiber rings.

**Theorem 3.2.** Let  $\phi : (R, \mathfrak{m}) \hookrightarrow (S, \mathfrak{n})$  be an injective local map of Noetherian local integral domains. Consider the following properties:

- (1)  $\mathfrak{m}S$  is  $\mathfrak{n}$ -primary, and  $S/\mathfrak{n}$  is finite algebraic over  $R/\mathfrak{m}$ .
- (2)  $R \hookrightarrow S$  is a TGF-extension and  $\dim R = \dim S$ .
- (3)  $R$  is analytically irreducible.
- (4)  $R$  is analytically normal and  $S$  is universally catenary.
- (5) All maximal ideals of  $\text{Gff}(R)$  have the same height.

If properties (1), (2) and (3) hold, then  $\dim(\text{Gff}(R)) = \dim(\text{Gff}(S))$ . If, in addition, properties (4) and (5) hold, then every maximal ideal of  $\text{Gff}(S)$  has height  $h = \dim(\text{Gff}(R))$ .

*Proof.* Let  $\hat{R}$  and  $\hat{S}$  denote the  $\mathfrak{m}$ -adic completion of  $R$  and  $\mathfrak{n}$ -adic completion of  $S$ , respectively, and let  $\hat{\phi} : \hat{R} \rightarrow \hat{S}$  be the natural extension of  $\phi$  as given in Discussion 2.2. Consider the commutative diagram

$$\begin{array}{ccc} \hat{R} & \xrightarrow{\hat{\phi}} & \hat{S} \\ \uparrow & & \uparrow \\ R & \xrightarrow{\phi} & S, \end{array} \tag{3.2.a}$$

where the vertical maps are the natural inclusion maps to the completion. Assume properties (1), (2) and (3) hold. Property (1) implies that  $\hat{S}$  is a finite  $\hat{R}$ -module

with respect to the map  $\hat{\phi}$  by [Matsumura 1989, Theorem 8.4]. By property (2), we have  $\dim \hat{R} = \dim R = \dim S = \dim \hat{S}$ . Item 3 says that  $\hat{R}$  is an integral domain. It follows that the map  $\hat{\phi} : \hat{R} \hookrightarrow \hat{S}$  is injective. Let  $Q \in \text{Spec } \hat{S}$  and let  $P = Q \cap \hat{R}$ . Since  $R \hookrightarrow S$  is a TGF-extension, by (2), commutativity of Diagram (3.2.a) implies that

$$Q \cap S = (0) \iff P \cap R = (0).$$

Therefore  $\hat{\phi}$  induces an injective finite map  $\text{Gff}(R) \hookrightarrow \text{Gff}(S)$ . We conclude that  $\dim(\text{Gff}(R)) = \dim(\text{Gff}(S))$ .

Assume in addition that properties (4) and (5) hold, and let  $h = \dim(\text{Gff}(R))$ . The assumption that  $S$  is universally catenary implies that  $\dim(\hat{S}/\mathfrak{q}) = \dim S$  for each minimal prime  $\mathfrak{q}$  of  $\hat{S}$  by [Matsumura 1989, Theorem 31.7]. Since

$$\frac{\hat{R}}{\mathfrak{q} \cap \hat{R}} \hookrightarrow \frac{\hat{S}}{\mathfrak{q}}$$

is an integral extension, we have  $\mathfrak{q} \cap \hat{R} = (0)$ . The assumption that  $\hat{R}$  is a normal domain implies that the going-down theorem holds for  $\hat{R} \hookrightarrow \hat{S}/\mathfrak{q}$  by [loc.cit., Theorem 9.4(ii)]. Therefore for each  $Q \in \text{Spec } \hat{S}$  we have  $\text{ht } Q = \text{ht } P$ , where  $P = Q \cap \hat{R}$ . Hence if  $\text{ht } P = h$  for each  $P \in \text{Spec } \hat{R}$  that is maximal with respect to  $P \cap R = (0)$ , then  $\text{ht } Q = h$  for each  $Q \in \text{Spec } \hat{S}$  that is maximal with respect to  $Q \cap S = (0)$ . This completes the proof of Theorem 3.2.  $\square$

**Remark 3.3.** We would like to thank Rodney Sharp and Roger Wiegand for their interest in Theorem 3.2. The hypotheses of Theorem 3.2 do not necessarily imply that  $S$  is a finite  $R$ -module, or even that  $S$  is essentially finitely generated over  $R$ . If  $\phi : (R, \mathfrak{m}) \hookrightarrow (T, \mathfrak{n})$  is an extension of rank one discrete valuation rings (DVRs) such that  $T/\mathfrak{n}$  is finite algebraic over  $R/\mathfrak{m}$ , then, for every field  $F$  that contains  $R$  and is contained in the field of fractions of  $\hat{T}$ , the ring  $S := \hat{T} \cap F$  is a DVR such that the extension  $R \hookrightarrow S$  satisfies the hypotheses of Theorem 3.2.

As a specific example where  $S$  is essentially finite over  $R$ , but not a finite  $R$ -module, let  $R = \mathbb{Z}_{5\mathbb{Z}}$ , the integers localized at the prime ideal generated by 5, and let  $A$  be the integral closure of  $\mathbb{Z}_{5\mathbb{Z}}$  in  $\mathbb{Q}[i]$ . Then  $A$  has two maximal ideals lying over  $5R$ , namely  $(1 + 2i)A$  and  $(1 - 2i)A$ . Let  $S = A_{(1+2i)A}$ . Then the extension  $R \hookrightarrow S$  satisfies the hypotheses of Theorem 3.2. Since  $S$  properly contains  $A$ , and every element in the field of fractions of  $A$  that is integral over  $R$  is contained in  $A$ , it follows that  $S$  is not finitely generated as an  $R$ -module. In Remark 4.5, we describe examples in higher dimension where  $S$  is not a finite  $R$ -module.

**Discussion 3.4.** As in the statement of Theorem 1.3, let  $S = k[z_1, \dots, z_r]_{\mathfrak{p}}$  be a local domain essentially finitely generated over a field  $k$ . We observe that  $S$  is a

localization at a maximal ideal of an integral domain that is a finitely generated algebra over an extension field  $F$  of  $k$ .

To see this, let  $A = k[x_1, \dots, x_r]$  be a polynomial ring in  $r$  variables over  $k$ , and let  $Q$  denote the kernel of the  $k$ -algebra homomorphism of  $A$  onto  $k[z_1, \dots, z_r]$  defined by mapping  $x_i \mapsto z_i$ , for each  $i$  with  $1 \leq i \leq r$ . Using permutability of localization and residue class formation, there exists a prime ideal  $N \supset Q$  of  $A$  such that  $S = A_N/QA_N$ . A version of Noether normalization as in [Matsumura 1980, Theorem 24 (14.F), p. 89] states that, if  $\text{ht } N = s$ , then there exist elements  $y_1, \dots, y_r$  in  $A$  such that  $A$  is integral over  $B = k[y_1, \dots, y_r]$  and  $N \cap B = (y_1, \dots, y_s)B$ . It follows that  $y_1, \dots, y_r$  are algebraically independent over  $k$  and  $A$  is a finitely generated  $B$ -module. Let  $F$  denote the field  $k(y_{s+1}, \dots, y_r)$ , and let  $U$  denote the multiplicatively closed set  $k[y_{s+1}, \dots, y_r] \setminus (0)$ . Then  $U^{-1}B$  is the polynomial ring  $F[y_1, \dots, y_s]$ , and  $U^{-1}A := C$  is a finitely generated  $U^{-1}B$ -module. Moreover  $NC$  is a prime ideal of  $C$  such that

$$NC \cap U^{-1}B = (y_1, \dots, y_s)U^{-1}B = (y_1, \dots, y_s)F[y_1, \dots, y_s]$$

is a maximal ideal of  $U^{-1}B$ , and  $(y_1, \dots, y_s)C$  is primary for the maximal ideal of  $C$ . Hence  $NC$  is a maximal ideal of  $C$  and  $S = C_{NC}/QC_{NC}$  is a localization of the finitely generated  $F$ -algebra  $D := C/QC$  at the maximal ideal  $NC/QC$ .

Therefore  $S$  is a localization of an integral domain  $D$  at a maximal ideal of  $D$  and  $D$  is a finitely generated algebra over an extension field  $F$  of  $k$ .

**Theorem 3.5.** *Let  $S$  be a local integral domain of dimension  $d$  that is essentially finitely generated over a field. Then every maximal ideal of the generic formal fiber ring  $\text{Gff}(S)$  has height  $d - 1$ .*

*Proof.* Using Discussion 3.4, we write  $S = D_N$ , where  $N$  is a maximal ideal of a finitely generated algebra  $D$  over a field  $F$ . Let  $\mathfrak{n} = NS$  be the maximal ideal of  $S$ . Choose  $x_1, \dots, x_d$  in  $\mathfrak{n}$  such that  $x_1, \dots, x_d$  are algebraically independent over  $F$  and  $(x_1, \dots, x_d)S$  is  $\mathfrak{n}$ -primary. Set  $R = F[x_1, \dots, x_d]_{(x_1, \dots, x_d)}$ , a localized polynomial ring over  $F$ , and let  $\mathfrak{m} = (x_1, \dots, x_d)R$ .

To prove Theorem 3.5, it suffices to show that the inclusion map  $\phi : R \hookrightarrow S$  satisfies properties (1)–(5) of Theorem 3.2. By construction,  $\phi$  is an injective local homomorphism and  $\mathfrak{m}S$  is  $\mathfrak{n}$ -primary. Also  $R/\mathfrak{m} = F$  and  $S/\mathfrak{n} = D/N$  is a field that is a finitely generated  $F$ -algebra and hence a finite algebraic extension field of  $F$ ; see [Matsumura 1989, Theorem 5.2]. Therefore property (1) holds. Since  $\dim S = d = \dim D$ , the field of fractions of  $S$  has transcendence degree  $d$  over the field  $F$ . Therefore  $S$  is algebraic over  $R$ . It follows that  $R \hookrightarrow S$  is a TGF extension. Thus property (2) holds. Since  $R$  is a regular local ring,  $R$  is

analytically irreducible and analytically normal. Since  $S$  is essentially finitely generated over a field,  $S$  is universally catenary. Therefore properties (3) and (4) hold. Since  $R$  is a localized polynomial ring in  $d$  variables, Theorem 1.1 implies that every maximal ideal of  $\text{Gff}(R)$  has height  $d - 1$ . By Theorem 3.2, every maximal ideal of  $\text{Gff}(S)$  has height  $d - 1$ .  $\square$

#### 4. Other results on generic formal fibers

The main theorem of [Heinzer et al. 2006a] includes results about the generic formal fiber ring of mixed polynomial-power series rings as in Theorem 4.1.

**Theorem 4.1** [Heinzer et al. 2006a, Theorem 1.1]. *Let  $m$  and  $n$  be positive integers, let  $k$  be a field, and let  $X = \{x_1, \dots, x_n\}$  and  $Y = \{y_1, \dots, y_m\}$  be sets of independent variables over  $k$ . Then, for  $R$  either the ring  $k[[X]][Y]_{(X,Y)}$  or the ring  $k[Y]_{(Y)}[[X]]$ , the dimension of the generic formal fiber ring  $\text{Gff}(R)$  is  $n + m - 2$  and every prime ideal  $P$  maximal in  $\text{Gff}(R)$  has  $\text{ht } P = n + m - 2$ .*

We use Theorem 3.2 and Theorem 4.1 to deduce Theorem 4.2.

**Theorem 4.2.** *Let  $R$  be either  $k[[X]][Y]_{(X,Y)}$  or  $k[Y]_{(Y)}[[X]]$ , where  $m$  and  $n$  are positive integers and  $X = \{x_1, \dots, x_n\}$  and  $Y = \{y_1, \dots, y_m\}$  are sets of independent variables over a field  $k$ . Let  $\mathfrak{m}$  denote the maximal ideal  $(X, Y)R$  of  $R$ . Let  $(S, \mathfrak{n})$  be a Noetherian local integral domain containing  $R$  such that:*

- (1) *the injection  $\varphi : (R, \mathfrak{m}) \hookrightarrow (S, \mathfrak{n})$  is a local map;*
- (2)  *$\mathfrak{m}S$  is  $\mathfrak{n}$ -primary, and  $S/\mathfrak{n}$  is finite algebraic over  $R/\mathfrak{m}$ ;*
- (3)  *$R \hookrightarrow S$  is a TGF-extension and  $\dim R = \dim S$ ;*
- (4)  *$S$  is universally catenary.*

*Then every maximal ideal of the generic formal fiber ring  $\text{Gff}(S)$  has height  $n + m - 2$ . Equivalently, if  $P$  is a prime ideal of  $\hat{S}$  maximal with respect to  $P \cap S = (0)$ , then  $\text{ht}(P) = n + m - 2$ .*

*Proof.* We check that the conditions 1–5 of Theorem 3.2 are satisfied for  $R$  and  $S$  and the injection  $\varphi$ . Since the completion of  $R$  is  $k[[X, Y]]$ ,  $R$  is analytically normal, and so also analytically irreducible. Items 1–4 of Theorem 4.2 ensure that the rest of conditions 1–4 of Theorem 3.2 hold. By Theorem 4.1, every maximal ideal of  $\text{Gff}(R)$  has height  $n + m - 2$ , and so condition 5 of Theorem 3.2 holds. Thus by Theorem 3.2, every maximal ideal of  $\text{Gff}(S)$  has height  $n + m - 2$ .  $\square$

**Remark 4.3.** Let  $k$ ,  $X$ ,  $Y$  and  $R$  be as in Theorem 4.2. Let  $A$  be a finite integral extension domain of  $R$  and let  $S$  be the localization of  $A$  at a maximal ideal. As observed in the proof of Theorem 4.2,  $R$  is a local analytically normal integral domain. Since  $S$  is a localization of a finitely generated  $R$ -algebra and  $R$  is



universally catenary, it follows that  $S$  is universally catenary. We also have that conditions 1–3 of Theorem 4.2 hold. Thus the extension  $R \hookrightarrow S$  satisfies the hypotheses of Theorem 4.2. Hence every maximal ideal of  $\text{Gff}(S)$  has height  $n + m - 2$ .

The next example is an application of Theorem 4.2 and Remark 4.3.

**Example 4.4.** Let  $k, X, Y$  and  $R$  be as in Theorem 4.2. Let  $K$  denote the field of fractions of  $R$ , and let  $L$  be a finite algebraic extension field of  $K$ . Let  $A$  be the integral closure of  $R$  in  $L$ , and let  $S$  be a localization of  $A$  at a maximal ideal. The ring  $R$  is a Nagata ring by [Marot 1975, Proposition 3.5]. Therefore  $A$  is a finite integral extension of  $R$  and the conditions of Remark 4.3 apply to show that every maximal ideal of  $\text{Gff}(S)$  has height  $n + m - 2$ .

**Remark 4.5.** With notation as in Example 4.4, since the sets  $X$  and  $Y$  are nonempty, the field  $K$  is a simple transcendental extension of a subfield. It follows that the regular local ring  $R$  is not Henselian; see [Berger et al. 1967, Satz 2.3.11, p. 60; Schmidt 1933]. Hence there exists a finite algebraic field extension  $L/K$  such that the integral closure  $A$  of  $R$  in  $L$  has more than one maximal ideal. It follows that the localization  $S$  of  $A$  at any one of these maximal ideals is not a finite  $R$ -module, and gives an example  $R \hookrightarrow S$  that satisfies the hypotheses of Theorem 3.2.

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