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Bounding the socles of powers of squarefree monomial ideals

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Let $S = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field K and $I \subset S$ a squarefree monomial ideal. In the present paper we are interested in the monomials $u \in S$ belonging to the socle $\mathrm{Soc}(S/I^k)$ of S/I^k, i.e., $u \notin I^k$ and $ux_i \in I^k$ for $1 \le i \le n$. We prove that if a monomial $x_1^{a_1} \cdots x_n^{a_n}$ belongs to $\mathrm{Soc}(S/I^k)$, then $a_i \le k-1$ for all $1 \le i \le n$. We then discuss squarefree monomial ideals $I \subset S$ for which $x_{[n]}^{k-1} \in \mathrm{Soc}(S/I^k)$, where $x_{[n]} = x_1 x_2 \cdots x_n$. Furthermore, we give a combinatorial characterization of finite graphs G on $[n] = \{1, \dots, n\}$ for which depth $S/(I_G)^2 = 0$, where I_G is the edge ideal of G.

Introduction

The depth of powers of an ideal (especially, a monomial ideal) of the polynomial ring has been studied by many authors. In the present paper, we are interested in the socle of powers of a squarefree monomial ideal.

Let K be a field, $S = K[x_1, \dots, x_n]$ the polynomial ring in n variables over K, and $I \subset S$ a graded ideal. We denote by $\mathfrak{m} = (x_1, \dots, x_n)$ the graded maximal ideal of S. An element $f + I \in S/I$ is called a *socle element* of S/I if $x_i f \in I$ for $i = 1, \dots, n$. Thus $f + I$ is a nonzero socle element of S/I if $f \in I : \mathfrak{m} \setminus I$. The set of socle elements $\mathrm{Soc}(S/I)$ of S/I is called the *socle* of S/I. Notice that $\mathrm{Soc}(S/I)$ is a K-vector space isomorphic to $(I : \mathfrak{m})/I$. One has depth $S/I = 0$ if and only if $\mathrm{Soc}(S/I) \neq \{0\}$.

In the case that I is a monomial ideal, a case which we mainly consider here, $\mathrm{Soc}(S/I)$ is generated by the residue classes of monomials. If u and v are monomials not belonging to I, then $u + I = v + I$, if and only if $u = v$. Thus, if u is a monomial, it is convenient to write $u \in \mathrm{Soc}(S/I)$ and to call u a socle element of S/I if $u + I \in \mathrm{Soc}(S/I)$ and $u + I \neq 0$. In other words, $u \in \mathrm{Soc}(S/I)$ if and only if $u \notin I$ and $ux_i \in I$ for all $1 \le i \le n$.

The present paper is organized as follows. In Section 1, we show that, for a squarefree monomial ideal $I \subset S$, if a monomial $x_1^{a_1} \cdots x_n^{a_n}$ is a socle element of

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S/I^k, then $a_i \leq k-1$ for all $1 \leq i \leq n$ (Corollary 2.2). In Section 2, the edge ideal I_G arising from a finite graph G is discussed. We give a combinatorial characterization of G for which depth $S/(I_G)^2 = 0$ (Theorem 3.1).

Let $I \subset S$ be a squarefree monomial ideal. If the monomial $u = x_{[n]}^{k-1}$ happens to be a socle element of S/I^k, then, according Corollary 2.2, u is a socle element of S/I^k of maximal degree. In Section 3, we study squarefree monomial ideals $I \subset S$ with $x_{[n]}^{k-1} \in \operatorname{Soc}(S/I^k)$. It is proved that, for a squarefree monomial ideal $I \subset S$ with $x_{[n]}^{k-1} \in \operatorname{Soc}(S/I^k)$, one has $k < n$ and depth $S/I^j > 0$ for $j < k$ (Corollary 4.2). Furthermore, for a squarefree monomial ideal $I \subset S$ generated in degree d with $x_{[n]}^{k-1} \in \operatorname{Soc}(S/I^k)$, we show that if $d > ((k-1)n+1)/k$, then depth $S/I^k > 0$ and that if $d = ((k-1)n+1)/k$ and depth $S/I^k = 0$, then $x_{[n]}^{k-1} \in \operatorname{Soc}(S/I^k)$ and depth $S/I^\ell = 0$ for all $\ell \geq k$ (Corollary 4.4).

2. Socles of powers of squarefree monomial ideals

Proposition 2.1. *Let I be a monomial ideal. For $i = 1, \ldots, n$ set*

$$c_i = \max\{\deg_{x_i}(u) : u \in G(I)\},$$

and let $x_1^{a_1} \cdots x_n^{a_n}$ be a socle element of S/I. Then $a_i \leq c_i - 1$ for $i = 1, \ldots, n$.

Proof. Let $u = x_1^{a_1} \cdots x_n^{a_n}$ be a socle element of S/I. Thus $u \notin I$ and $u \in I : \mathfrak{m}$. Suppose that $a_i \geq c_i$ for some i. Since $x_i u \in I$, there exists $v \in G(I)$ which divides $x_i u$.

It follows that $\deg_{x_j}(v) \leq \deg_{x_j}(x_i u) = \deg_{x_j}(u)$ for $j \neq i$, and $\deg_{x_i}(v) \leq c_i \leq \deg_{x_i}(u)$. Therefore, v divides u, and hence $u \in I$, a contradiction. □

Corollary 2.2. *Let I be a squarefree monomial ideal, and let $x_1^{a_1} \cdots x_n^{a_n}$ be a socle element of S/I^k. Then*

$$a_i \leq k-1 \quad \text{for } i = 1, \ldots, n.$$

3. Edge ideals whose square has depth zero

We consider the case of edge ideals.

Theorem 3.1. *Let $I = I_G \subset S = K[x_1, \ldots, x_n]$ be the edge ideal of graph G on the vertex set $[n]$. The following conditions are equivalent*:

(a) depth $S/I^2 = 0$;

(b) *G is a connected graph containing a cycle C of length* 3, *and any vertex of G is a neighbor of C.*

Moreover, $x_{[n]} \in \operatorname{Soc}(S/I^2)$ *if and only if G is a cycle of length* 3.

Proof. (b) $\Rightarrow$ (a): Suppose that G has a cycle of length 3, say, $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$ are edges of G and that, for each $4 \leq j \leq n$, one of $\{1, j\}$, $\{2, j\}$ and $\{3, j\}$ is an edge of G. It then follows immediately that the monomial $u = x_1x_2x_3$ satisfies $u \notin I^2$ and $u \in I^2 : \mathfrak{m}$. Hence depth $S/I^2 = 0$, as required. This argument also shows that $x_{[n]} \in \mathrm{Soc}(S/I^2)$ if and only if G is cycle of length 3

(a) $\Rightarrow$ (b): Let $I = I_G$ be the edge ideal of a finite graph G with depth $S/I^2 = 0$. Then there exists a monomial u with $u \notin I^2$ such that $u \in I^2 : \mathfrak{m}$. Let H denote the induced subgraph of G whose vertices are those $i \in [n]$ such that x_i divides u. Since $u \notin I^2$ it follows that H cannot possess two disjoint edges. If H possesses an isolated vertex i, then $x_iu \notin I^2$. This contradict $u \in I^2 : \mathfrak{m}$. Hence H is connected without disjoint edges. Thus H must be either a cycle of length 3, or a line of length at most 2.

First, if H is a line of length 1, i.e., H is an edge of G, then we may assume that $u = x_1^{a_1}x_2^{a_2}$ with each $a_i \geq 1$. If each $a_i \geq 2$, then $u \in I^2$, a contradiction. Let $a_1 = 1$ and $u = x_1x_2^{a_2}$. Then $ux_2 \notin I^2$. This contradicts $u \in I^2 : \mathfrak{m}$.

Now, let H be either a cycle of length 3, or a line of length 2. Thus we may assume that $u = x_1^{a_1}x_2^{a_2}x_3^{a_3}$ with each $a_i \geq 1$, where $\{1, 2\}$ and $\{1, 3\}$ are edges of G. Since $u \notin I^2$, it follows that $a_1 = 1$. Thus $u = x_1x_2^{a_2}x_3^{a_3}$. If $\{2, 3\}$ is not an edge of G, then $x_2u \notin I^2$, a contradiction. Hence $\{2, 3\}$ is an edge of G. Then, since $u \notin I^2$, it follows that $a_2 = a_3 = 1$. Thus $u = x_1x_2x_3$ and $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$ are edges of G. Let $j \geq 4$. Since $x_ju \in I^2$, it follows that one of $\{1, j\}$, $\{2, j\}$ and $\{3, j\}$ must be an edge of G, as desired. □

This result has been shown independently by Terai and Trung [2014].

4. Powers of squarefree monomial ideals with maximal socle

Let $I \subset S = K[x_1, \dots, x_n]$ be a squarefree monomial ideal. If the monomial $u = x_{[n]}^{k-1}$ happens to be a socle element of S/I^k, then, by Corollary 2.2, u is a socle element of S/I^k of maximal degree. The next proposition characterizes those squarefree monomial ideals for which $x_{[n]}^{k-1}$ is indeed a socle element of S/I^k.

We consider I as the facet ideal of a simplicial complex Δ. Thus $I = I(\Delta)$ where the set of facets $\mathcal{F}(\Delta)$ of Δ is given as

$$\mathcal{F}(\Delta) = \{\mathrm{supp}(u) : u \in G(I)\}.$$

In other words, $G(I(\Delta)) = \{x_F : F \in \mathcal{F}(\Delta)\}$ where we set $x_F = \prod_{i \in F} x_i$ for $F \subset [n]$.

Proposition 4.1. *Let Δ be a simplicial complex on the vertex set $[n]$, and*

$$I = I(\Delta) \subset S = K[x_1, \dots, x_n]$$

its facet ideal.

(a) *The following conditions are equivalent*:

(i) $x_{[n]}^{k-1} \notin I^k$.

(ii) $\bigcap_{i=1}^k F_i \neq \varnothing$ *for all* $F_1, \ldots, F_k \in \mathcal{F}(\Delta)$.

(b) *Assuming that* $x_{[n]}^{k-1} \notin I^k$, *the following conditions are equivalent*:

(i) $x_j x_{[n]}^{k-1} \in I^k$ *for all* j.

(ii) *For each* $j = 1, \ldots, n$, *there exist* $F_1, \ldots, F_k \in \mathcal{F}(\Delta)$ *such that* $\bigcap\limits_{i=1}^{k} F_i = \{j\}$.

In particular, $x_{[n]}^{k-1} \in \operatorname{Soc}(S/I^k)$ *if and only if* (a)(ii) *and* (b)(ii) *hold.*

Proof. (a) $x_{[n]}^{k-1} \in I^k$ if and only if there exist $F_1, \ldots, F_k \in \mathcal{F}(\Delta)$ such that $x_{F_1} x_{F_2} \cdots x_{F_k}$ divides $x_{[n]}^{k-1}$. This is the case, if and only if no x_i^k divides $x_{F_1} x_{F_2} \cdots x_{F_k}$. This is equivalent to saying that $\bigcap_{i=1}^k F_i = \varnothing$. Thus the desired conclusion follows.

(b) $x_j x_{[n]}^{k-1} \in I^k$ if and only if $x_{F_1} x_{F_2} \cdots x_{F_k}$ divides $x_j x_{[n]}^{k-1}$ for some $F_1, \ldots, F_k \in \mathcal{F}(\Delta)$. By (a), $\bigcap_{i=1}^k F_i \neq \varnothing$. Therefore, $x_{F_1} x_{F_2} \cdots x_{F_k}$ divides $x_j x_{[n]}^{k-1}$ if and only if $\bigcap_{i=1}^k F_i = \{j\}$. ☐

Corollary 4.2. *Let* $I \subset S = K[x_1, \ldots, x_n]$ *be a squarefree monomial ideal. Let* $n > 1$ *and suppose that* $x_{[n]}^{k-1} \in \operatorname{Soc}(S/I^k)$. *Then* $k < n$, *and* depth $S/I^j > 0$ *for* $j < k$.

Proof. The condition (b)(ii) of Proposition 4.1 guarantees the existence of $F^{(j)} \in \mathcal{F}(\Delta)$ with $j \in F^{(j)}$ and $j+1 \notin F^{(j)}$ for each $1 \leq j < n$ and the existence of $F^{(n)} \in \mathcal{F}(\Delta)$ with $n \in F^{(n)}$ and $1 \notin F^{(n)}$. Then $\bigcap_{j=1}^n F^{(j)} = \varnothing$. Thus if $k \geq n$, then condition (a)(ii) of Proposition 4.1 is violated, and hence $k < n$.

Let $j < k$ and suppose that depth $S/I^j = 0$. Then $j \geq 2$, since I is squarefree. Let $u \in \operatorname{Soc}(S/I^j)$; then $ux_i \in I^j$ for all i and hence also $x_{[n]}^{j-1} x_i \in I^j$ for all i. Since $n > 1$, the ideal I cannot be a principal ideal, because otherwise depth $S/I^j > 0$ for all j. Hence we may assume that $x_2 x_3 \cdots x_n \in I$. Then

$$x_{[n]}^j = \big(x_{[n]}^{j-1} x_1\big)(x_2 x_3 \cdots x_n) \in I^{j+1}.$$

It follows that

$$x_{[n]}^{k-1} = x_{[n]}^j x_{[n]}^{k-j-1} = \big(x_{[n]}^j x_1^{k-j-1}\big)(x_2 x_3 \cdots x_n)^{k-j-1} \in I^k,$$

a contradiction. ☐

Examples 4.3. (a) The ideal

$$I = (x_1 x_2 \cdots x_{n-1}, x_1 x_n, x_2 x_n, \ldots, x_{n-1} x_n)$$

in $S = K[x_1, \ldots, x_n]$ satisfies conditions (a)(ii) and (b)(ii) of Proposition 4.1 for $k = 2$. Hence depth$(S/I^2) = 0$.

(b) Let $n = 2d - 1$ and I a monomial ideal of $S = K[x_1, \dots, x_n]$ generated by squarefree monomials of degree d. Then condition (a)(ii) in Proposition 4.1 is satisfied for $k = 2$. Thus if a squarefree monomial w belongs to $\mathrm{Soc}(S/I^2)$, then w must be $x_{[n]}$. Hence depth $S/I^2 = 0$ if and only if I satisfies for $k = 2$ condition (b)(ii) in Proposition 4.1.

For example, if I is generated by the following squarefree monomials

$$x_1x_2\cdots x_d, \quad x_1x_{d+1}x_{d+2}\cdots x_{2d-1},$$
$$x_ix_{d+1}x_{d+2}\cdots x_{2d-1} \quad \text{with } 2 \le i \le d,$$
$$x_2x_3\cdots x_dx_j \quad \text{with } d+1 \le j \le 2d-1,$$

then depth $S/I^2 = 0$.

Examples 4.3(b) shows that for any odd integer $n > 1$ there exists a squarefree monomial ideal $I \subset K[x_1, \dots, x_n]$ generated in degree $d = (n+1)/2$ such that depth $S/I^2 = 0$.

On the other hand for a squarefree monomial ideal generated in degree $d > (n+1)/2$ one has depth $S/I^2 > 0$, as follows from Corollary 4.4.

Corollary 4.4. *Let $I \subset K[x_1, \dots, x_n]$ be a squarefree monomial ideal generated in the single degree d.*

(a) *If $d > ((k-1)n+1)/k$, then* depth $S/I^k > 0$.

(b) *For all positive integer d, k and n such that $d = ((k-1)n+1)/k$, there exists a squarefree monomial ideal $I \subset K[x_1, \dots, x_n]$ generated in degree d such that* depth $S/I^k = 0$.

(c) *If $d = ((k-1)n+1)/k$ and* depth $S/I^k = 0$, *then $x_{[n]}^{k-1} \in \mathrm{Soc}(S/I^k)$ and* depth $S/I^\ell = 0$ *for all $\ell \ge k$.*

Proof. (a) Let $F_1, \dots, F_k$ subset of $[n]$ of cardinality d. We first show by induction on i that

$$\left|\bigcap_{j=1}^{i} F_j\right| > ((k-i)n+i)/k.$$

The assertion is trivial for $i = 1$. By using the induction hypothesis, we see that

$$\begin{aligned}\left|\bigcap_{j=1}^{i} F_j\right| &\ge \left|\bigcap_{j=1}^{i-1} F_j\right| + |F_i| - n \\ &> \frac{(k-i+1)n+(i-1)}{k} + \frac{(k-1)n+1}{k} - n = \frac{(k-i)n+i}{k},\end{aligned}$$

as desired.

It follows that any intersection of k subsets of $[n]$ of cardinality d admits more than one element. Therefore I satisfies condition (a)(ii) of Proposition 4.1, but violates condition (b)(ii).

Since condition (a)(ii) is satisfied, it follows from Proposition 4.1 that $x_{[n]}^{k-1}$ is not in I^k. Thus, if we assume that depth $S/I^k = 0$, Corollary 2.2 implies that $x_{[n]}^{k-1} \in \mathrm{Soc}(S/I^k)$. However, since condition (b)(ii) is violated, this is not possible.

(b) Suppose that $d = ((k-1)n+1)/k$. Then $n \equiv 1 \bmod k$, say, $n = (r+1)k+1$ for an integer $r \geq 0$. It then follows that $d = (r+1)k - r$. Consider the monomial ideal I generated by all squarefree monomials of degree d in $K[x_1, \dots, x_n]$. By [Herzog and Hibi 2005, Corollary 3.4] one has

$$\operatorname{depth} S/I^k = \max\{0, n - k(n-d) - 1\}.$$

Since $n - k(n-d) - 1 = (r+1)k + 1 - k(r-1) - 1 = 0$, the assertion follows.

(c) Let $u \in \mathrm{Soc}(S/I^k)$, $u = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$. Then, by Corollary 2.2, $a_i \leq k-1$ for all i, and hence $\deg u \leq (k-1)n = kd - 1$. On the other hand, since $ux_i \in I^k$, it follows that $\deg u + 1 \geq kd$. Thus we conclude that $\deg u = kd - 1 = (k-1)n$, which is only possible if $u = x_{[n]}^{k-1}$. Let $\ell > k$ and let v be a generator of $I^{\ell-k}$. Then $uvx_i \in I^{\ell+1}$, but $uv \notin I^\ell$, because

$$\deg uv = (kd-1) + (\ell - k) \leq kd - 1 + (\ell - k)d = \ell d - 1 < \ell d.$$

This shows that $uv \in \mathrm{Soc}(S/I^\ell)$, and consequently depth $S/I^\ell = 0$, as required. □

Example 4.5. Let $k \geq 2$, and assume that $d = ((k-1)n+1)/k$. Then $n = (kd-1)/(k-1)$, and this is an integer if and only if $d \equiv 1 \bmod (k-1)$. One solution is $d = k$. Then $n = k+1$. With these data we may choose the ideal $I \subset S = K[x_1, \dots, x_n]$ generated by all squarefree monomials of degree $d = k = n-1$. Then obviously I satisfies conditions (a)(i) and (b)(i) of Proposition 4.1. Thus $x_{[n]}^{k-1} \in \mathrm{Soc}(S/I^k)$. In particular, depth $S/I^k = 0$. It is shown in [Herzog and Hibi 2005] that depth $S/I^j > 0$ for $j < k$. (This also follows from Corollary 4.2). This example shows that arbitrary high powers of a squarefree monomial ideal may have a maximal socle.

It is known by a result of Brodmann [1979] (see also [Herzog and Hibi 2005]) that the depth function $f(k) = \operatorname{depth} S/I^k$ is eventually constant. In [Herzog et al. 2013] the smallest number k for which depth $S/I^k = \operatorname{depth} S/I^j$ for all $j \geq k$, is denoted by dstab(I). In [Herzog and Asloob Qureshi 2015] it is conjectured that dstab$(I) < n$ for all graded ideals in $K[x_1, \dots, x_n]$. Corollary 4.2 together with Corollary 4.4(c) show that this conjecture holds true for a squarefree monomial ideal $I \subset K[x_1, \dots, x_n]$ generated in degree $d = ((k-1)n+1)/k$ for which depth $S/I^k = 0$.

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