Singularities with respect to Mather–Jacobian discrepancies

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As is well known, the "usual discrepancy" is defined for a normal $\mathbb{Q}$-Gorenstein variety. By using this discrepancy we can define a canonical singularity and a log canonical singularity. In the same way, by using a new notion, Mather–Jacobian discrepancy introduced in recent papers we can define a "canonical singularity" and a "log canonical singularity" for not necessarily normal or $\mathbb{Q}$-Gorenstein varieties. In this paper, we show basic properties of these singularities, behavior of these singularities under deformations and determine all these singularities of dimension up to 2.

1. Introduction

In birational geometry, canonical, log canonical, terminal and log terminal singularities play important roles. These singularities are all normal $\mathbb{Q}$-Gorenstein singularities and each step of the minimal model program is performed inside the category of normal $\mathbb{Q}$-Gorenstein singularities. But in turn, from a purely singularity theoretic view point, the normal $\mathbb{Q}$-Gorenstein property seems, in some sense, to be an unnecessary restriction for a singularity to be considered as a good singularity, because there are many "good" singularities without normal $\mathbb{Q}$-Gorenstein property (for example, the cone over the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$).

In this paper, we take off the restriction normal $\mathbb{Q}$-Gorenstein, give definitions of "good" singularities which have some compatibilities with the usual canonical, log canonical, terminal and log terminal singularities and study our "good" singularities. To contrast, remember the definition of the usual canonical, log canonical, terminal and log terminal singularities. We say that a pair $(X, \mathfrak{a}^t)$ consisting of a normal $\mathbb{Q}$-Gorenstein variety X, a nonzero coherent ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$ and $t \in \mathbb{R}_{\geq 0}$ has canonical (resp. log canonical, terminal, log terminal)

singularities if, for a log resolution $\varphi : Y \to X$ of $(X, \mathfrak{a})$, the log discrepancy $a(E; X, \mathfrak{a}^t)$ satisfies the inequality

$$a(E; X, \mathfrak{a}^t) := \operatorname{ord}_E(K_{Y/X}) - t \operatorname{val}_E(\mathfrak{a}) + 1 \geq 1 \quad (\text{resp. } \geq 0, > 1, > 0)$$

for every exceptional prime divisor E. We say that $(X, \mathfrak{a}^t)$ has klt singularities if the above inequality holds for every prime divisor on Y. Here we note that the discrepancy divisor $K_{Y/X} = K_Y - \frac{1}{r}\varphi^*(rK_X)$ is well defined if there is an integer r such that rK_X is a Cartier divisor, which means that X is a $\mathbb{Q}$-Gorenstein variety.

Now, consider a pair $(X, \mathfrak{a}^t)$ under a more general setting. Let X be a connected reduced equidimensional affine scheme of finite type over an algebraically closed field k of characteristic zero. Let $\mathfrak{a}$ be a coherent ideal sheaf of $\mathcal{O}_X$ nonvanishing identically on any component. For a log resolution $\varphi : Y \to X$ of $(X, \mathfrak{a})$ which factors through the Nash blow-up, we can define the Mather discrepancy divisor $\widehat{K}_{Y/X}$ (Definition 2.1). For the Jacobian ideal $\mathcal{J}_X \subset \mathcal{O}_X$ we define the Jacobian discrepancy divisor $J_{Y/X}$ by $\mathcal{O}_Y(-J_{Y/X}) = \mathcal{J}_X \mathcal{O}_Y$. The combination $\widehat{K}_{Y/X} - J_{Y/X}$ is called the Mather–Jacobian discrepancy divisor and plays a central role in this paper. The basic idea is just to replace the usual discrepancy $K_{Y/X}$ by the Mather–Jacobian discrepancy, i.e., we define the Mather–Jacobian log discrepancy

$$a_{\mathrm{MJ}}(E; X, \mathfrak{a}^t) := \operatorname{ord}_E(\widehat{K}_{Y/X} - J_{Y/X}) - t \operatorname{val}_E(\mathfrak{a}) + 1,$$

and by $a_{\mathrm{MJ}}(E; X, \mathfrak{a}^t) \geq 1$ (resp. $\geq 0, > 1, > 0$) for every exceptional prime divisor E, we define that $(X, \mathfrak{a}^t)$ is MJ-canonical (resp. MJ-log canonical, MJ-terminal, MJ-log terminal). We say that $(X, \mathfrak{a}^t)$ is MJ-klt if $a_{\mathrm{MJ}}(E; X, \mathfrak{a}^t) > 0$ for every prime divisor on Y. Here, we should be careful about the difference between just a prime divisor over X and an exceptional divisor over X. The definition of an exceptional divisor over X is given in Definition 2.15.

According to the basic idea of the replacement by Mather–Jacobian discrepancy, the invariants the minimal log discrepancy mld and the multiplier ideal $\mathcal{J}(X, \mathfrak{a}^t)$ defined by using the usual discrepancy divisor, can be modified to the Mather–Jacobian versions $\mathrm{mld}_{\mathrm{MJ}}$ and $\mathcal{J}_{\mathrm{MJ}}(X, \mathfrak{a}^t)$.

In some points, the Mather–Jacobian discrepancy behaves better than the usual discrepancy divisor. One of the most distinguished properties of the Mather–Jacobian discrepancy is the inversion of adjunction which was proved in [de Fernex and Docampo 2014] and [Ishii 2013] independently:

Proposition 1.1 (inversion of adjunction [de Fernex and Docampo 2014; Ishii 2013]). *Let X be a connected reduced equidimensional scheme of finite type over k. Let A be a nonsingular variety containing X as a closed subscheme of*

codimension c and W a strictly proper closed subset of X. Let $\tilde{\mathfrak{a}} \subset \mathcal{O}_A$ be a nonzero coherent ideal sheaf such that its image $\mathfrak{a} := \tilde{\mathfrak{a}}\mathcal{O}_X \subset \mathcal{O}_X$ is nonzero on each irreducible component. Denote the defining ideal of X in A by I_X. Then,

$$\mathrm{mld}_{\mathrm{MJ}}(W; X, \mathfrak{a}^t) = \mathrm{mld}_{\mathrm{MJ}}(W; A, \tilde{\mathfrak{a}}^t I_X^c) = \mathrm{mld}(W; A, \tilde{\mathfrak{a}}^t I_X^c).$$

This theorem was proved by using the discussion of arc spaces and jet schemes. Many good properties follows from this formula.

In this paper we study basic properties of MJ-canonical, MJ-log canonical singularities and determine these singularities of dimension up to 2. Concretely we obtain the following. The first one below is about the relation of singularities of MJ-version and singularities of the usual version.

Proposition 1.2 (Proposition 2.21). *Let X be a normal $\mathbb{Q}$-Gorenstein variety, $\mathfrak{a} \subset \mathcal{O}_X$ a nonzero coherent ideal sheaf of $\mathcal{O}_X$ and t a nonnegative real number. If $(X, \mathfrak{a}^t)$ is MJ-canonical (resp. MJ-log canonical, MJ-terminal, MJ-log terminal, MJ-klt), then it is canonical (resp. log canonical, terminal, log terminal, klt) in the usual sense.*

We call MJ-canonical singularities, MJ-log canonical singularities and so on by the generic name "MJ-singularities". As MJ-singularities are not necessarily normal, it is reasonable to compare these with existing nonnormal singularities which is considered as "good" singularities. The following gives the relation of MJ-log canonical singularities and semilog canonical singularities.

Proposition 1.3 (Proposition 3.16). *Assume X is S_2 and $\mathbb{Q}$-Gorenstein. If $(X, \mathfrak{a}^t)$ is MJ-log canonical, then it is semilog canonical.*

We also obtain the relation of MJ-singularities and the singularities appeared recently in [de Fernex and Hacon 2009].

Theorem 1.4 (Theorem 3.19). *Assume that X is normal. If a pair $(X, \mathfrak{a}^t)$ is MJ-klt (resp. MJ-canonical, MJ-log canonical), then it is log terminal (resp. canonical, log canonical) in the sense of de Fernex and Hacon.*

By the property of de Fernex and Hacon's singularities we obtain:

Corollary 1.5 (Corollary 3.20). *If a pair $(X, \mathfrak{a}^t)$ is MJ-klt (resp. MJ-log canonical), then there is a boundary Δ on X such that $((X, \Delta), \mathfrak{a}^t)$ is klt (resp. log canonical) in the usual sense.*

By the proof of the above theorem, the relation of MJ-multiplier ideals and de Fernex–Hacon's multiplier ideals.

Theorem 1.6 (Theorem 3.21). *Let $(X, \mathfrak{a}^t)$ be a pair with a normal variety X, a nonzero coherent ideal sheaf $\mathfrak{a}$ on X and $t \in \mathbb{R}_{\geq 0}$. Then*

$$\mathcal{J}_{\mathrm{MJ}}(X, \mathfrak{a}^t) \subset \mathcal{J}_m(X, \mathfrak{a}^t) \quad \textit{for every } m \in \mathbb{N};$$

in particular

$$\mathcal{J}_{\mathrm{MJ}}(X, \mathfrak{a}^t) \subset \mathcal{J}(X, \mathfrak{a}^t).$$

It is known that canonical (resp. log canonical) singularities are stable under a small flat deformation. We obtain the similar results for MJ-singularities. Here, we do not need the flatness of the deformation. We define that $\{(X_\tau, \mathfrak{a}_\tau^t)\}_{\tau \in T}$ is a deformation of $(X_0, \mathfrak{a}_0^t)$, if there is a surjective morphism $\pi : X \to T$ with equidimensional reduced fibers $X_\tau = \pi^{-1}(\tau)$ of common dimension r for all closed points $\tau \in T$ and there exists a coherent ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$ nonvanishing on any component of the total space X such that $\mathfrak{a}_\tau^t = \mathfrak{a}^t \mathcal{O}_{X_\tau}$ are not zero on any component of X_τ for all $\tau \in T$.

Theorem 1.7 (Theorem 4.4, 4.9). *Let* $\{(X_\tau, \mathfrak{a}_\tau^t)\}_{\tau \in T}$ *be a deformation of* $(X_0, \mathfrak{a}_0^t)$. *Assume* $(X_0, \mathfrak{a}_0^t)$ *is MJ-canonical* (*resp. MJ-log canonical*) *at* $x \in X_0$. *Then there are neighborhoods* $X^* \subset X$ *of* x *and* $T^* \subset T$ *of* 0 *such that* X_τ^* *is MJ-canonical* (*resp. MJ-log canonical*) *for every closed point* $\tau \in T^*$.

The lower semicontinuity of MJ-minimal log discrepancies is also proved:

Proposition 1.8 (Proposition 4.11). *Let* $\{(X_\tau, \mathfrak{a}_\tau^t)\}_{\tau \in T}$ *be a deformation of* $(X_0, \mathfrak{a}_0^t)$ *and let* $\pi : X \to T$ *is the morphism giving the deformation. Let* $\sigma : T \to X$ *a section of* π. *Then, the map* $T \to \mathbb{R}, \tau \mapsto \mathrm{mld}_{\mathrm{MJ}}(\sigma(\tau), X_\tau, \mathfrak{a}_\tau^t)$ *is lower semicontinuous.*

In the last section we determine all MJ-canonical, MJ-log canonical singularities up to dimension 2.

Proposition 1.9 (Proposition 5.1). *Let* (X, x) *be a singularity on a one-dimensional reduced scheme.*

  (i) (X, x) *is MJ-canonical if and only if it is nonsingular.*

 (ii) (X, x) *is MJ-log canonical if and only if it is nonsingular or ordinary node.*

Theorem 1.10 (Theorem 5.3). *Let* (X, x) *be a singularity on a* 2*-dimensional reduced scheme. Then* (X, x) *is MJ-canonical if and only if it is nonsingular or rational double.*

The following theorem gives the total list of 2-dimensional MJ-log canonical singularities. Here, we should note that the embedding dimension of MJ-log canonical singularities are at most 4 (see Proposition 3.3).

Theorem 1.11 (Theorem 5.4, 5.6). *Let* $(X, 0)$ *be a singularity on a* 2*-dimensional reduced scheme with* $\mathrm{emb}(X, 0) = 3$. *Then,* $(X, 0)$ *is an MJ-log canonical singularity if and only if* X *is defined by* $f(x, y, z) \in k[\![x, y, z]\!]$ *as follows*:

  (i) $\mathrm{mult}_0 f = 3$ *and the projective tangent cone of* X *at* 0 *is a reduced curve with at worst ordinary nodes.*

(ii) $\text{mult}_0 f = 2$.

(a) $f = x^2 + y^2 + g(z)$, $\deg g \geq 2$.

(b) $f = x^2 + g_3(y, z) + g_4(y, z)$, $\deg g_i \geq i$, g_3 *is homogeneous of degree*
  3 *and* $g_3 \neq l^3$ *(l linear)*.

(c) $f = x^2 + y^3 + yg(z) + h(z)$, $\text{mult}_0 g \leq 4$ *or* $\text{mult}_0 h \leq 6$.

(d) $f = x^2 + g(y, z) + h(y, z)$, g *is homogeneous of degree* 4 *and it does
  not have a linear factor with multiplicity more than* 2.

Let $(X, 0)$ *be a singularity on a* 2-*dimensional reduced scheme with*
$\text{emb}(X, 0) = 4$. *Then, the following hold*:

(iii) *In case* $(X, 0)$ *is locally a complete intersection*: X *is MJ-log canonical
  at* 0 *if and only if* $\widehat{\mathcal{O}_{X,,0}} \simeq k[\![x_1, x_2, x_3, x_4]\!]/(f, g)$, *where* f, g *satisfy the
  conditions that* $\text{mult}_0 f = \text{mult}_0 g = 2$ *and* $V(\text{in}(f), \text{in}(g)) \subset \mathbb{P}^3$ *is a
  reduced curve with at worst ordinary double points*.

(iv) *In case* $(X, 0)$ *is not locally a complete intersection*: X *is MJ-log canonical
  at* 0 *if and only if* X *is a subscheme of a locally complete intersection surface
  M which is MJ-log canonical at* 0.

2. Preliminaries

In this paper X is always a connected reduced equidimensional affine scheme of
finite type over an uncountable algebraically closed field k of characteristic zero.
Sometimes we put some additional conditions on X, but in that case it is always
stated clearly. Denote the dimension $\dim X = d$. A variety in this paper always
means an irreducible reduced separated scheme of finite type over k. A nonzero
ideal $\mathfrak{a}$ on X always means a coherent ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$ that does not vanish
on any irreducible component of X.

Let $\widehat{X} \to X$ be the Nash blow-up (for the definition, see, for example, [de Fer-
nex et al. 2008]). The Nash blow-up has the following property:
If a resolution $\varphi : Y \to X$ factors through the Nash blow-up $\widehat{X} \to X$, the
canonical homomorphism $\varphi^*(\Omega_X^d) \to \Omega_Y^d$ has the invertible image [de Fernex
et al. 2008].

Definition 2.1 [de Fernex et al. 2008]. Let $\varphi: Y \to X$ be a resolution of singular-
ities of X that factors through the Nash blow-up of X. By the above comment,
the image of the canonical homomorphism

$$\varphi^*(\Omega_X^d) \to \Omega_Y^d$$

is an invertible sheaf of the form $J\Omega_Y^d$, where J is the invertible ideal sheaf on
Y that defines an effective divisor supported on the exceptional locus of φ. This
divisor is called the *Mather discrepancy divisor* and denoted by $\widehat{K}_{Y/X}$.

Definition 2.2. Recall that the *Jacobian ideal* $\mathcal{J}_X$ of a variety X is the d-th Fitting ideal $\mathrm{Fitt}_d(\Omega_X)$ of Ω_X. If $\varphi : Y \to X$ is a log resolution of $\mathcal{J}_X$, we denote by $J_{Y/X}$ the effective divisor on Y such that $\mathcal{J}_X \mathcal{O}_Y = \mathcal{O}_Y(-J_{Y/X})$. This divisor is called the *Jacobian discrepancy divisor*.

Here, we note that every log-resolution of $\mathcal{J}_X$ factors through the Nash blow-up [Ein et al. 2011, Remark 2.3].

Definition 2.3. Let $\mathfrak{a} \subseteq \mathcal{O}_X$ be a coherent ideal sheaf on X nonvanishing on any component of X, and $t \in \mathbb{R}_{\geq 0}$. Given a log resolution $\varphi : Y \to X$ of $\mathcal{J}_X \mathfrak{a}$, we denote by $Z_{Y/X}$ the effective divisor on Y such that $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-Z_{Y/X})$. For a prime divisor E over X, we define the Mather–Jacobian-log discrepancy (MJ-log discrepancy for short) at E as

$$a_{\mathrm{MJ}}(E; X, \mathfrak{a}^t) := \mathrm{ord}_E(\hat{K}_{Y/X} - J_{Y/X} - tZ_{Y/X}) + 1.$$

Remark 2.4. For nonzero ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ on X, one can similarly define a mixed MJ-log discrepancy $a_{\mathrm{MJ}}(E; X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r})$ for every $t_1, \ldots, t_r \in \mathbb{R}_{\geq 0}$. With the notation in Definition 2.3, if f is a log resolution of $\mathcal{J}_X \mathfrak{a}_1 \cdots \mathfrak{a}_r$, and if we put $\mathfrak{a}_i \mathcal{O}_Y = \mathcal{O}_Y(-Z_i)$, then

$$a_{\mathrm{MJ}}(E; X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r}) = \mathrm{ord}_E(\hat{K}_{Y/X} - J_{Y/X} - t_1 Z_1 - \ldots - t_r Z_r) + 1.$$

For simplicity, we will mostly state the results for a pair $(X, \mathfrak{a}^t)$ with one ideal, but all statements have obvious generalizations to the mixed case.

Remark 2.5. If X is normal and locally a complete intersection, then

$$a_{\mathrm{MJ}}(E; X, \mathfrak{a}^t) = a(E; X, \mathfrak{a}^t),$$

where the right-hand side is the usual log discrepancy $\mathrm{ord}_E(K_{Y/X} - tZ_{Y/X}) + 1$. Indeed, in this case the image of the canonical map $\Omega_X^d \to \omega_X$ is $\mathcal{J}_X \omega_X$, hence $\hat{K}_{Y/X} - J_{Y/X} = K_{Y/X}$. In particular, we see that $a_{\mathrm{MJ}}(E; X, \mathfrak{a}^t) = a(E; X, \mathfrak{a}^t)$ if X is smooth.

Definition 2.6. Let X be a normal and $\mathbb{Q}$-Gorenstein variety. Let W be a proper closed subset of X. The *minimal log-discrepancy* of $(X, \mathfrak{a}^t)$ along W is defined as follows: if $\dim X \geq 2$,

$$\mathrm{mld}(W; X, \mathfrak{a}^t) = \inf\{a(E; X, \mathfrak{a}^t) \mid E \text{ prime divisor over } X \text{ with center in } W\}.$$

When $\dim X = 1$, we use the same definition as above, unless the infimum is negative, in which case we make the convention that $\mathrm{mld}(W; X, \mathfrak{a}^t) = -\infty$.

Now returning to the general setting on X, we define a modified invariant.

Definition 2.7. Let W be a closed subset of X such that it does not contain an irreducible component of X. (We call such a closed subset a "strictly proper closed subset" in this paper.) Let η be a point of X such that its closure is a strictly proper closed subset of X. The *Mather–Jacobian minimal log-discrepancy* of $(X, \mathfrak{a}^t)$ along W (resp. at η) are defined as follows: if $\dim X \geq 2$,

$\mathrm{mld}_{\mathrm{MJ}}(W; X, \mathfrak{a}^t)$

$\quad = \inf\{a_{\mathrm{MJ}}(E; X, \mathfrak{a}^t) \mid E \text{ prime divisor over } X \text{ with center in } W\}$,

$\mathrm{mld}_{\mathrm{MJ}}(\eta; X, \mathfrak{a}^t)$

$\quad = \inf\{a_{\mathrm{MJ}}(E; X, \mathfrak{a}^t) \mid E \text{ prime divisor over } X \text{ with center } \overline{\{\eta\}}\}$.

(Note that we strictly distinguish between "center in Z" and "center Z".)

When $\dim X = 1$, we use the same definition as above, unless the infimum is negative, in which case we make the convention that $\mathrm{mld}_{\mathrm{MJ}}(W; X, \mathfrak{a}^t) = -\infty$ (resp. $\mathrm{mld}_{\mathrm{MJ}}(\eta; X, \mathfrak{a}^t) = -\infty$).

Remark 2.8.   (i) By Remark 2.5, we have

$$\mathrm{mld}(W; X, \mathfrak{a}^t) = \mathrm{mld}_{\mathrm{MJ}}(W; X, \mathfrak{a}^t),$$

if X is normal and locally a complete intersection.

 (ii) In case $\dim X \geq 2$, if there is a prime divisor E with the center in W such that $a_{\mathrm{MJ}}(E; X, \mathfrak{a}^t) < 0$, then $\mathrm{mld}_{\mathrm{MJ}}(W; X, \mathfrak{a}^t) = -\infty$. This is proved by using $\hat{K}_{Y'/X} - J_{Y'/X} = K_{Y'/Y} + \psi^*(K_{Y/X} - J_{Y/X})$ for another resolution $Y' \to X$ factoring through $Y \to X$, in the similar way as the usual discrepancy case [Kollár and Mori 1998, Section 2.3].

(iii) There are some conflicts of notation in [de Fernex and Docampo 2014; Ein et al. 2011; Ishii 2013; Ishii and Reguera 2013], since these papers are working on the same materials and some of these papers were done independently of others. Here, we propose the notation $\mathrm{mld}_{\mathrm{MJ}}(W; X, \mathfrak{a}^t)$ for Mather–Jacobian minimal log discrepancy, while in [de Fernex and Docampo 2014] it is denoted as $\mathrm{mld}^\diamond(W; X, \mathfrak{a}^t)$ and in [Ishii and Reguera 2013] as $\widehat{\mathrm{mld}}(W; X, \mathcal{J}_X \mathfrak{a}^t)$. We hope the new notation here is appropriate to unify the notation.

Proposition 2.9 (inversion of adjunction [de Fernex and Docampo 2014; Ishii 2013]). *Let A be a nonsingular variety containing X as a closed subscheme of codimension c and W a strictly proper closed subset of X. Let $\tilde{\mathfrak{a}} \subset \mathcal{O}_A$ be an ideal such that its image $\mathfrak{a} := \tilde{\mathfrak{a}}\mathcal{O}_X \subset \mathcal{O}_X$ is nonzero on each irreducible component of X. Denote the defining ideal of X in A by I_X. Then,*

$$\mathrm{mld}_{\mathrm{MJ}}(W; X, \mathfrak{a}^t) = \mathrm{mld}_{\mathrm{MJ}}(W; A, \tilde{\mathfrak{a}}^t I_X^c) = \mathrm{mld}(W; A, \tilde{\mathfrak{a}}^t I_X^c).$$

The second equality is trivial by Remark 2.8(1). The inversion of adjunction is proved by discussions of jet schemes and we also use them in this paper. Here, we introduce the basic notion of jet schemes.

Definition 2.10. Let $K \supset k$ be a field extension and $m \in \mathbb{Z}_{\geq 0}$. A morphism $\operatorname{Spec} K[t]/(t^{m+1}) \to X$ is called an *m-jet* of X and $\operatorname{Spec} K[\![t]\!] \to X$ is called an *arc* of X.

2.11. Let $\mathcal{S}ch/k$ be the category of k-schemes and $\mathcal{S}et$ the category of sets. Define a contravariant functor $F_m : \mathcal{S}ch/k \to \mathcal{S}et$ by

$$F_m(Y) = \operatorname{Hom}_k(Y \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}), X).$$

Then, F_m is representable by a scheme $\mathcal{L}^m(X)$ of finite type over k, i.e., $\mathcal{L}^m(X)$ is the fine moduli scheme of m-jets of X.

The scheme $\mathcal{L}^m(X)$ is called the *scheme of m-jets* of X.

In the same way, the fine moduli scheme $\mathcal{L}^\infty(X)$ of arcs of X also exists and it is called the *scheme of arcs* of X. We should note that $\mathcal{L}^\infty(X)$ is not necessarily of finite type over k. The canonical surjection $k[t]/(t^{m+1}) \to k[t]/(t^{n+1})$ ($n < m \leq \infty$) induces a morphism $\psi_{mn} : \mathcal{L}^m(X) \to \mathcal{L}^n(X)$. In particular for $m = \infty$, we write $\psi_m : \mathcal{L}^\infty(X) \to \mathcal{L}^m(A)$.

If $X = \operatorname{Spec} k[x_1, \ldots, x_N]/(f_1, \ldots, f_r)$, then

$$\mathcal{L}^m(X) = \operatorname{Spec} k[\boldsymbol{x}^{(0)}, \boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(m)}]/(F_i^{(j)})_{1 \leq i \leq r, 0 \leq j \leq m},$$

where $\boldsymbol{x}^{(j)} = (x_1^{(j)}, \ldots, x_N^{(j)})$ and $\sum_{j=0}^\infty F^{(j)} t^j$ is the Taylor expansion of $f\left(\sum_j \boldsymbol{x}^{(j)} t^j\right)$; hence $F^{(j)} \in k[\boldsymbol{x}^{(0)}, \ldots, \boldsymbol{x}^{(j)}]$. If $0 \in X \subset \mathbb{A}^N$, we have

$$\psi_{m0}^{-1}(0) = \operatorname{Spec} k[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(m)}]/(\overline{F}_i^{(j)})_{1 \leq i \leq r, 0 \leq j \leq m}, \tag{1}$$

where $\overline{F}_i^{(j)}$ is the image of $F_i^{(j)}$ by the canonical projection map

$$k[\boldsymbol{x}^{(0)}, \boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(m)}] \to k[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(m)}],$$

which sends $\boldsymbol{x}^{(0)}$ to 0.

Remark 2.12. Under the notation above, for a polynomial $f \in k[x_1, \ldots, x_N]$, let

$$f\left(\sum x_1^{(j)} t^j, \ldots, \sum x_N^{(j)} t^j\right) = F^{(0)} + F^{(1)} t + F^{(2)} t^2 + \cdots$$

be the Taylor expansion. Then a monomial in $F^{(j)}$ is of the type

$$x_{i_1}^{(e_1)} \cdots x_{i_r}^{(e_r)} \left(e_l \geq 0, i_l \in \{1, \ldots, N\}, \sum_l e_l = j\right).$$

Here, if $r > j$, the monomial must contain a factor $x_{i_l}^{(0)}$; therefore the image of this monomial by the projection map

$$k[x^{(0)}, x^{(1)}, \ldots, x^{(m)}] \to k[x^{(1)}, \ldots, x^{(m)}]$$

is zero. By this observation we obtain that if $j < \mathrm{mult}_0 \, f$, then $\overline{F}^{(j)} = 0$ and if $j = \mathrm{mult}_0 \, f$, then $\overline{F}^{(j)} = \mathrm{in} \, f(x^{(1)})$, where $\mathrm{in} \, f$ is the initial term of f with the usual grading in $k[x_1, \ldots, x_N]$.

Definition 2.13 [Ein et al. 2004]. For an ideal $\mathfrak{a}$ on a variety X, we define

$$\mathrm{Cont}^m(\mathfrak{a}) = \{\alpha \in \mathcal{L}^\infty(X) \mid \mathrm{ord}_\alpha(\mathfrak{a}) = m\}$$

and

$$\mathrm{Cont}^{\geq m}(\mathfrak{a}) = \{\alpha \in \mathcal{L}^\infty(X) \mid \mathrm{ord}_\alpha(\mathfrak{a}) \geq m\}.$$

These subsets are called *contact loci* of the ideal $\mathfrak{a}$. The subset $\mathrm{Cont}^{\geq m}(\mathfrak{a})$ is closed and $\mathrm{Cont}^m(\mathfrak{a})$ is locally closed; both are cylinders.

For a contact locus, we define the codimension in the arc space $\mathcal{L}^m(X)$ [de Fernex et al. 2008, Section 3].

By the inversion of adjunction, we can describe Mather–Jacobian discrepancy in terms of the jet schemes of A as follows:

Proposition 2.14. *Let X, A, c, $\mathfrak{a}$ and $\tilde{\mathfrak{a}}$ be as in Proposition 2.9. Let $N = d + c$ and $Z = V(\tilde{\mathfrak{a}})$. Let $\psi_m : \mathcal{L}^\infty(A) \to \mathcal{L}^m(A)$ and $\psi_{mn} : \mathcal{L}^m(A) \to \mathcal{L}^n(A)$ be the canonical projections of jet schemes of A (not for X). Then,*

$$\mathrm{mld}_{\mathrm{MJ}}(W; X, \mathfrak{a}^t) = \inf_{m,n \in \mathbb{Z}_{\geq 0}} \{(M + 1)N - (m + 1)t - (n + 1)c$$
$$- \dim(\psi_{Mm}^{-1}(\mathcal{L}^m(Z)) \cap \psi_{Mn}^{-1}(\mathcal{L}^n(X)) \cap \psi_{M0}^{-1}(W))\},$$

where $M = \max\{m, n\}$.

In particular for $\mathfrak{a}^t = \mathcal{O}_X$ we obtain

$$\mathrm{mld}_{\mathrm{MJ}}(W; X, \mathcal{O}_X) = \inf_{n \in \mathbb{Z}_{\geq 0}} \{(n + 1)d - \dim(\psi_{n0}^X)^{-1}(W)\}, \tag{2}$$

where $\psi_{n0}^X : \mathcal{L}^n(X) \to \mathcal{L}^0(X) = X$ is the canonical projection of jet schemes of X.

Proof. By the inversion of adjunction, we can represent

$$\mathrm{mld}_{\mathrm{MJ}}(W; X, \mathfrak{a}^t) = \mathrm{mld}_{\mathrm{MJ}}(W; A, \tilde{\mathfrak{a}}^t I_X^c) = \mathrm{mld}(W; A, \tilde{\mathfrak{a}}^t I_X^c).$$

By [Ishii 2013, Remark 3.8], this is represented as

$$\mathrm{mld}(W; A, \tilde{\mathfrak{a}}^t I_X^c)$$
$$= \inf_{m,n \in \mathbb{N}} \{\mathrm{codim}(\mathrm{Cont}^{\geq m}(\mathfrak{a}) \cap \mathrm{Cont}^{\geq n}(I_X) \cap \mathrm{Cont}^{\geq 1}(I_W)) - mt - nc\},$$

where codim is the codimension in the arc space $\mathcal{L}^{\infty}(A)$. By shifting m to $m+1$ and n to $n+1$, this is represented as

$$\inf_{m,n \in \mathbb{Z}_{\geq 0}} \{ \mathrm{codim}(\mathrm{Cont}^{\geq m+1}(\mathfrak{a}) \cap \mathrm{Cont}^{\geq n+1}(I_X) \cap \mathrm{Cont}^{\geq 1}(I_W)) \\ -(m+1)t - (n+1)c \},$$

Now noting that

$$\mathrm{Cont}^{\geq m+1}(\mathfrak{a}) = \psi_m^{-1}(\mathcal{L}^m(Z)) \quad \text{and} \quad \mathrm{Cont}^{\geq n+1}(I_X) = \psi_n^{-1}(\mathcal{L}^n(X)),$$

we obtain the equality

$$\mathrm{codim}(\mathrm{Cont}^{\geq m+1}(\mathfrak{a}) \cap \mathrm{Cont}^{\geq n+1}(I_X) \cap \mathrm{Cont}^{\geq 1}(I_W)) \\ = \mathrm{codim}(\psi_{Mm}^{-1}(\mathcal{L}^m(Z)) \cap \psi_{Mn}^{-1}(\mathcal{L}^n(X)) \cap \psi_{M0}^{-1}(W), \mathcal{L}_M(A)),$$

where $M = \max\{m, n\}$. As $\dim A = N$, we have $\dim \mathcal{L}_M(A) = (M+1)N$ which yields the required equality. $\qquad\square$

Now we define an exceptional divisor over X, which is a generalization of an exceptional divisor for normal variety (Note that if X is normal, an exceptional divisor is defined as a divisor over X with the center of codimension ≥ 2 on X.)

Definition 2.15. Let E be a prime divisor over X. Let $\varphi : Y \to X$ be a proper birational morphism such that Y is normal and E appears on Y. Then E is called an exceptional divisor over X if φ is not isomorphic at the generic point of E. Here, we note that this definition is independent of the choice of φ.

Definition 2.16. We call a pair $(X, \mathfrak{a}^t)$ consisting of a connected reduced equidimensional scheme X of finite type over k and a nonzero ideal $\mathfrak{a} \subset \mathcal{O}_X$ with a nonnegative real number t is *MJ-canonical* (resp. *MJ-log canonical*) if for every exceptional prime divisor E over X, the inequality $a_{\mathrm{MJ}}(E; X, \mathfrak{a}^t) \geq 1$ (resp. ≥ 0) holds.

We say that $(X, \mathfrak{a}^t)$ is MJ-canonical (resp. MJ-log canonical) at a point $x \in X$, if there is an open neighborhood $U \subset X$ of x such that $(U, \mathfrak{a}^t|_U)$ is MJ-canonical (resp. MJ-log canonical).

If $(X, \mathcal{O}_X)$ is MJ-canonical (resp. MJ-log canonical), we say that X is MJ-canonical (resp. MJ-log canonical), or X has MJ-canonical (resp. MJ-log canonical) singularities.

In the similar way, we can define *MJ-terminal* and *MJ-log terminal* by the conditions for all exceptional prime divisors. In addition, we say that $(X, \mathfrak{a}^t)$ is *MJ-klt* if for every prime divisor E over X, the inequality $a_{\mathrm{MJ}}(E; X, \mathfrak{a}^t) > 0$ holds.

Definition 2.17. Let $(X, \mathfrak{a}^t)$ be a pair consisting of X and a nonzero ideal $\mathfrak{a} \subset \mathcal{O}_X$ with a nonnegative real number t. Let $\varphi : Y \to X$ be a log resolution

of $(X, \mathfrak{a}\mathcal{J}_X)$. Define a divisor $Z_{Y/X}$ by $\mathcal{O}_Y(-Z_{Y/X}) = \mathfrak{a}\mathcal{O}_Y$. Then we can define the *Mather–Jacobian multiplier ideal* (or *MJ-multiplier ideal* for short) as follows:

$$\mathcal{J}_{\mathrm{MJ}}(X, \mathfrak{a}^t) = \varphi_*\big(\mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} - [tZ_{Y/X}])\big),$$

where $[D]$ is the round down of the real divisor D.

Remark 2.18. At the stage of the definition, this multiplier "ideal" is only a fractional ideal sheaf for nonnormal X. But in [Ein et al. 2011] we proved that it is really an ideal sheaf of $\mathcal{O}_X$ in general. In [Ein et al. 2011], the MJ-multiplier ideal is proved to have good properties which a "multiplier ideal" is expected to have.

In [Ein et al. 2011] this multiplier ideal is called Mather multiplier ideal and denoted by $\widehat{\mathcal{J}}(X, \cdots)$. On the other hand, in [de Fernex and Docampo 2014] MJ-canonical (resp. MJ-log canonical) are called J-canonical (resp. log J-canonical). Here we think that it is more appropriate to call these notions with both M and J.

Remark 2.19. Fix a log resolution $Y \to X$ of $(X, \mathcal{J}_X\mathfrak{a})$. Then $(X, \mathfrak{a}^t)$ is MJ-canonical (resp. MJ-log canonical, MJ-terminal, MJ-log terminal) if and only if $a_{\mathrm{MJ}}(E; X, \mathfrak{a}^t) \geq 1$ (resp. ≥ 0, > 1, > 0) for all exceptional prime divisor E on Y. Also $(X, \mathfrak{a}^t)$ is MJ-klt if and only if $a_{\mathrm{MJ}}(E; X, \mathfrak{a}^t) > 0$ for every prime divisor E on Y. This is proved by using the fact that

$$\widehat{K}_{Y'/X} - J_{Y'/X} = K_{Y'/Y} + \psi^*(\widehat{K}_{Y/X} - J_{Y/X})$$

for another resolution $Y' \to X$ factoring through $Y \to X$.

Remark 2.20. Assume that X is normal and locally a complete intersection. Then by Remark 2.5, MJ-canonical (resp. MJ-log canonical) are equivalent to canonical (resp. log canonical). For normal and $\mathbb{Q}$-Gorenstein case, we have the following:

Proposition 2.21. *Let X be a normal $\mathbb{Q}$-Gorenstein variety, $\mathfrak{a} \subset \mathcal{O}_X$ a ideal sheaf and t a nonnegative real number. If $(X, \mathfrak{a}^t)$ is MJ-canonical (resp. MJ-log canonical, MJ-terminal, MJ-log terminal, MJ-klt), then it is canonical (resp. log canonical, terminal, log terminal, klt) in the usual sense.*

Proof. Let the index of X be r. Then the image of the canonical map

$$\big(\textstyle\bigwedge^d \Omega_X\big)^{\otimes r} \to \omega_X^{[r]}$$

is written as $I_r \omega_X^{[r]}$ with an ideal I_r since $\omega_X^{[r]}$ is invertible. Then, by the definition of the Mather discrepancy and the usual discrepancy, we have

$$I_r \mathcal{O}_Y(r\widehat{K}_{Y/X}) = \mathcal{O}_Y(rK_{Y/X})$$

for a log resolution $Y \to X$ of $(X, \mathcal{J}_X \mathfrak{a})$. Let $J_r = \mathcal{J}_X{}^r : I_r$; then $J_r I_r$ and $\mathcal{J}_X{}^r$ have the same integral closures by [Ein and Mustață 2009, Corollary 9.4]. Therefore if we write $\mathcal{O}_Y(-Z_r) = I_r \mathcal{O}_Y$ and $\mathcal{O}_Y(-Z'_r) = J_r \mathcal{O}_Y$, then $r J_{Y/X} = Z_r + Z'_r$ and

$$r \widehat{K}_{Y/X} - r J_{Y/X} = r \widehat{K}_{Y/X} - Z_r - Z'_r = r K_{Y/X} - Z'_r \leq r K_{Y/X},$$

which gives our assertions.   □

Proposition 2.22. (i) *A pair* $(X, \mathfrak{a}^t)$ *is MJ-log canonical at a* (*not necessarily closed*) *point* $x \in X$ *if and only if*

$$\mathrm{mld}_{\mathrm{MJ}}(x; X, \mathfrak{a}^t) \geq 0.$$

(ii) *If a pair* $(X, \mathfrak{a}^t)$ *is MJ-canonical at a* (*not necessarily closed*) *point* $x \in X$ *then*

$$\mathrm{mld}_{\mathrm{MJ}}(x; X, \mathfrak{a}^t) \geq 1.$$

Proof. It is clear that if a pair $(X, \mathfrak{a}^t)$ is MJ-log canonical (resp. MJ-canonical) at a point $x \in X$ then $\mathrm{mld}_{\mathrm{MJ}}(x; X, \mathfrak{a}^t) \geq 0$ (resp. $\mathrm{mld}_{\mathrm{MJ}}(x; X, \mathfrak{a}^t) \geq 1$) by the definitions. For the proof of the converse statement in (i), we have only to note that

$$\widehat{K}_{Y'/X} = K_{Y'/Y} + \varphi^* \widehat{K}_{Y/X}$$

for another resolution Y' of X that dominates Y by $\varphi : Y' \to Y$. The proof of the proposition is the same as the corresponding statement for the usual minimal log discrepancy.   □

The converse of the statement of (ii) in Proposition 2.22 does not hold. The following is an example for that.

Example 2.23. Let X be a hypersurface in $\mathbb{A}^3$ defined by $x_1 x_2 = 0$, where x_1, x_2, x_3 are the coordinates of $\mathbb{A}^3$. Then the x_3-axis C is the singular locus of X. By the inversion of adjunction, we have

$$\mathrm{mld}_{\mathrm{MJ}}(C; X, \mathcal{O}_X) = \mathrm{mld}(C; \mathbb{A}^3, (x_1 x_2)),$$

where the right-hand side is known to be zero. Therefore X is not MJ-canonical at the origin 0. On the other hand, again by the inversion of adjunction,

$$\mathrm{mld}_{\mathrm{MJ}}(0; X, \mathcal{O}_X) = \mathrm{mld}(0; \mathbb{A}^3, (x_1 x_2)),$$

where the right-hand side is known to be 1.

In the Definition 2.16 of MJ-log canonical singularities, the conditions are for exceptional prime divisors over X. But we can replace them by prime divisors over X.

Proposition 2.24. *A pair* $(X, \mathfrak{a}^t)$ *is MJ-log canonical if and only if* $a_{\mathrm{MJ}}(E; X, \mathfrak{a}^t) \geq 0$ *holds for every prime divisor* E *over* X.

Proof. The "if" part of the proof is obvious. For the converse, we have only to note that

$$\widehat{K}_{Y'/X} = K_{Y'/Y} + \varphi^* \widehat{K}_{Y/X}$$

for another resolution Y' of X that dominates Y by $\varphi : Y' \to Y$. The proof of the statement is the same as the corresponding statement for the usual log discrepancy. □

3. Basic properties of the MJ-singularities

In this section, we show some basic properties on MJ-singularities. Before that, we recall two known properties:

Proposition 3.1 [de Fernex and Docampo 2014; Ein et al. 2011]. *If* X *is MJ-canonical*, *then it is normal and has rational singularities.*

Proposition 3.2 [de Fernex and Docampo 2014]. *If* $k = \mathbb{C}$ *and* X *is MJ-log canonical*, *then* X *has Du Bois singularities.*

We will see that the class of Du Bois singularities is much wider than that of MJ-log canonical singularities (see Example 5.2).

Proposition 3.3. *Let* $x \in X$ *be a closed point. If* X *is MJ-canonical at* x, *then the embedding dimension* $\mathrm{emb}(X, x) \leq 2d - 1$. *If* X *is MJ-log canonical at* x, *then the embedding dimension* $\mathrm{emb}(X, x) \leq 2d$.

Proof. By (2) in Proposition 2.14, with putting $W = \{x\}$ we have

$$\mathrm{mld}_{\mathrm{MJ}}(x; X, \mathcal{O}_X) = \inf_{n \in \mathbb{Z}_{\geq 0}} \left\{ (n+1)d - \dim(\psi_{n0}^X)^{-1}(x) \right\}.$$

If X is MJ-canonical at x, then $\mathrm{mld}_{\mathrm{MJ}}(x; X, \mathcal{O}_X) \geq 1$ and this implies that $\dim(\psi_{n0}^X)^{-1}(x) \leq (n+1)d - 1$ holds for every $n \in \mathbb{N}$. Therefore, in particular for $n = 1$, we have

$$\dim(T_{X,x}) = \dim(\psi_{10}^X)^{-1}(x) \leq 2d - 1,$$

where $T_{X,x}$ is the Zariski tangent space of X at x. Hence the embedding dimension of X at x is $2d - 1$. The proof for the statement on MJ-log canonical singularities follows in the same way. □

Definition 3.4. Let X be embedded in a nonsingular variety A and I_X the defining ideal of X in A. Let $\Phi : \bar{A} \to A$ be a proper birational morphism which is isomorphic on the generic point of each irreducible component of X. Let $\overline{X}$ be the strict transform of X in $\bar{A}$ and $I_{\overline{X}}$ be the defining ideal of $\overline{X}$ in $\bar{A}$. Then, we call Φ a factorizing resolution of X in A if the following hold:

(i) Φ is an embedded resolution of X in A;

(ii) There is an effective divisor R on $\bar{A}$ such that

$$I_X \mathcal{O}_{\bar{A}} = I_{\bar{X}} \mathcal{O}_{\bar{A}}(-R).$$

The existence of factorizing resolution of a given embedding $X \subset A$ is proved by A. Bravo and O. Villamayor [2003], and E. Eisenstein [2010] obtained a modified version which can be applied to the case of log resolutions. The following is an easy corollary of [Eisenstein 2010, Lemma 3.1].

Proposition 3.5. *Let $X \subset A$ be a closed embedding into a nonsingular variety A and let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of $\mathcal{O}_X$ and $\mathcal{O}_A$, respectively. Let $\tilde{\mathfrak{a}}$ be an ideal such that $\tilde{\mathfrak{a}} \mathcal{O}_X = \mathfrak{a}$. Assume that $\mathfrak{a}$ and $\mathfrak{b}$ are not zero on the generic point of each irreducible component of X. Then, there exists a factorizing resolution $\Phi : \bar{A} \to A$ of X in A such that Φ is a log resolution of $(A, \tilde{\mathfrak{a}}\mathfrak{b})$ and the restriction $\Phi|_{\bar{X}}$ of Φ onto the strict transform $\bar{X}$ is a log resolution of $(X, \mathfrak{a})$.*

We sometimes come across the situation to compare the MJ-discrepancies of two schemes connected by a proper birational morphism. The following gives some information on that.

Theorem 3.6. *Let $\varphi : X' \to X$ be a proper birational morphism which can be extended to a proper birational morphism $\Phi : A' \to A$ of nonsingular varieties such that $X' \subset A'$, $X \subset A$ with codimension c and Φ is isomorphic at the generic point of each irreducible component of X. Let I_X and $I_{X'}$ be defining ideals of X and X' in A and A', respectively.*

If $I_{X'} \mathfrak{b}' \subset I_X \mathcal{O}_{A'} \subset I_{X'} \mathfrak{b}$ holds for some coherent ideal sheaves $\mathfrak{b}$, $\mathfrak{b}'$ in $\mathcal{O}_{A'}$ that do not vanish on any irreducible component of X', then there exists an embedded resolution $\Psi : \bar{A} \to A'$ of X' in A' such that the restriction $(\Phi \circ \Psi)|_{\bar{X}} : \bar{X} \to X$ is a log resolution of $(X, \mathfrak{a} \mathfrak{J}_X)$ and satisfying

$$\hat{K}_{\bar{X}/X'} - J_{\bar{X}/X'} - cR'|_{\bar{X}} \le \hat{K}_{\bar{X}/X} - J_{\bar{X}/X} - \Psi^* K_{A'/A}|_{\bar{X}}$$

$$\le \hat{K}_{\bar{X}/X'} - J_{\bar{X}/X'} - cR|_{\bar{X}},$$

where R and R' are effective divisors on $\bar{A}$ such that $\mathfrak{b} \mathcal{O}_{\bar{A}} = \mathcal{O}_{\bar{A}}(-R)$ and $\mathfrak{b}' \mathcal{O}_{\bar{A}} = \mathcal{O}_{\bar{A}}(-R')$.

For the proof of the theorem, we need the following lemma which is a generalization of [Eisenstein 2010, Lemma 4.3]:

Lemma 3.7. *Let X be embedded into a nonsingular variety A with codimension c, $\Phi : \bar{A} \to A$ a proper birational morphism of nonsingular varieties isomorphic at the generic points of the irreducible components of X and $\bar{X}$ the strict transform*

*of X in $\bar{A}$. Assume that $\bar{X}$ is nonsingular. Denote the ideal of X and $\bar{X}$ by I_X
and $I_{\bar{X}}$, respectively. Assume*

$$I_{\bar{X}}\mathcal{O}_{\bar{A}}(-R') \subset I_X\mathcal{O}_{\bar{A}} \subset I_{\bar{X}}\mathcal{O}_{\bar{A}}(-R), \tag{3}$$

for some effective divisors R, R' on $\bar{A}$ that do not contain any irreducible component of $\bar{X}$ in their supports. Then, we have

$$(K_{\bar{A}/A} - cR')|_{\bar{X}} \le \hat{K}_{\bar{X}/X} - J_{\bar{X}/X} \le (K_{\bar{A}/A} - cR)|_{\bar{X}}. \tag{4}$$

In particular, if $I_X\mathcal{O}_{\bar{A}} = I_{\bar{X}}\mathcal{O}_{\bar{A}}(-R)$, then

$$\hat{K}_{\bar{X}/X} - J_{\bar{X}/X} = (K_{\bar{A}/A} - cR)|_{\bar{X}}.$$

Proof. We use the notation in [Eisenstein 2010]. The notation $[a_{ij}]_c$ means the ideal generated by c-minors of the matrix (a_{ij}). Now since the problem is local, it is sufficient to show the statement at a neighborhood of a point $P \in \bar{A}$. Let I_X be generated by $h_1, \ldots, h_m$ around $\Phi(P)$. Let $(z_1, \ldots, z_N)$ be local coordinates of A at $\Phi(P)$ and $(w_1, \ldots, w_d, w_{d+1}, \ldots, w_N)$ local coordinates of $\bar{A}$ at P such that $(w_1, \ldots, w_d)$ is local coordinates of $\bar{X}$. Then, by [Eisenstein 2010, Lemma 4.3], it follows:

$$\mathcal{O}_{\bar{X}}(-\hat{K}_{\bar{X}/X})\left(\left[\frac{\partial(h_i \circ \Phi)}{\partial w_j}\right]_c\right)\mathcal{O}_{\bar{X}} = \mathcal{O}_{\bar{A}}(-K_{\bar{A}/A})\left(\left[\frac{\partial h_i}{\partial z_j}\right]_c \mathcal{O}_{\bar{A}}\right)\mathcal{O}_{\bar{X}}, \tag{5}$$

where the right-hand side coincides with

$$\mathcal{O}_{\bar{X}}(-K_{\bar{A}/A}|_{\bar{X}} - J_{\bar{X}/X}).$$

Let g and g' be local generators of $\mathcal{O}_{\bar{A}}(-R)$ and $\mathcal{O}_{\bar{A}}(-R')$ at P, respectively. As $I_{\bar{X}}$ is generated by $w_{d+1}, \ldots, w_N$, the condition of the lemma implies:

$$(g'w_{d+1}, \ldots, g'w_N) \subset I_X\mathcal{O}_{\bar{A}} = (h_1 \circ \Phi, \ldots, h_m \circ \Phi) \subset (gw_{d+1}, \ldots, gw_N).$$

Then, we obtain:

$$\left[\frac{\partial(g'w_i)}{\partial w_j}\right]_c\Big|_{\bar{X}} \subset \left[\frac{\partial(h_i \circ \Phi)}{\partial w_j}\right]_c\Big|_{\bar{X}} \subset \left[\frac{\partial(gw_i)}{\partial w_j}\right]_c\Big|_{\bar{X}}. \tag{6}$$

Here, we used a general fact: If $I = (g_1, \ldots, g_n) \subset J = (f_1, \ldots, f_m)$ are ideals. Then for a closed subscheme $Z \subset Z(J)$, it holds that

$$\left[\frac{\partial g_i}{\partial w_j}\right]_c\Big|_Z \subset \left[\frac{\partial f_i}{\partial w_j}\right]_c\Big|_Z.$$

Note that

$$\frac{\partial(gw_i)}{\partial w_j}\Big|_{\bar{X}} = \left(g\frac{\partial w_i}{\partial w_j} + w_i\frac{\partial g}{\partial w_j}\right)\Big|_{\bar{X}} = g\frac{\partial w_i}{\partial w_j},$$

since $w_i = 0$ on $\overline{X}$ for $i = d + 1, \ldots, N$. Here, we obtain

$$\left[\frac{\partial(g w_i)}{\partial w_j}\right]_c \bigg|_{\overline{X}} = g^c|_{\overline{X}},$$

and similarly

$$\left[\frac{\partial(g' w_i)}{\partial w_j}\right]_c \bigg|_{\overline{X}} = g'^c|_{\overline{X}}.$$

Therefore, the inclusions of (6) turn out to be

$$(g'^c)|_{\overline{X}} \subset \left[\frac{\partial(h_i \circ \Phi)}{\partial w_j}\right]_c \bigg|_{\overline{X}} \subset (g^c)|_{\overline{X}}.$$

Substituting this into (5) we obtain

$$\mathcal{O}_{\overline{X}}(-\widehat{K}_{\overline{X}/X} - cR') \subset \mathcal{O}_{\overline{X}}(-K_{\overline{A}/A} - J_{\overline{X}/X}) \subset \mathcal{O}_{\overline{X}}(-\widehat{K}_{\overline{X}/X} - cR),$$

which proves the required inequalities. $\qquad\square$

Proof of Theorem 3.6. Applying Proposition 3.5 to $X' \subset A'$, we obtain a factorizing resolution $\Psi : \overline{A} \to A'$ of X' in A', such that it is a log resolution of $(A', \mathfrak{bb}'\widetilde{\mathcal{J}_X}\widetilde{\mathcal{J}_{X'}})$, where $\widetilde{\mathcal{J}_X}$ and $\widetilde{\mathcal{J}_{X'}}$ are ideals of $\mathcal{O}_{A'}$ such that $\widetilde{\mathcal{J}_X}\mathcal{O}_{X'} = \mathcal{J}_X \mathcal{O}_{X'}$ and $\widetilde{\mathcal{J}_{X'}}\mathcal{O}_{X'} = \mathcal{J}_{X'}$, respectively. Let $\mathfrak{b}\mathcal{O}_{\overline{A}} = \mathcal{O}_{\overline{A}}(-R)$ and $\mathfrak{b}'\mathcal{O}_{\overline{A}} = \mathcal{O}_{\overline{A}}(-R')$. As Ψ is a factorizing resolution of X' in A', there exists an effective divisor G on $\overline{A}$ such that

$$I_{X'}\mathcal{O}_{\overline{A}} = I_{\overline{X}}\mathcal{O}_{\overline{A}}(-G).$$

By the assumption of the proposition, we have

$$I_{X'}\mathcal{O}_{\overline{A}}(-R') \subset I_X\mathcal{O}_{\overline{A}} = (I_X\mathcal{O}_{A'})\mathcal{O}_{\overline{A}} \subset I_{X'}\mathcal{O}_{\overline{A}}(-R),$$

which yields

$$I_{\overline{X}}\mathcal{O}_{\overline{A}}(-G - R') \subset I_X\mathcal{O}_{\overline{A}} \subset I_{\overline{X}}\mathcal{O}_{\overline{A}}(-G - R).$$

Now by Lemma 3.7, we obtain

$$(K_{\overline{A}/A} - cG - cR')|_{\overline{X}} \leq \widehat{K}_{\overline{X}/X} - J_{\overline{X}/X} \leq (K_{\overline{A}/A} - cG - cR')|_{\overline{X}}.$$

By substituting

$$K_{\overline{A}/A} = K_{\overline{A}/A'} + \Psi^* K_{A'/A} \quad \text{and} \quad (K_{\overline{A}/A'} - cG)|_{\overline{X}} = \widehat{K}_{\overline{X}/X'} - J_{\overline{X}/X'},$$

which follows from the second statement of Lemma 3.7, we conclude the inequalities:

$$\widehat{K}_{\overline{X}/X'} - J_{\overline{X}/X'} - cR'|_{\overline{X}} \leq \widehat{K}_{\overline{X}/X} - J_{\overline{X}/X} - \Psi^* K_{A'/A}|_{\overline{X}}$$

$$\leq \widehat{K}_{\overline{X}/X'} - J_{\overline{X}/X'} - cR|_{\overline{X}}. \qquad\square$$

Remark 3.8. Let us make a comment about a condition of Theorem 3.6. Locally on X, every projective birational morphism $X' \to X$ can be extended to a projective birational morphism $A' \to A$ of nonsingular varieties. This is proved as follows. We can assume that X is embedded in $\mathbb{A}^N$ and $X' \to X$ is a blow-up by an ideal $\mathfrak{I} = (f_1, \dots, f_r)$ of $\mathcal{O}_X$. Extend the canonical surjective homomorphism $k[x_1, \dots, x_N] \to \Gamma(X, \mathcal{O}_X)$ to a homomorphism

$$k[x_1, \dots, x_N, y_1, \dots, y_r] \to \Gamma(X, \mathcal{O}_X)$$

by $y_i \mapsto f_i$ for $i = 1, \dots, r$. Let $X \subset \mathbb{A}^{N+r}$ be the embedding corresponding to this homomorphism. Then the blow-up $\Phi : A' \to A$ by the ideal $(y_1, \dots, y_r)$ gives the blow-up by the ideal $\mathfrak{I}$ on X. Since the center of the blow-up Φ is nonsingular, A' is also nonsingular.

The most effective application of Theorem 3.6 is for the case that $X' \to X$ is the blow-up at a closed point.

Corollary 3.9. *Let $X \subset A$ be a closed embedding into a nonsingular variety A with codimension c and $\mathfrak{a}$ an ideal of $\mathcal{O}_X$. Let $\Phi : A' \to A$ be the blow-up of A at a closed point $x \in X$ and X' the strict transform of X. Let E be the exceptional divisor for Φ and nonnegative integers a, b as*

$$I_{X'} \mathcal{O}_{A'}(-aE) \subset I_X \mathcal{O}_{A'} \subset I_{X'} \mathcal{O}_{A'}(-bE).$$

Then, there is a proper birational morphism $\Psi : \bar{A} \to A'$ with the strict transform $\bar{X}$ of X in $\bar{A}$ such that the restriction $\Phi \circ \Psi|_{\bar{X}} : \bar{X} \to X$ is a log resolution of $(X, \mathfrak{a}\mathfrak{J}_X)$ and $\Psi|_{\bar{X}} : \bar{X} \to X'$ is a log resolution of $(X', \mathfrak{J}_{X'})$ satisfying

$$\hat{K}_{\bar{X}/X'} - J_{\bar{X}/X'} - (ac - c - d + 1)\Psi^* E|_{\bar{X}} \leq \hat{K}_{\bar{X}/X} - J_{\bar{X}/X}$$
$$\leq \hat{K}_{\bar{X}/X'} - J_{\bar{X}/X'} - (bc - c - d + 1)\Psi^* E|_{\bar{X}}.$$

In particular if $I_{X'} \mathcal{O}_{A'}(-aE) = I_X \mathcal{O}_{A'}$, then

$$\hat{K}_{\bar{X}/X'} - J_{\bar{X}/X'} - (ac - c - d + 1)\Psi^* E|_{\bar{X}} = \hat{K}_{\bar{X}/X} - J_{\bar{X}/X}.$$

Proof. As $\dim X = d$, note that $K_{A'/A} = (c + d - 1)E$ and apply Theorem 3.6. $\quad\square$

Example 3.10. Let (X, x) be a singularity on a reduced 2-dimensional scheme X and let $\varphi : X' \to X$ be the blow-up at x. If (X, x) is MJ-canonical (MJ-log canonical) singularity, then X' has MJ-canonical (MJ-log canonical) singularities.

Here, if (X, x) is nonsingular, then X' is also nonsingular and the above statement is trivial. Therefore we may assume that (X, x) is a singular point. For the both statements of the example, it is sufficient to prove

$$\hat{K}_{\bar{X}/X} - J_{\bar{X}/X} \leq \hat{K}_{\bar{X}/X'} - J_{\bar{X}/X'}$$

for a log resolution $\Psi : \overline{X} \to X'$ of $\mathcal{J}_{X'}\mathcal{J}_X \mathcal{O}_{X'}$. As (X, x) is singular, we have $c \geq 1$ and

$$I_X \mathcal{O}_{A'} \subset I_{X'} \mathcal{O}_{A'}(-2E),$$

under the notation of Corollary 3.9. Let $b = 2$ and note that

$$bc - c - d + 1 = c - 1 \geq 0.$$

Then apply the corollary, we obtain the required inequality

$$\hat{K}_{\overline{X}/X} - J_{\overline{X}/X} \leq \hat{K}_{\overline{X}/X'} - J_{\overline{X}/X'}.$$

Example 3.11. Let (X, x) be a singular point in a 3-dimensional reduced scheme. Assume (X, x) is not a hypersurface double point. Let X' be the same as in Example 3.10. If (X, x) is MJ-canonical (MJ-log canonical), then X' has MJ-canonical (MJ-log canonical) singularities.

As in Example 3.10, it is sufficient to prove that $bc - c - d + 1 \geq 0$. If (X, x) is not a hypersurface singularity, then $c \geq 2$ and we can take $b = 2$ and obtain $bc - c - d + 1 = c - 2 \geq 0$. If (X, x) is a hypersurface singularity of multiplicity ≥ 3, then we can take $b \geq 3$; therefore $bc - c - d + 1 \geq 2 - 3 + 1 = 0$.

Example 3.12. Let $S \subset \mathbb{P}^{N-1}$ be a $(d-1)$-dimensional nonsingular projectively normal closed subvariety defined by polynomials of common degree a. Let $X \subset \mathbb{A}^N$ be its affine cone. Then,

  (i) X is MJ-canonical if and only if $a \leq \frac{N-1}{N-d}$,

 (ii) X is MJ-log canonical if and only if $a \leq \frac{N}{N-d}$.

Let us check the MJ-log canonicity and MJ-canonicity of X. Let $\Phi : A' \to \mathbb{A}^N$ be the blow-up at the origin, E the exceptional divisor and X' the strict transform of X in A'. Then, by the defining equations of X in $\mathbb{A}^N$, we have

$$I_X \mathcal{O}_{X'} = I_{X'} \mathcal{O}_{X'}(-aE).$$

By Corollary 3.9, we have

$$\hat{K}_{\overline{X}/X'} - J_{\overline{X}/X'} - (ac - N + 1)\Psi^* E|_{\overline{X}} = \hat{K}_{\overline{X}/X} - J_{\overline{X}/X},$$

with $c = N - d$ for an appropriate log resolution $\Psi : \bar{A} \to A'$. Therefore we obtain

$$\hat{K}_{\overline{X}/X'} - J_{\overline{X}/X'} - (a(N-d) - N + 1)\Psi^* E|_{\overline{X}} = \hat{K}_{\overline{X}/X} - J_{\overline{X}/X}. \qquad (7)$$

Here, we note that $(X', E|_{X'})$ is nonsingular pair and $(X', \alpha E|_{X'})$ is log MJ-canonical if and only if $\alpha \leq 1$. Then by the equality (7) we have X is MJ-log canonical if and only if $a(N-d) - N + 1 \leq 1$ which is equivalent to $a \leq \frac{N}{N-d}$. On the other hand, if $a(N-d) - N + 1 \leq 0$ which implies $a \leq \frac{N-1}{N-d}$, we have that

X is MJ-canonical by the equality (7). If $a(N-d)-N+1=1$, then the equality (7) implies $a_{\mathrm{MJ}}(E; X, \mathcal{O}_X) = 0$, which yields that X is not MJ-canonical.

Example 3.13. Under the same setting as in the previous example, let $a = 2$. Then,

  (i) X is MJ-canonical if and only if $N \leq 2d - 1$,

 (ii) X is MJ-log canonical if and only if $N \leq 2d$.

Note that these conditions on N and d are only the necessary conditions for a general X to be MJ-canonical and MJ-log canonical as are seen in Proposition 3.3.

We can see that the cones of many homogeneous spaces enjoy these conditions. For example, the cones of $G(2, 5) \subset \mathbb{P}^9$, $E_6 \subset \mathbb{P}^{26}$ [Lazarsfeld and Van de Ven 1984] and 10-dimensional Spinor variety in $\mathbb{P}^{15}$ [Ein 1986] are all MJ-canonical.

Let $S_{rm} = \mathbb{P}^r \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{N-1}$ be the Segre embedding, i.e., the correspondence of the homogeneous coordinates is $(x_i) \times (y_j) \mapsto (x_i y_j)$. Then the subscheme S_{rm} is defined in $\mathbb{P}^N$ by the equations

$$z_{ij} z_{kl} - z_{il} z_{kj} = 0, (i = 0, \ldots, r, j = 0, \ldots, m),$$

where z_{ij}'s are homogeneous coordinates of $\mathbb{P}^{N-1}$ ($N = (r + 1)(m + 1)$). Let $X_{rm} \subset \mathbb{A}^N$ be the affine cone over S_{rm}. Then, as $d = r + m + 1$, we have the following:

  (i) X_{rm} is MJ-log canonical if an only if $(r - 1)(m - 1) \leq 2$,

 (ii) X_{rm} is MJ-canonical if and only if $(r - 1)(m - 1) \leq 1$.

In particular, X_{1m} and X_{r1} are all MJ-canonical. Here, we note that X_{rm} is $\mathbb{Q}$-Gorenstein if and only if $r = m$. Thus, if $r \neq 1$ or $m \neq 1$, then X_{1m} and X_{r1} are examples of MJ-canonical singularities which are not $\mathbb{Q}$-Gorenstein.

Example 3.14. Three-dimensional terminal quotient singularities have been determined as $\frac{1}{r}(s, -s, 1)$ ($0 < s < r, \gcd(s, r) = 1$) by Morrison and Stevens [1984]. If $s \neq 1, r-1$, then the singularity $\frac{1}{r}(s, -s, 1)$ is not MJ-log canonical. Indeed, the singularity is at the origin of $X = \operatorname{Spec} k[x^r, y^r, z^r, xy, xz^{r-s}, yz^s] = k[x_1, \ldots, x_6]/I$, where $I = (x_3 x_4 - x_5 x_6, x_1 x_2 - x_4^r, x_1 x_3^{r-s} - x_5^r, x_2 x_3^s - x_6^r)$. Here, we note that the number of generators with order 2 is two.

Assume that X has MJ-log canonical singularity at 0; then

$$\mathrm{mld}_{\mathrm{MJ}}(0, X, \mathcal{O}_X) \geq 0,$$

and therefore by the formula (2) in Proposition 2.14 we have

$$\dim(\psi_{n0}^X)^{-1}(0) \leq d(n + 1) = 3(n + 1).$$

In particular for $n = 2$, it follows that $\dim(\psi_{20}^X)^{-1}(0) \leq 9$. Under the notation

in 2.11, we have by Remark 2.12

$(\psi_{20}^X)^{-1}(0)$
$$= \operatorname{Spec} k[x_1^{(1)}, \ldots, x_6^{(1)}, x_1^{(2)}, \ldots, x_6^{(2)}]/(x_3^{(1)}x_4^{(1)} - x_5^{(1)}x_6^{(1)}, x_1^{(1)}x_2^{(1)})$$

whose dimension is greater than 9, a contradiction. Therefore X is not MJ-log canonical at 0.

Remark 3.15. The MJ-discrepancy has good properties: inversion of adjunction on minimal log discrepancies, lower semicontinuity of MJ-minimal log discrepancies [de Fernex and Docampo 2014; Ishii 2013], ascending chain condition (ACC) of MJ-log canonical thresholds [de Fernex and Docampo 2014]. So, if every step in a minimal model program (MMP) would preserve MJ-log canonicity, we could prove MMP simply. But actually a divisorial contraction does not preserve MJ-log canonicity. Kawamata [1996] determined the divisorial contraction to a three-dimensional terminal quotient singularity as a certain weighted blow-up. By this we can prove that every three-dimensional terminal quotient singularity can be resolved by the successive weighted blow-ups which are divisorial contractions. This gives a counterexample to the expectation that MJ-log canonicity would be preserved under divisorial contractions.

Proposition 3.16. *Assume X is S_2 and $\mathbb{Q}$-Gorenstein. If $(X, \mathfrak{a}^t)$ is MJ-log canonical, then it is semilog canonical.*

Proof. The definition of a semilog canonical singularity requires S_2 and $\mathbb{Q}$-Gorenstein property. The additional conditions for a semilog canonical singularity are [Kollár 2013]:

(i) X is nonsingular or has normal crossing double singularities in codimension one.

(ii) Let $\nu : X_\nu \to X$ be the normalization, $\mathfrak{a}_\nu$ the pull back of $\mathfrak{a}$ on X_ν and D_ν the divisor on X_ν defined by the conductor $(\mathcal{O}_X : \nu_*(\mathcal{O}_{X_\nu}))$. Then, $(X_\nu, \mathfrak{a}_\nu^t \mathcal{O}_{X_\nu}(-D_\nu))$ is log canonical in the usual sense.

Let W be an irreducible component of singular locus of X of codimension 1. Then $\operatorname{mld}_{\mathrm{MJ}}(W; X, \mathcal{O}_X) \geq 0$ implies $(\psi_{m0}^X)^{-1}(W) \leq d(m+1)$ by (2) in Proposition 2.14. As $\dim W = d - 1$, for a general point $x \in W$ we have $(\psi_{m0}^X)^{-1}(x) \leq dm + 1$; then again by (2) in Proposition 2.14, it follows that

$$\operatorname{mld}_{\mathrm{MJ}}(x; X, \mathcal{O}_X) \geq d - 1.$$

In this case, $\operatorname{mld}_{\mathrm{MJ}}(x; X, \mathcal{O}_X) = d - 1$ holds by [Ishii 2013, Corollary 3.15; de Fernex and Docampo 2014, Corollary 4.15] and such (X, x) is classified in [Ishii and Reguera 2013] as to be normal crossing double or a pinch point when it is nonnormal. As the pinch point locus is of codimension 2, we have the

assertion (i). The condition (ii) is equivalent to that the usual log discrepancy $a(E; X_\nu, \mathfrak{a}_\nu^t \mathcal{O}_{X_\nu}(-D_\nu)) \geq 0$ for every prime divisor E over X_ν. As $\nu^* K_X \sim_{\mathbb{Q}} K_{X_\nu} + D_\nu$, it is equivalent to $a(E, X, \mathfrak{a}^t) \geq 0$. By the same argument as in the proof of Proposition 2.21, we obtain $a_{\mathrm{MJ}}(E, X, \mathfrak{a}^t) \leq a(E, X, \mathfrak{a}^t)$, which yields the assertion (ii). Here, we note that the proof of Proposition 2.21 used Corollary 9.4 of [Ein and Mustaţă 2009], which was stated under the condition that X is normal. But the proof of the corollary works also for nonnormal case. $\square$

Corollary 3.17. *Let X be locally a complete intersection. Then, $(X, \mathfrak{a}^t)$ is MJ-log canonical if and only if it is semilog canonical.*

Proof. As X is locally a complete intersection, it is S_2. Then, by Proposition 3.16, if $(X, \mathfrak{a}^t)$ is MJ-log canonical, it is semilog canonical. Conversely, if $(X, \mathfrak{a}^t)$ is semilog canonical, then by the condition (ii) of semilog canonical in the proof of Proposition 3.16, we obtain

$$a_{\mathrm{MJ}}(E, X, \mathfrak{a}^t) = a(E, X, \mathfrak{a}^t) \geq 0$$

for every prime divisor E over X in the same way as in the proof above. This yields the required equivalence. $\square$

Here we note that the S_2 condition is necessary for a MJ-log canonical singularity to be semilog canonical. Actually there is an example of MJ-log canonical singularity which does not satisfy S_2 condition (see Example 5.7).

De Fernex and Hacon [2009] introduced in the notions log canonical, log terminal singularities on an arbitrary normal variety. These are direct generalizations of usual log canonical, klt singularities for $\mathbb{Q}$-Gorenstein case. Actually they defined that $(X, \mathfrak{a}^t)$ is log terminal (resp. log canonical) if there is $m \in \mathbb{N}$ such that

$$a_m(F; X, \mathfrak{a}^t) := \mathrm{ord}_F(K_{m,Y/X}) - t\,\mathrm{val}_F(\mathfrak{a}) + 1 > 0 \quad (\mathrm{resp.} \geq 0)$$

for every prime divisor F over X. Note that the log terminal an log canonical in their sense are not determined by finite number of exceptional divisors.

Here, in a local situation, as we can take an effective divisor mK_X, we can think a divisorial sheaf $\mathcal{O}_X(-mK_X)$ as an ideal sheaf. Let $Y \to X$ be a log resolution of an ideal $\mathcal{O}_X(-mK_X)$ and define the effective divisor D_m on Y by $\mathcal{O}_X(-mK_X)\mathcal{O}_Y = \mathcal{O}_Y(-D_m)$. Note that an arbitrary prime divisor F over X can appear on such a resolution Y. Under this notation we define the divisor

$$K_{m,Y/X} = K_Y - \frac{1}{m} D_m$$

with the support on the exceptional divisor.

On the other hand de Fernex and Hacon also introduce "canonical singularities" for a normal variety X in a slightly different line from log terminal and log canonical case. Let X be a normal variety and $f : Y \to X$ a resolution of the singularities of X. The relative canonical divisor $K_{Y/X}$ is defined as follows:

$$K_{Y/X} := K_Y + f^*(-K_X),$$

where $f^*(-K_X)$ is the pull-back of a Weil divisor $-K_X$ introduced in [de Fernex and Hacon 2009]. Here, we note that $f^*(-K_X) \neq -f^*(K_X)$ in general. They define that a pair $(X, \mathfrak{a}^t)$ has canonical singularities if

$$a_{dfh}(F; X, \mathfrak{a}^t) := \operatorname{ord}_F(K_{Y/X}) - t \operatorname{val}_F(\mathfrak{a}) + 1 \geq 1$$

holds for every exceptional prime divisor F over X.

We will see the relation of MJ-singularities and de Fernex–Hacon's singularities. First the following gives the relation of the divisor $K_{m,Y/X}$ and our MJ-discrepancy divisor.

Lemma 3.18. *Let X be an affine normal variety and m a positive integer. Then, there is a log resolution $Y \to X$ of $\mathfrak{J}_X \mathcal{O}_X(-mK_X)$ such that*

$$\widehat{K}_{Y/X} - J_{Y/X} \leq K_{m,Y/X}.$$

Proof. Fix a log resolution $\varphi : Y \to X$ of $\mathfrak{J}_X \mathcal{O}_X(-mK_X)$. Take a reduced complete intersection scheme $M \subset \mathbb{A}^N$ of codimension c such that M contains X as an irreducible component. Then we have a sequence of homomorphisms of $\mathcal{O}_X$-modules:

$$\left(\textstyle\bigwedge^d \Omega_X\right)^{\otimes m} \xrightarrow{\eta} \omega_X^{[m]} \xrightarrow{u} (\omega_M|_X)^m. \tag{8}$$

By [Ein and Mustață 2009, Proposition 9.1] the image of $u \circ \eta$ is written as

$$(\mathfrak{J}_M|_X)^m (\omega_M|_X)^m. \tag{9}$$

Then take a pull-back of the sequence (8):

$$\varphi^*\left(\textstyle\bigwedge^d \Omega_X\right)^{\otimes m} \xrightarrow{\eta} \varphi^* \omega_X^{[m]} \xrightarrow{u} \varphi^*(\omega_M|_X)^m. \tag{10}$$

Define a divisor D_m on Y as $\mathcal{O}_Y(-D_m) = \mathcal{O}_X(-mK_X)\mathcal{O}_Y$.

Then, we claim that

$$\varphi_*(\mathcal{O}_Y(D_m)) = \mathcal{O}_X(mK_X). \tag{11}$$

The inclusion $\subset$ holds because outside of the singular locus the both sheaves coincide and the right hand side is reflective, For the opposite inclusion, regard $\mathcal{O}_X(mK_X)$ as $\mathcal{O}_X(-mK_X)^* = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(-mK_X), \mathcal{O}_X)$. For the claim, it is sufficient to prove that every homomorphism $f : \mathcal{O}_X(-mK_X) \to \mathcal{O}_X$ comes

from a homomorphism $\mathcal{O}_X(-mK_X)\mathcal{O}_Y \to \mathcal{O}_Y$. The homomorphism f is lifted to $f' : \varphi^*(\mathcal{O}_X(-mK_X)) \to \mathcal{O}_Y$. Here, the torsion elements are mapped to zero by f'. Therefore f' factors through $\varphi^*(\mathcal{O}_X(-mK_X))/\operatorname{Tor} \to \mathcal{O}_Y$, where Tor is the subsheaf consisting of the torsion elements of $\varphi^*(\mathcal{O}_X(-mK_X))$. But we can prove that $\varphi^*(\mathcal{O}_X(-mK_X))/\operatorname{Tor} = \mathcal{O}_Y(-D_m)$. This completes the proof of the claim (11).

By (11), the sequence (10) factors as

$$\varphi^*\big(\textstyle\bigwedge^d \Omega_X\big)^{\otimes m} \xrightarrow{\eta'} \mathcal{O}_Y(D_m) \xrightarrow{u'} \varphi^*(\omega_M|_X)^m, \tag{12}$$

where u' is the dual map of

$$\mathcal{O}_X(-mK_X)\mathcal{O}_Y \leftarrow \mathcal{O}_X(-mK_M|_X)\mathcal{O}_Y = (\varphi^*(\omega_M|_X)^m)^*.$$

As the second and the third sheaves in the sequence (12) are invertible, we can write

$$\operatorname{Im}\eta' = I\mathcal{O}_Y(D_m), \quad \operatorname{Im}u' = J_M\varphi^*(\omega_M|_X)^m, \tag{13}$$

with the ideals $I, J_M \subset \mathcal{O}_Y$. Then, by (9), we obtain

$$IJ_M = (\mathcal{J}_M|_X)^m\mathcal{O}_Y. \tag{14}$$

Consider all M and define $J = \sum_M J_M$; then $IJ = \big(\sum_M \mathcal{J}_M^m\big)\mathcal{O}_Y$. By taking the integral closure of the both sides, we get

$$\overline{IJ} = \overline{\mathcal{J}_X^m\mathcal{O}_Y}. \tag{15}$$

Now, given a prime divisor F over X, it appears on a log resolution $\nu : Y' \to Y$ of IJ. Let $\psi : Y' \to X$ be the composite $\nu \circ \varphi$. Define effective divisors B, C on Y' such that $\mathcal{O}_{Y'}(-B) = I\mathcal{O}_{Y'}$ and $\mathcal{O}_{Y'}(-C) = J\mathcal{O}_{Y'}$; then

$$B + C = mJ_{Y'/X}. \tag{16}$$

As ψ factors through the Nash blow-up, the torsion-free sheaf

$$\big((\psi^*\textstyle\bigwedge^d \Omega_X)/\operatorname{Tor}\big)^{\otimes m}$$

is invertible; therefore it is written as $\mathcal{O}_{Y'}(G)$ by a divisor G on Y'. Then, by the definition of $\widehat{K}_{Y'/X}$, we have $m\widehat{K}_{Y'/X} = mK_{Y'} - G$. On the other hand, by (13) we have $G = \nu^*D_m - B$ and by (15) we have

$$m\widehat{K}_{Y'/X} - mJ_{Y'/X}$$
$$= mK_{Y'} - G - (B+C) = mK_{Y'} - \nu^*D_m - C \le mK_{Y'} - \nu^*D_m,$$

which completes the proof of the lemma. $\qquad\square$

The following shows the relation of our MJ-singularities and the singularities of de Fernex and Hacon.

Theorem 3.19. *Assume that X is normal. If a pair $(X, \mathfrak{a}^t)$ is MJ-klt (resp. MJ-canonical, MJ-log canonical), then it is log terminal (resp. canonical, log canonical) in the sense of de Fernex and Hacon.*

Proof. Since the problem is local, we may assume that X is a closed subvariety of the affine space $\mathbb{A}^N$ of codimension c. It is sufficient to prove for a fixed $m \in \mathbb{N}$ that

$$a_{\mathrm{MJ}}(F; X, \mathcal{O}_X) \le a_m(F; X, \mathcal{O}_X) \le a_{dfh}(F; X, \mathcal{O}_X)$$

for every prime divisor F over X. The last inequality is given in [de Fernex and Hacon 2009, Remark 3.3]. We will show the first inequality. As noted above, we may assume that $\mathcal{O}_X(-mK_X)$ is an ideal sheaf of $\mathcal{O}_X$. By the lemma we have a log resolution $\varphi : Y \to X$ of $\mathcal{J}_X \mathcal{O}_X(-mK_X)$ such that the inequality

$$\hat{K}_{Y/X} - J_{Y/X} \le K_{m,Y/X}$$

holds. Then, note that every resolution $\psi : Y' \to X$ factoring through φ satisfies the inequality. Therefore, every prime divisor F over X appears on a resolution on which the inequality holds, which yields $a_{\mathrm{MJ}}(F; X, \mathcal{O}_X) \le \mathfrak{a}_m(F; X, \mathcal{O}_X)$. $\qquad\square$

By [de Fernex and Hacon 2009, Theorem 1.2] a pair $(X, \mathfrak{a}^t)$ is log terminal (resp. log canonical) in de Fernex and Hacon's sense if and only if there is a boundary Δ (it means that Δ is a $\mathbb{Q}$-divisor such that $[\Delta] = 0$ and $K_X + \Delta$ is a $\mathbb{Q}$-Cartier divisor) such that $((X, \Delta), \mathfrak{a}^t)$ is klt (resp. log canonical) in the usual sense. Here, we note that X is not necessarily affine. Therefore we obtain the following corollary.

Corollary 3.20. *Assume that X is normal. If a pair $(X, \mathfrak{a}^t)$ is MJ-klt (resp. MJ-log canonical), then there is a boundary Δ on X such that $((X, \Delta), \mathfrak{a}^t)$ is klt (resp. log canonical) in the usual sense.*

De Fernex and Hacon [2009] also introduced a multiplier ideal for a pair $(X, \mathfrak{a}^t)$ with a normal variety X and an ideal $\mathfrak{a}$ on X. First for $m \in \mathbb{N}$ they defined m-th "multiplier ideal" as

$$\mathcal{J}_m(X, \mathfrak{a}^t) = \varphi_*(\mathcal{O}_Y(\ulcorner K_{m,Y/X} - tZ \urcorner)),$$

where $\varphi : Y \to X$ is a log resolution of $\mathfrak{a}\mathcal{O}_X(-mK_X)$ and let $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-Z)$. They proved that the family of ideals $\{\mathcal{J}_m(X, \mathfrak{a}^t)\}_m$ has the unique maximal element and call it the multiplier ideal of $(X, \mathfrak{a}^t)$ and denote it by $\mathcal{J}(X, \mathfrak{a}^t)$. The following is the relation between their multiplier ideal and our MJ-multiplier ideal, which follows immediately from Lemma 3.18

Theorem 3.21. *Let $(X, \mathfrak{a}^t)$ be a pair with a normal variety X, an ideal $\mathfrak{a}$ on X and $t \in \mathbb{R}_{\geq 0}$. Then*

$$\mathcal{J}_{\mathrm{MJ}}(X, \mathfrak{a}^t) \subset \mathcal{J}_m(X, \mathfrak{a}^t) \quad \text{for every } m \in \mathbb{N};$$

in particular

$$\mathcal{J}_{\mathrm{MJ}}(X, \mathfrak{a}^t) \subset \mathcal{J}(X, \mathfrak{a}^t).$$

The following proposition is an application of inversion of adjunction, where the first result is contained in Corollary 3.20, but we think that it makes sense to give a direct proof without using the result of [de Fernex and Hacon 2009].

Proposition 3.22. (i) *Let X be an MJ-canonical variety. Then there exists an effective $\mathbb{Q}$-divisor Δ on X such that (X, Δ) is klt (i.e., X is normal, $K_X + \Delta$ is $\mathbb{Q}$-Cartier and $K_Y = f^*(K_X + \Delta) + \sum a_i E_i$ with $a_i > -1$ for a log resolution $f : Y \to X$.)*

(ii) *Let X be MJ-log canonical and W be a minimal MJ-log canonical center. Then there exists an effective $\mathbb{Q}$-divisor Δ on W such that (W, Δ) is klt.*

Proof. As X is MJ-canonical, it is irreducible and normal by [de Fernex and Docampo 2014] or [Ein et al. 2011]. If there exist an open covering $\{U_i\}$ of X (resp. W) and an effective $\mathbb{Q}$-divisor Δ_i on U_i such that (U_i, Δ_i) is klt for each i, then by [de Fernex and Hacon 2009, Theorem 1.2] there exists a global $\mathbb{Q}$-divisor Δ on X (resp. W) such that (X, Δ) (resp. (W, Δ)) is klt. So we may assume that X is affine for both statement (i) and (ii). Let X be embedded in a nonsingular affine variety A with codimension c and the defining ideal I_X.

(i) As X is MJ-canonical, we have $\mathrm{mld}_{\mathrm{MJ}}(Z; X, \mathcal{O}_X) \geq 1$ for every proper closed subset $Z \subset X$. By inversion of adjunction we have

$$\mathrm{mld}(Z; A, I_X^c) \geq 1.$$

On the other hand, for any point $\eta \notin X$ in A

$$\mathrm{mld}(\eta; A, I_X^c) = \mathrm{mld}(\eta; A, \mathcal{O}_A) \geq 1,$$

because A is nonsingular. Finally for the generic point η of X, we have

$$\mathrm{mld}(\eta, A, I_X^c) = 0.$$

Hence, X is the unique log canonical center of (A, I_X^c).

Now, take a log resolution $\varphi : \bar{A} \to A$ of (A, I_X). Take a general element $g \in I_X^{2c}$ and let D_0 be the zero locus of g on A and then let $D = \frac{1}{2} D_0$. Then, by the generality of g, the morphism φ is also a log resolution of (A, D_0) and for every exceptional prime divisor E_i on $\bar{A}$ we have

$$a(E_i; A, I_X^c) = a(E_i; A, D).$$

As (A, D) is klt outside of X, (A, D) has also unique log canonical center X. Then, by local subadjunction formula by Fujino and Gongyo [2010], there exists a $\mathbb{Q}$-divisor Δ on X such that (X, Δ) is klt.

(ii) By inversion of adjunction we have

$$\mathrm{mld}_{\mathrm{MJ}}(W; A, I_X^c) = 0 \quad \text{and} \quad \mathrm{mld}_{\mathrm{MJ}}(Z; A, I_X^c) \geq 0$$

for every strictly proper closed subset Z of X. By the minimality of W we have

$$\mathrm{mld}_{\mathrm{MJ}}(Z; A, I_X^c) > 0$$

for every strictly proper closed subset $Z \subset W$. We also have $\mathrm{mld}_{\mathrm{MJ}}(\eta; A, I_X^c) = 0$ for the generic point η of an irreducible component of X and $\mathrm{mld}_{\mathrm{MJ}}(\eta; A, I_X^c) \geq 1$ for any point $\eta \notin X$ in A. Therefore, (A, I_X^c) is log canonical and W is a minimal log canonical center of (A, I_X^c). Then, by the same argument as in (i), we have (W, Δ) is klt for some boundary Δ. $\square$

4. Deformations

In this section we prove that MJ-canonical singularities and MJ-log canonical singularities are preserved under small deformations. First we start with the strengthening of inversion of adjunction. Proposition 2.9 does not hold for singular A in general (see, [Ishii 2013, Example 3.13]), but if X is a complete intersection in a singular A, then it holds.

Corollary 4.1 (strong inversion of adjunction). *Let A be an affine connected reduced equidimensional scheme of finite type over k of dimension $d + c$ containing X as a complete intersection, that is, X is defined by c equations $f_1 = f_2 = \cdots = f_c = 0$ in A.*

 (i) *Assume X is reduced and let W be a strictly proper closed subset of X. Let $\tilde{\mathfrak{a}} \subset \mathcal{O}_A$ be an ideal such that its image $\mathfrak{a} := \tilde{\mathfrak{a}}\mathcal{O}_X \subset \mathcal{O}_X$ is nonzero on each irreducible component of X. Then,*

$$\mathrm{mld}_{\mathrm{MJ}}(W; X, \mathfrak{a}^t) = \mathrm{mld}_{\mathrm{MJ}}(W; A, \tilde{\mathfrak{a}}^t (f_1, \ldots, f_c)^c).$$

 (ii) *If A satisfies S_2, $c = 1$ and $(A, (f_1))$ is MJ-log canonical, then automatically X is reduced and the formula in (i) holds.*

Proof. We may assume that A is embedded into the affine space $\mathbb{A}^N$. By using the same idea as in Remark 3.8, we can construct an embedding $A \subset \mathbb{A}^{N+c}$ such that there exists a linear subspace L of codimension c in $\mathbb{A}^{N+c}$ satisfying $L \cap A = X$. Denote $B = \mathbb{A}^{N+c}$. Let $\bar{\mathfrak{a}} \subset \mathcal{O}_B$ be an ideal such that $\tilde{\mathfrak{a}} = \bar{\mathfrak{a}}\mathcal{O}_A$ and let $\mathfrak{a}' = \bar{\mathfrak{a}}\mathcal{O}_L$. Then we have $\mathfrak{a} = \mathfrak{a}'\mathcal{O}_X$. Let $I_{X/L}$, $I_{A/B}$, $I_{L/B}$ be the defining

ideals of X in L, A in B, L in B, respectively. Then $L \cap A = X$ implies that $I_{X/A} = I_{L/B}\mathcal{O}_A$ and $I_{X/L} = I_{A/B}\mathcal{O}_L$.

Noting that B and L are nonsingular, apply Proposition 2.9 for $X \subset L$, $L \subset B$ and $A \subset B$. Then we obtain

$$\mathrm{mld}_{\mathrm{MJ}}(W; X, \mathfrak{a}^t) = \mathrm{mld}(W; L, \mathfrak{a}'^t (I_{X/L})^{N-d}),$$

$$\mathrm{mld}(W; L, \mathfrak{a}'^t (I_{X/L})^{N-d}) = \mathrm{mld}(W; B, \bar{\mathfrak{a}}^t (I_{A/B})^{N-d}(I_{L/B})^c),$$

$$\mathrm{mld}_{\mathrm{MJ}}(W, A, \tilde{\mathfrak{a}}^t (I_{X/A})^c) = \mathrm{mld}(W; B, \bar{\mathfrak{a}}^t (I_{L/B})^c (I_{A/B})^{N-d}).$$

The required equality in (i) follows from just composing these equalities.

For the proof of (ii), first we see that A is smooth at the generic point of every irreducible component of X. This is proved as follows: Assume A is not smooth at the generic point η of an irreducible component of X. Then as $\mathrm{mld}_{\mathrm{MJ}}(\eta; A, (f_1)) \geq 0$, we have $\mathrm{mld}_{\mathrm{MJ}}(\eta; A, \mathcal{O}_A) \geq 1$, which implies that A is MJ-canonical around general points of $\overline{\{\eta\}}$. But MJ-canonical singularities are normal, a contradiction. By restricting A to a neighborhood of X we may assume that $Z = \mathrm{Sing}\,A$ is of codimension ≥ 2. Let $A_0 = A \setminus Z$; then $(A_0, (f_1))$ is MJ-log canonical, but it is equivalent to that $(A_0, (f_1))$ is log canonical in the usual sense, because A_0 is nonsingular. Therefore f_1 is reduced on A_0. For every open subset $U \subset A$, define $U_0 := U \cap A_0$ Consider the exact sequence

$$0 \to \Gamma(U, (f_1)) \to \Gamma(U_0, (f_1)) \to H^1_{Z \cap U}(U, (f_1)).$$

Since the ideal sheaf (f_1) is principal, the last term $H^1_{Z \cap U}(U, (f_1))$ is isomorphic to $H^1_{Z \cap U}(U, \mathcal{O}_U)$ and this is 0, because A is S_2. Therefore by the exact sequence, we obtain $(f_1) = i_*(f_1|_{A_0}) = i_*(\sqrt{(f_1|_{A_0})}) \supset \sqrt{(f_1)}$, where $i : A_0 \hookrightarrow A$ is the inclusion. This shows that the ideal (f_1) is reduced on A. Once we know that X is reduced we can apply (i) to obtain the formula of $\mathrm{mld}_{\mathrm{MJ}}$. $\square$

Definition 4.2. Let T be a reduced scheme of finite type over k and $0 \in T$ a closed point. Let $\pi : X \to T$ be a surjective morphism with equidimensional reduced fibers $X_\tau = \pi^{-1}(\tau)$ of common dimension r for all closed points $\tau \in T$. Then $\pi : X \to T$ is called a deformation of X_0 with the parameter space T.

If moreover a pair $(X, \mathfrak{a}^t)$ is given, $\mathfrak{a}_\tau^t = \mathfrak{a}^t \mathcal{O}_{X_\tau}$ are not zero on each irreducible component of X_τ for all $\tau \in T$ and $\pi : X \to T$ is a deformation of X_0, then the family $\{(X_\tau, \mathfrak{a}_\tau^t)\}_{\tau \in T}$ is called a deformation of $(X_0, \mathfrak{a}_0^t)$.

From now on, for a morphism $\pi : Z \to T$ from some scheme Z to the parameter space T, we denote the fiber $\pi^{-1}(\tau)$ by Z_τ.

Lemma 4.3. *Let $\pi : X \to T$ be a deformation of $(X_0, \mathfrak{a}_0^t)$ $(0 \in T)$ given by a nonzero ideal $\mathfrak{a} \subset \mathcal{O}_X$. Then, there exists an open dense subset $T_0 \subset T$ and a log resolution $\varphi : Y \to X$ of $(X, \mathfrak{a}\mathcal{J}_X)$ such that for every $\tau \in T_0$ the following hold*:

(i) $\varphi_\tau : Y_\tau \to X_\tau$ is a log resolution of $(X_\tau, \mathfrak{a}_\tau \mathcal{J}_{X_\tau})$.

(ii) $(\hat{K}_{Y/X} - J_{Y/X} - tZ)|_{Y_\tau} = \hat{K}_{Y_\tau/X_\tau} - J_{Y_\tau/X_\tau} - tZ_\tau$,

where φ_τ is the restriction of φ onto the fiber $Y_\tau = (\pi \circ \varphi)^{-1}(\tau)$ and

$$\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-Z) \quad and \quad \mathfrak{a}_\tau \mathcal{O}_{Y_\tau} = \mathcal{O}_{Y_\tau}(-Z_\tau).$$

In particular, $(X|_{T_0}, \mathfrak{a}^t)$ *is MJ-log canonical* (*resp. MJ-canonical*) *if and only if* $(X_\tau, \mathfrak{a}_\tau^t)$ *is MJ-log canonical* (*resp. MJ-canonical*) *for every* $\tau \in T_0$.

Proof. As it is sufficient to prove the existence of such an open subset of T on each irreducible component, we may assume that T is irreducible. Let r be the common dimension of the fiber X_τ for closed points $\tau \in T$. Let $\mathcal{J}_{X/T}$ be the r-th Fitting ideal of $\Omega_{X/T}$. By Proposition 3.5, we can take a factorizing resolution $\Phi : \bar{A} \to A$ of X in A with the strict transform Y of X in $\bar{A}$ such that the restriction $\varphi : Y \to X$ of Φ is a log resolution of $(X, \mathfrak{a}\mathcal{J}_X \mathcal{J}_{X/T})$. Let E_i $(i = 1, \ldots, s)$ be an exceptional prime divisor of Φ. Then, by the generic smoothness theorem, there is an open dense subset T_0 of T such that $E_{i_1} \cap \cdots \cap E_{i_j}$, $E_{i_1} \cap \cdots \cap E_{i_j} \cap Y$, Y, $\bar{A}$, A are smooth over T_0 for all collections $\{i_1, \ldots, i_j\}$ if they are not empty. On the other hand, since Φ is a factorizing resolution of X in A, we have an effective divisor R on $\bar{A}$ such that $I_X \mathcal{O}_{\bar{A}} = I_Y \mathcal{O}_{\bar{A}}(-R)$. Replacing T_0 by a smaller open subset if necessary, we may assume that the support of R does not contain $\bar{A}_\tau$ $(\tau \in T_0)$. By restricting this equality on the fiber of τ, we have

$$I_X \mathcal{O}_{\bar{A}_\tau} = I_{Y_\tau} \mathcal{O}_{\bar{A}_\tau}(-R|_{A_\tau}).$$

Because of this, $\Phi_\tau : \bar{A}_\tau \to A_\tau$ is a factorizing resolution of X_τ in A_τ for every $\tau \in T_0$.

Then, by $\Omega_{X/T} \otimes \mathcal{O}_{X_\tau} = \Omega_{X_\tau}$ and the functoriality of Fitting ideals, we have $\mathcal{J}_{X/T} \mathcal{O}_{X_\tau} = \mathcal{J}_{X_\tau}$ for every $\tau \in T_0$. This shows that φ_τ is a log resolution of $(X_\tau, \mathfrak{a}_\tau \mathcal{J}_{X_\tau})$.

By the Lemma 3.7 we have

$$\hat{K}_{Y/X} - J_{Y/X} = (K_{\bar{A}/A} - cR)|_Y,$$

where $c = \mathrm{codim}(X, A)$. Noting that c is also the codimension of X_τ in A_τ for a closed point $\tau \in T_0$, we have

$$\hat{K}_{Y_\tau/X_\tau} - J_{Y_\tau/X_\tau} = (K_{\bar{A}_\tau/A_\tau} - cR|_{\bar{A}_\tau})|_{Y_\tau}.$$

Since $(K_{\bar{A}/A})|_{\bar{A}_\tau} = K_{\bar{A}_\tau/A_\tau}$, we obtain for $\tau \in T_0$

$$(\hat{K}_{Y/X} - J_{Y/X})|_{Y_\tau} = \hat{K}_{Y_\tau/X_\tau} - J_{Y_\tau/X_\tau}.$$

For the statement (ii) we have only to note that $Z|_{Y_\tau} = Z_\tau$ for $\tau \in T_0$. $\square$

Theorem 4.4. *Let $\{(X_\tau, \mathfrak{a}_\tau^t)\}_{\tau \in T}$ be a deformation of $(X_0, \mathfrak{a}_0^t)$. Assume $(X_0, \mathfrak{a}_0^t)$ is MJ-log canonical at $x \in X_0$. Then there are neighborhoods $X^* \subset X$ of x and $T^* \subset T$ of 0 such that $(X_\tau^*, \mathfrak{a}_\tau^t|_{X_\tau^*})$ is MJ-log canonical for every closed point $\tau \in T^*$.*

Proof. The statement is reduced to the case that T is a nonsingular curve. Then X_0 is defined by one equation, say $f = 0$, and $\dim X_0$ is one less than $\dim X = d$. By applying Corollary 4.1, we have

$$\mathrm{mld}_{\mathrm{MJ}}(x; X_0, \mathfrak{a}_0^t) = \mathrm{mld}_{\mathrm{MJ}}(x; X, \mathfrak{a}^t(f)).$$

By the assumption we have $\mathrm{mld}_{\mathrm{MJ}}(x; X_0, \mathfrak{a}_0^t) \geq 0$ which implies $\mathrm{mld}_{\mathrm{MJ}}(x; X, \mathfrak{a}^t(f)) \geq 0$ and therefore $\mathrm{mld}_{\mathrm{MJ}}(x; X, \mathfrak{a}^t) \geq 0$. Then, by Proposition 2.22 there is an open neighborhood $X^* \subset X$ of x such that $(X^*, \mathfrak{a}^t|_{X^*})$ is MJ-log canonical. Then, by the last statement of Lemma 4.3, there exists an open subset $T*$ such that $(X_\tau^*, \mathfrak{a}_\tau^t|_{X_\tau^*})$ is MJ-log canonical for every $\tau \in T^*$. $\square$

Remark 4.5. Replacing X by a small neighborhood of x, we can assume that $X \subset T \times \mathbb{A}^N$, since the morphism $X \to T$ is of finite type. If T is nonsingular, then $A = T \times \mathbb{A}^N \to T$ is a smooth morphism of nonsingular varieties. For $(X, \mathfrak{a}^t)$, take $\tilde{\mathfrak{a}} \subset A$ as the pull back of $\mathfrak{a}$ by the canonical surjective map $\mathcal{O}_A \to \mathcal{O}_X$. Then, we can prove that $(X_\tau, \mathfrak{a}_\tau^t)$ is MJ-log canonical if and only if $(A_\tau, \tilde{\mathfrak{a}}(I_{X_\tau})^c)$ is log canonical. By using this fact, Theorem 4.4 can also be proved by discussions only on A and A_τ.

For the similar statement as Theorem 4.4 for MJ-canonical singularities we need some notions and a lemma.

Definition 4.6. Let A be a nonsingular variety and $\eta \in A$ a (not necessarily closed) point. For a cylinder $C \subset \mathcal{L}^\infty(A)$ we define the codimension of $C \cap \psi_{\infty 0}^{-1}(\eta)$ as

$$\mathrm{codim}\, C \cap \psi_{\infty 0}^{-1}(\eta) := \mathrm{codim}(\overline{\psi_m(C \cap \psi_0^{-1}(\eta))}, \mathcal{L}^m(A)),$$

for $m \gg 0$, where $\psi_m : \mathcal{L}^\infty(A) \to \mathcal{L}^m(A)$ is the canonical projection.

Here, note that the value of the right-hand side is constant for $m \gg 0$, where $C = \psi_n^{-1}(S)$ for $S \subset \mathcal{L}^n(A)$.

Lemma 4.7. *Let A be a nonsingular variety, $\eta \in A$ a (not necessarily closed) point and $\mathfrak{a} \subset \mathcal{O}_A$ $(i = 1, \ldots, r)$ a nonzero ideal. Then*

$$\mathrm{mld}(\eta; A, \mathfrak{a}^t) = \inf\{\mathrm{codim}(\mathrm{Cont}^m(\mathfrak{a}) \cap \psi_0^{-1}(\eta)) - mt\}.$$

Proof. First we prove the inequality $\geq$. Let E be a prime divisor over A with the center $\overline{\{\eta\}}$ and let $v = \mathrm{val}_E$. Let $m = v(\mathfrak{a})$; then there exists a open dense subset $C \subset C_A(v)$ such that $C \subset \mathrm{Cont}^m(\mathfrak{a}) \cap \psi_0^{-1}(\eta)$, where $C_A(v)$ is the

maximal divisorial set (for definition see, for example, [Ishii 2013]) in $\mathcal{L}^\infty(X)$ corresponding to v. This is because the generic point $\alpha \in C_A(v)$ has $\mathrm{ord}_\alpha(\mathfrak{a}) = m$ by [de Fernex et al. 2008] and the center of α is η. Then

$$\mathrm{ord}_E(K_{A'/A}) - t v(\mathfrak{a}) + 1 = \mathrm{codim}(C_A(v)) - mt$$
$$\geq \mathrm{codim}(\mathrm{Cont}^m(\mathfrak{a}) \cap \psi_0^{-1}(\eta)) - mt,$$

where $Y \to X$ is a log resolution of $\mathfrak{a}$ such that E appears on Y. Here, note that we use the equality $\mathrm{ord}_E(K_{A'/A}) + 1 = \mathrm{codim}(C_A(v))$ proved in [de Fernex et al. 2008]. This completes the proof of $\geq$.

Next we prove the opposite inequality $\leq$. We may assume that

$$\mathrm{ord}_E(K_{A'/A}) - t \, \mathrm{val}_E(\mathfrak{a}) + 1 \geq 0$$

for every prime divisor E over X with the center $\overline{\{\eta\}}$, because otherwise the claimed inequality is trivial. For an arbitrary $m \in \mathbb{N}$ take

$$\zeta \in \mathrm{Cont}^m(\mathfrak{a}) \cap \psi_0^{-1}(\eta)$$

such that $\overline{\{\zeta\}}$ is an irreducible component of $\overline{\mathrm{Cont}^m(\mathfrak{a}) \cap \psi_0^{-1}(\eta)}$ and

$$\overline{\psi_s(\zeta)} \subset \overline{\psi_s(\mathrm{Cont}^m(\mathfrak{a}) \cap \psi_0^{-1}(\eta))}, \quad s \geq m$$

gives the codimension of $\mathrm{Cont}^m(\mathfrak{a}) \cap \psi_0^{-1}(\eta)$. Then

$$\overline{\{\zeta\}} = \psi_s^{-1}(\overline{\psi_s(\zeta)}),$$

which is an irreducible cylinder. Then, a divisorial valuation $v = q \, \mathrm{val}_E$ over A corresponds to this cylinder [de Fernex et al. 2008, Propositions 2.12, 3.10]. Here, we note that E is a prime divisor with the center $\overline{\{\eta\}}$ and $m = q \, \mathrm{val}_E(\mathfrak{a})$. By the maximality of $C_A(v)$, we have $\overline{\{\zeta\}} \subset C_A(v)$. Hence

$$\mathrm{codim}(\mathrm{Cont}^m(\mathfrak{a}) \cap \psi_0^{-1}(\eta)) - tm \geq \mathrm{codim}\, C_A(v) - tm$$
$$= q(\mathrm{ord}_E(K_{A'/A}) + 1) - q \, \mathrm{val}_E(\mathfrak{a})$$
$$\geq \mathrm{ord}_E(K_{A'/A}) - t \, \mathrm{val}_E(\mathfrak{a}) + 1,$$

which gives the inequality $\leq$ in the lemma as required. $\qquad\square$

Remark 4.8. Let A and η be as above. Let $\mathfrak{a}_i \subset \mathcal{O}_A$ $(i = 1, \ldots, r)$ be nonzero ideals and t_i $(i = 1, \ldots, r)$ nonnegative real numbers. Then

$\mathrm{mld}(\eta; A, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r})$

$$= \inf\left\{\mathrm{codim}\big(\mathrm{Cont}^{m_1}(\mathfrak{a}_1) \cap \cdots \cap \mathrm{Cont}^{m_r}(\mathfrak{a}_r) \cap \psi_0^{-1}(\eta)\big) - \sum_i m_i t_i\right\}$$
$$= \inf\left\{\mathrm{codim}\big(\mathrm{Cont}^{\geq m_1}(\mathfrak{a}_1) \cap \cdots \cap \mathrm{Cont}^{\geq m_r}(\mathfrak{a}_r) \cap \psi_0^{-1}(\eta)\big) - \sum_i m_i t_i\right\}.$$

Here, the first equality is proved in the similar way as in Lemma 4.7 and the second equality follows from the same argument as the proof of [Ishii 2013, Proposition 3.7].

Theorem 4.9. *Let* $\{(X_\tau, \mathfrak{a}_\tau^t)\}_{\tau \in T}$ *be a deformation of* $(X_0, \mathfrak{a}_0^t)$. *Assume* $(X_0, \mathfrak{a}_0^t)$ *is MJ-canonical at* $x \in X_0$. *Then there are neighborhoods* $X^* \subset X$ *of* x *and* $T^* \subset T$ *of* 0 *such that* $(X_\tau^*, \mathfrak{a}_\tau^t|_{X_\tau^*})$ *is MJ-canonical for every* $\tau \in T^*$.

Proof. As in Theorem 4.4, we reduce to the case that T is a nonsingular curve. If the statement does not hold, then there is a horizontal irreducible closed subset W (i.e., W dominates T) such that $x \in W$ and $\mathrm{mld}_{\mathrm{MJ}}(W; X, \mathfrak{a}^t) < 1$. Replacing X by a small neighborhood of x we can assume that $X \subset T \times \mathbb{A}^N = A$. Then, by inversion of adjunction, we have $\mathrm{mld}(W; A, \tilde{\mathfrak{a}}^t I_X) < 1$, where $\tilde{\mathfrak{a}} \subset \mathcal{O}_A$ is an ideal such that $\mathfrak{a} = \tilde{\mathfrak{a}} \mathcal{O}_X$. Then,

$$\mathrm{mld}(\eta; A, \tilde{\mathfrak{a}}^t I_X) < 1.$$

Therefore, there exists a prime divisor E over A with the center W and satisfying $a(E; A, \tilde{\mathfrak{a}}^t I_X) < 1$. Then, by Lemma 4.3, there is an open dense subset $T_0 \subset T$ such that

$$\mathrm{mld}(\eta_\tau^{(i)}; A_\tau, \tilde{\mathfrak{a}}_\tau^t I_{X_\tau}) < 1 \quad \text{for } \tau \in T_0 \tag{17}$$

where $\eta_\tau^{(i)}$ is the generic point of an irreducible component $W_\tau^{(i)}$ of W_τ.

$\mathrm{mld}(W_\tau; A_\tau, \tilde{\mathfrak{a}}_\tau^t I_{X_\tau})$

$\quad = \mathrm{mld}_{\mathrm{MJ}}(W_\tau; X_\tau, \tilde{\mathfrak{a}}_\tau^t)$

$\quad = \inf_{m,n} \{(M+1)N - (m+1)t - (n+1)c$

$\qquad\qquad - \dim(\psi_{Mm}^{-1}(\mathcal{L}^m(Z_\tau)) \cap \psi_{Mn}^{-1}(\mathcal{L}^n(X_\tau)) \cap \psi_{M0}^{-1}(W_\tau))\}, \quad (18)$

where $M = \max\{m, n\}$ and $\psi_{Mn} : \mathcal{L}^M(A) \to \mathcal{L}^n(A)$ and so on. Now, fix m, n. For simplicity let us assume $M = n$. (for the other case $M = m$, the proof is similar). Let $\mathcal{L}^n(X/T)$ be the relative n-jet scheme with respect to $\pi : X \to T$. It is defined as

$$\mathcal{L}^n(X/T) := \pi_n^{-1}(\Sigma^n(T)) \subset \mathcal{L}^n(X),$$

where $\pi_n : \mathcal{L}^n(X) \to \mathcal{L}^n(T)$ is the morphism of n-jet schemes induced from $\pi : X \to T$ and $\Sigma^n(T) \subset \mathcal{L}^n(T)$ is the locus of trivial n-jets on T. Note that $(\mathcal{L}^n(X/T))_\tau = \mathcal{L}^n(X_\tau)$. Denote by ρ_{nm}^X the canonical projection $\mathcal{L}^n(X/T) \to \mathcal{L}^m(X/T)$; then $\rho_{nm}^X|_{(\mathcal{L}^n(X/T))_\tau}$ is the canonical projection $\mathcal{L}^n(X_\tau) \to \mathcal{L}^m(X_\tau)$.

The description in (18) is then

$$\mathrm{mld}(W_\tau; A_\tau, \tilde{\mathfrak{a}}_\tau^t I_{X_\tau}) = \inf_{m,n}\{(M+1)N - (m+1)t - (n+1)c - R_{n,m,\tau}\},$$

where we have set

$$R_{n,m,\tau} = \dim\big((\rho_{nm}^X)^{-1}(\mathcal{L}^m(Z_\tau)) \cap \mathcal{L}^n(X_\tau) \cap (\rho_{n0}^X)^{-1}(W_\tau)\big).$$

Let

$$\mathcal{R} := (\rho_{nm}^X)^{-1}(\mathcal{L}^m(Z/T)) \cap \mathcal{L}^n(X/T) \cap (\rho_{n0}^X)^{-1}(W)$$

and consider the restricted morphism $\rho : \mathcal{R} \to W$ of $\rho_{n0}^X : \mathcal{L}^n(X/T) \to X$.

Here, note that $R_{n,m,\tau} = \dim \rho^{-1}(W_\tau)$ for every $\tau \in T$. Assume $\dim W = s$; then $\dim W_\tau = s - 1$ since T is a nonsingular curve and therefore W_τ is a hypersurface in W. Therefore

$$R_{n,m,0} = \dim \rho^{-1}(W_0) \geq \dim \rho^{-1}(y) + s - 1$$

for general closed point $y \in W$. Take $\tau \in T$ such that $y \in W_\tau^{(i)} \subset W_\tau$; then

$$\dim \rho^{-1}(y) + s - 1 = \dim \overline{\rho^{-1}(\eta_\tau^{(i)})}.$$

Note that

$$\mathrm{mld}(\eta_\tau^{(i)}; A_\tau, \tilde{\mathfrak{a}} I_{X_\tau}) = \inf_{n,m}\{(M+1)N - (m+1)t - (n+1)c - \dim \overline{\rho^{-1}(\eta_\tau^{(i)})}\}$$

by Lemma 4.7. From (17) we obtain

$$1 \leq \mathrm{mld}(W_0; A_0, \tilde{\mathfrak{a}}_0 I_{X_0}) \leq \mathrm{mld}(\eta_\tau^{(i)}; A_\tau, \tilde{\mathfrak{a}} I_{X_\tau}) < 1,$$

which is a contradiction. □

As a corollary, we obtain a sufficient condition for a hypersurface singularity not to be MJ-log canonical or MJ-canonical. Terminologies "nondegenerate", "Newton polygon" in the corollary can be referred in [Ishii 1996].

Corollary 4.10. *Let* $(X, 0) \subset (\mathbb{A}^{d+1}, 0)$ *be a reduced hypersurface singularity defined by an equation* $f = 0$. *Let* $\Gamma(f)$ *be the Newton polygon of* f *in* $\mathbb{R}^{d+1}$.

(i) *If* $(1, \ldots, 1) \notin \Gamma(f)$, *then* $(X, 0)$ *is not MJ-log canonical.*

(ii) *If* $(1, \ldots, 1) \notin \Gamma(f)^0$, *then* $(X, 0)$ *is not MJ-canonical. Here,* $\Gamma(f)^0$ *means the interior of* $\Gamma(f)$.

Proof. It is known that the statements hold for nondegenerate f (see [Ishii 1996, Corollary 1.7]), since in this case MJ-canonical (resp. MJ-log canonical) is equivalent to canonical (resp. log canonical) in the usual sense. Let f be possibly degenerate and assume $\mathbf{1} := (1, \ldots, 1) \notin \Gamma(f)$. Perturb the coefficients of f to obtain f_ϵ with $\Gamma(f_\epsilon) = \Gamma(f)$. Let $\epsilon \in T := \mathbb{A}^r$ and $f = f_0$. Then

f_ϵ ($\epsilon \in T$) gives a deformation of hypersurfaces X_ϵ. Then for general ϵ, f_ϵ is nondegenerate; therefore $\mathbf{1} \notin \Gamma(f_\epsilon)$ implies that X_ϵ is not log canonical. Hence, $X_0 = X$ is not MJ-log canonical by Theorem 4.4. For the statement of MJ-canonical follows by using Theorem 4.9 in the similar way as above. $\square$

Proposition 4.11 (lower semicontinuity of MJ-minimal log discrepancy). *Let* $\{(X_\tau, \mathfrak{a}_\tau^t)\}_{\tau \in T}$ *be a deformation of* $(X_0, \mathfrak{a}_0^t)$ *and let* $\pi : X \to T$ *be the morphism giving the deformation. Let* $\sigma : T \to X$ *be a section of* π. *Then, the map* $T \to \mathbb{R}, \tau \mapsto \mathrm{mld}_{\mathrm{MJ}}(\sigma(\tau), X_\tau, \mathfrak{a}_\tau^t)$ *is lower semicontinuous.*

Proof. For the statement of the proposition, we may assume that T is irreducible. We use the same notation as in the proof of Theorem 4.9. First note that there is a nonempty open subset $T^* \subset T$ such that $\mathrm{mld}_{\mathrm{MJ}}(\sigma(\tau), X_\tau, \mathfrak{a}_\tau^t)$ is constant for all $\tau \in T^*$. This is proved as follows: Take a log resolution $\varphi : Y \to X$ of $(X, \mathfrak{a}\mathcal{J}_X \mathcal{J}_{X/T} I_\Sigma)$, where I_Σ is the defining ideal of the section $\Sigma := \mathrm{Im}\,\sigma$. Then, by Lemma 4.3, there exists a nonempty open subset $T^* \subset T$ such that for every $\tau \in T^*$ the restriction $\varphi_\tau : Y_\tau \to X_\tau$ is a log resolution of $(X_\tau, \mathfrak{a}_\tau \mathcal{J}_{X_\tau} \mathfrak{m}_{X_\tau, \sigma(\tau)})$ and

$$(\hat{K}_{Y/X} - J_{Y/X} - tZ)|_{Y_\tau} = \hat{K}_{Y_\tau/X_\tau} - J_{Y_\tau/X_\tau} - tZ_\tau,$$

where $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-Z)$ and $\mathfrak{a}_\tau \mathcal{O}_{Y_\tau} = \mathcal{O}_{Y_\tau}(-Z_\tau)$. Now take an exceptional prime divisor E over $X|_{T^*}$ with the center Σ. Then E_τ is the disjoint sum of nonsingular exceptional divisors $E_\tau^{(i)}$ with the center $\sigma(\tau)$ and

$$\mathrm{ord}_E(\hat{K}_{Y/X} - J_{Y/X} - tZ) = \mathrm{ord}_{E_\tau^{(i)}}(\hat{K}_{Y_\tau/X_\tau} - J_{Y_\tau/X_\tau} - Z_\tau).$$

Hence, the constancy of the MJ-minimal log discrepancy follows as required.

For the lower semicontinuity of MJ-minimal log discrepancy follows just by showing

$$\mathrm{mld}_{\mathrm{MJ}}(\sigma(0), X_0, \mathfrak{a}_0^t) \leq \mathrm{mld}_{\mathrm{MJ}}(\sigma(\tau), X_\tau, \mathfrak{a}_\tau^t), \tag{19}$$

for some $\tau \in T^*$.

In the same way as to get (18) in the proof of Theorem 4.9, we obtain

$$\mathrm{mld}_{\mathrm{MJ}}(\sigma(\tau); X_\tau, \tilde{\mathfrak{a}}_\tau^t) = \inf_{m,n}\{(M + 1)N - (m + 1)t - (n + 1)c$$
$$- \dim\big(\psi_{Mm\tau}^{-1}(\mathcal{L}^m(Z_\tau)) \cap \psi_{Mn\tau}^{-1}(\mathcal{L}^n(X_\tau)) \cap \psi_{M0\tau}^{-1}(\sigma(\tau))\big)\},$$

where $M = \max\{m, n\}$ and $\psi_{mn\tau} : \mathcal{L}^m(A_\tau) \to \mathcal{L}^n(A_\tau)$ is the canonical projection. For simplicity, let us assume $M = n$. (For the other case $M = m$, the proof is the same.) Then the scheme $\psi_{Mm}^{-1}(\mathcal{L}^m(Z_\tau)) \cap \psi_{Mn}^{-1}(\mathcal{L}^n(X_\tau)) \cap \psi_{M0}^{-1}(\sigma(\tau))$ is the fiber of the point $\sigma(\tau)$ by the canonical projection

$$\rho_{nm} : \mathcal{W}_{nm} := \psi_{nm}^{-1}(\mathcal{L}^m(Z)) \cap \mathcal{L}^n(X/T) \to \Sigma \simeq T,$$

where $\psi_{nm} : \mathcal{L}^n(A) \to \mathcal{L}^m(A)$ is the canonical projection.

Here, note that the space $\mathcal{W}_{nm}$ is $\mathbb{G}_m$-invariant and also the subspace

$$S_r := \{ Q \in \mathcal{W}_{nm} \mid \dim \rho_{nm}^{-1} \rho_{nm}(Q) \geq r \}$$

is $\mathbb{G}_m$-invariant for every $r \in \mathbb{N}$. For every $r \in \mathbb{N}$, the subset S_r is known to be a closed subset (see, for example, [Mumford 1999, Chapter 1, Section 8]). Therefore by [Ishii 2007, Proposition 3.2],

$$\{ \tau \in T \mid \dim \rho_{nm}^{-1}(\tau) \geq r \} = \rho_{nm}(S_r)$$

is a closed subset of T. Therefore, for fixed $m, n \in \mathbb{N}$

$$\tau \mapsto d_{nm}(\tau) := (M+1)N - (m+1)t - (n+1)c - \dim \rho_{nm}^{-1}(\tau)$$

is lower semicontinuous. Therefore, there is a nonempty open subset $U_{nm} \subset T^*$ such that $d_{nm}(0) \leq d_{nm}(\tau)$ for all $\tau \in U_{nm}$. As k is uncountable, $\bigcap_{nm} U_{nm} \neq \varnothing$ which completes the proof of (19). $\square$

5. Low-dimensional MJ-singularities

In this section we determine MJ-canonical and MJ-log canonical singularities of dimension 1 and 2.

Proposition 5.1. *Let (X, x) be a singularity on a one-dimensional reduced scheme.*

 (i) *(X, x) is MJ-canonical if and only if it is nonsingular.*

 (ii) *(X, x) is MJ-log canonical if and only if it is nonsingular or ordinary node.*

Proof. It is clear that a nonsingular point is MJ-canonical. On the contrary if (X, x) is MJ-canonical, then it must be normal by Proposition 3.1. We can see the nonsingularity of (X, x) also by emb $\leq 2 \dim X - 1 = 1$ (Proposition 3.3).

   For (ii), assume (X, x) is singular. Then it is MJ-log canonical if and only if $\mathrm{mld}_{\mathrm{MJ}}(x; X, \mathcal{O}_X) = 0$ by [Ishii 2013, Corollary 3.15] and it is equivalent to that (X, x) is ordinary node by [Ishii and Reguera 2013]. $\square$

Example 5.2. It is known that the union of the three axes in the three-dimensional affine space is a Du Bois curve. But it is not an MJ-log canonical curve by Proposition 5.1(ii).

Theorem 5.3. *Let (X, x) be a singularity on two-dimensional reduced scheme. Then (X, x) is MJ-canonical if and only if it is nonsingular or rational double.*

Proof. First note that for a complete intersection singularity, canonicity and MJ-canonicity are equivalent. As a two-dimensional rational double point (X, x) is a hypersurface singularity and canonical; therefore it is MJ-canonical. Conversely, if (X, x) is MJ-canonical, then $\mathrm{mld}_{\mathrm{MJ}}(x; X, \mathcal{O}_X) \geq 1$. Such singularities are

classified in [Ishii and Reguera 2013] to be nonsingular or rational double or normal crossing double or a pinch point. As an MJ-canonical singularity is normal by Proposition 3.1, only rational double points among them can be MJ-canonical. $\qquad\square$

Next we will characterize MJ-log canonical singularities of dimension 2. By Proposition 3.3, for an MJ-log canonical singularity (X, x) of dimension 2, we have

$$\mathrm{emb}(X, x) \leq 4.$$

First we will determine the case $\mathrm{emb}(X, x) = 3$. Many of the singularities listed in the following theorem can be observed to be MJ-log canonical singularities by the calculation in [Kuwata 1999]. But we give a self contained proof below.

Theorem 5.4. *Let $(X, 0)$ be a singularity on a two-dimensional reduced scheme with $\mathrm{emb}(X, 0) = 3$. Then, $(X, 0)$ is an MJ-log canonical singularity if and only if X is defined by $f(x, y, z) \in k[\![x, y, z]\!]$ as follows:*

(i) $\mathrm{mult}_0 f = 3$ *and the projective tangent cone of X at 0 is a reduced curve with at worst ordinary nodes.*

(ii) $\mathrm{mult}_0 f = 2$:

    (a) $f = x^2 + y^2 + g(z)$, $\deg g \geq 2$.

    (b) $f = x^2 + g_3(y, z) + g_4(y, z)$, $\deg g_i \geq i$, g_3 *is homogeneous of degree 3 and $g_3 \neq l^3$ (l linear).*

    (c) $f = x^2 + y^3 + yg(z) + h(z)$, $3 \leq \mathrm{mult}_0 g \leq 4$ *or* $\mathrm{mult}_0 h \leq 6$.

    (d) $f = x^2 + g(y, z) + h(y, z)$, g *is homogeneous of degree 4 and it does not have a linear factor with multiplicity more than 2 and* $\mathrm{mult}_0 h \geq 5$.

Proof. Let $(X, 0)$ be an MJ-log canonical singularity defined by $f \in k[\![x, y, z]\!]$. By (2) in Proposition 2.14, we have

$$\mathrm{mld}_{\mathrm{MJ}}(0; X, \mathcal{O}_X) = \inf_n \{(n + 1)2 - \dim(\psi_{n0}^X)^{-1}(0)\} \geq 0;$$

therefore in particular for $n = 3$, we have

$$\dim(\psi_{3,0}^X)^{-1}(0) \leq 8.$$

Since $(\psi_{3,0}^X)^{-1}(0) = \mathrm{Spec}\, k[x^{(i)}, y^{(j)}, z^{(k)} \mid i, j, k = 1, 2, 3]/(F^{(1)}, F^{(2)}, F^{(3)})$, at least one $F^{(j)}$ ($j = 1, 2, 3$) must be nonzero in $k[x^{(i)}, y^{(j)}, z^{(k)}]$. By Remark 2.12, this implies that $\mathrm{mult}_0 f \leq 3$.

Case I: $\mathrm{mult}_0 f = 3$. Let $(X, 0) \subset (A, 0)$ be the embedding into the 3-dimensional nonsingular variety, and let $\Phi : A' \to A$ be the blow-up at 0. Let E be the exceptional divisor on A', X' the strict transform of X in A', $\Psi : \bar{A} \to A'$ a factorizing resolution of X' in A' and $\bar{X}$ the strict transform of X' in $\bar{A}$. We

can take Ψ such that the restriction $\psi = \Psi|_Y : Y \to X'$ is a log resolution of $\mathcal{J}_{X'}\mathcal{J}_X\mathcal{O}_{X'}$. As X is a hypersurface of multiplicity 3 at 0, we have

$$I_X\mathcal{O}_{A'} = I_Y\mathcal{O}_{A'}(-3E).$$

Then, by Corollary 3.9, it follows

$$\widehat{K}_{\overline{X}/X'} - J_{\overline{X}/X'} - \psi^*(E|_{X'}) = \widehat{K}_{\overline{X}/X} - J_{\overline{X}/X}.$$

Therefore, $(X, 0)$ is MJ-log canonical if and only if $(X', E|_{X'})$ is MJ-log canonical around $E|_{X'}$. Since X' is a hypersurface, it is S_2. Then, by Corollary 4.1(ii), MJ-log canonicity of $(X', E|_{X'})$ is equivalent to that $E|_{X'}$ is reduced and MJ-log canonical. As $\dim(E|_{X'}) = 1$ we can apply Proposition 5.1(ii), and obtain that $E|_{X'}$ has ordinary nodes. Note that $E|_{X'}$ is a hypersurface in $\mathbb{P}^2$ defined by $\mathrm{in}(f)$.

Case II: $\mathrm{mult}_0 f = 2$. Let $\Phi : A' \to A$ be the blow-up at 0, X' the strict transform of X in A' and E the exceptional divisor with respect to Φ. Then as the same discussion using Corollary 3.9 as in (I), it follows that X has MJ-log canonical singularities if and only if X' has MJ-log canonical singularities along E.

Here we introduce an invariant for a hypersurface singularity. The smallest possible dimension $\tau(f)$ of a linear subspace V_0 of $V = kx + ky + kz$ such that $\mathrm{in}(f)$ lies in the subalgebra $k[V_0]$ of $k[x, y, z]$ is an invariant of the germ $(X, 0)$ [Ishii and Reguera 2013, 3.15]. (For $\mathrm{mult}_0 f = 2$, in particular, τ is just the rank of the quadratic forms defining the tangent cone; therefore it is clear that τ is an invariant of (X, x).)

(II-1) $\tau(f) \geq 2$. In this case, by Weierstrass preparation theorem and a coordinate transformation (for example, see [Ishii and Reguera 2013]) the equation $f = 0$ is written as

$$x^2 + y^2 + g(z) = 0,$$

where $\mathrm{mult}_0 g \geq 2$ (if $g = 0$ we define $\mathrm{mult}_0 g = \infty$). Then $\mathrm{mld}_{\mathrm{MJ}}(0; X, \mathcal{O}_X) = 1$ by [Ishii and Reguera 2013]; therefore $(X, 0)$ is MJ-log canonical.

(II-2) $\tau(f) = 1$. In this case the equation $f = 0$ is written as

$$x^2 + g(y, z) = 0,$$

where $\mathrm{mult}_0 g \geq 3$. Now let us consider the germ of the hypersurface $g(y, z) = 0$ at 0 in $\mathrm{Spec}\, k[\![y, z]\!]$. Although this germ depends on the choice of the coordinates, its multiplicity $m_2 := \mathrm{mult}\, g$, and its τ-invariant at 0, let it be τ_2, only depends on $(X, 0)$ (this follows from [Hironaka 1967]. See [Ishii and Reguera 2013, Remark 3.19]).

(II-2-1) $\tau(f) = 1, m_2 \geq 5$. In this case (X, x) is not MJ-log canonical. Indeed we can see that $(1, 1, 1) \notin \Gamma(f)$, which implies that $(X, 0)$ is not MJ-log canonical by Corollary 4.10.

(II-2-2) $\tau(f) = 1, m_2 = 4$. In this case the equation f is written as

$$x^2 + g_4(y, z) + g_5(y, z) = 0,$$

where g_4 is homogeneous of degree 4 and $\mathrm{mult}_0 g_5 \geq 5$. Then, we can see that the singular locus C of X' lying on E is isomorphic to $\mathbb{P}^1$. Let $\Phi' : A'' \to A'$ is the blow-up with the center C, X'' the strict transform of X in A'' and F the exceptional divisor with respect to Φ'. Then, as $I_{X'}\mathcal{O}_{A''} = I_{X''}\mathcal{O}_{A''}(-2F)$ and $K_{A''/A'} = F$, by Theorem 3.6 we obtain

$$\hat{K}_{\overline{X}/X''} - J_{\overline{X}/X''} - \Psi'^*(F|_{X''}) = \hat{K}_{\overline{X}/X'} - J_{\overline{X}/X'},$$

where $\Psi' : \bar{A} \to A''$ is a factorizing resolution of X'' in A'' and $\overline{X}$ is the strict transform of X'' in $\bar{A}$. The above equality yields the X' has MJ-log canonical singularities if and only if $(X'', F|_{X''})$ is MJ-log canonical. Here, as X'' is a hypersurface, so in particular satisfies S_2 condition, by Corollary 4.1 the curve $F|_{X''}$ is reduced and MJ-log canonical. We can see that $F|_{X''}$ has at worst ordinary nodes if and only if g_4 does not have a linear factor with multiplicity more than 2.

(II-2-3) $\tau(f) = 1, m_2 = 3$.

(II-2-3-a) $\tau(f) = 1, m_2 = 3, \tau_2 > 1$. Proposition 3.21 of [Ishii and Reguera 2013] then gives $\mathrm{mld}_{\mathrm{MJ}}(0; X, \mathcal{O}_X) = 1$. Therefore $(X, 0)$ is MJ-log canonical.

(II-2-3-b), $\tau(f) = 1, m_2 = 3, \tau_2 = 1$. In this case the equation f is written as

$$f = x^2 + y^3 + yg(z) + h(z),$$

where $\mathrm{mult}_0 g \geq 3$ and $\mathrm{mult}_0 h \geq 4$.

If $\mathrm{mult}_0 g = 3$ or $\mathrm{mult}_0 h \leq 5$, then $\mathrm{mld}_{\mathrm{MJ}}(0; X, \mathcal{O}_X) = 1$ by [Ishii and Reguera 2013, Proposition 3.23]. Therefore $(X, 0)$ is MJ-log canonical.

If $\mathrm{mult}_0 g = 4$ or $\mathrm{mult}_0 h = 6$, by a coordinate transformation we may assume $g(z) = az^4$ and $h(z) = bz^6 +$ (higher degree term in z) $(a, b \in k)$. Here, note that the condition "$\mathrm{mult}_0 g = 4$ or $\mathrm{mult}_0 h = 6$" implies "$a \neq 0$ or $b \neq 0$". Take a blow-up $\Phi : A' \to A$ and look at the equation defining X' on each canonical affine chart of A', we can see that on two affine charts X' is nonsingular and on one affine chart X' is defined by

$$u^2 + v^3w + avw^3 + bw^4 + h'(w) = 0,$$

where $\mathrm{mult}_0 \, h' \geq 5$. Here, as $a \neq 0$ or $b \neq 0$, the degree 4 part $v^3 w + a v w^3 + b w^4$ does not have a linear factor with multiplicity 3. Therefore, by (II-2-2) the singularity is MJ-log canonical at the point with the coordinate $(u, v, w) = (0, 0, 0)$ and the other points are nonsingular. Thus, in this case $(X, 0)$ is MJ-log canonical.

If $\mathrm{mult}_0 \, g \geq 5$ and $\mathrm{mult}_0 \, h \geq 7$, then the Newton polygon $\Gamma(f)$ does not contain the point $\mathbf{1} = (1, 1, 1)$. Therefore by Corollary 4.10 the singularity $(X, 0)$ is not MJ-log canonical. $\qquad\square$

Next we consider the case $\mathrm{emb}(X, 0) = 4$.

Lemma 5.5. *Assume that X is two-dimensional MJ-log canonical at a point $0 \in X$ with $\mathrm{emb}(X, 0) = 4$.*

(i) *When we write $\widehat{\mathcal{O}_{X,,0}} \simeq k[\![x_1, x_2, x_3, x_4]\!]/I$, the ideal I contains two elements f, g with $\mathrm{mult}_0 \, f = \mathrm{mult}_0 \, g = 2$ and $\mathrm{in}(f), \mathrm{in}(g)$ form a regular sequence in $k[x_1, x_2, x_3, x_4]$.*

(ii) *The projective scheme $E_X := V(\mathrm{in}(I)) \subset \mathbb{P}^3$ is a reduced curve with at worst ordinary nodes.*

Proof. By (2) in Proposition 2.14, we have

$$\mathrm{mld}_{\mathrm{MJ}}(0; X, \mathcal{O}_X) = \inf_n \{(n+1)2 - \dim(\psi_{n0}^X)^{-1}(0)\} \geq 0;$$

therefore in particular for $n = 2$, we have

$$\dim(\psi_{20}^X)^{-1}(0) \leq 6. \tag{20}$$

Here, note that

$$(\psi_{20}^X)^{-1}(0) = \mathrm{Spec} \, k[x_1^{(i)}, x_2^{(j)}, x_3^{(k)}, x_4^{(l)} \mid i, j, k, l = 1, 2]/(F^{(1)}, F^{(2)} \mid f \in I)$$

under the notation in Remark 2.12. Since 4 is the embedding dimension of $(X, 0)$, it follows that $\mathrm{mult}_0 \, f \geq 2$ for all $f \in I$; therefore $F^{(1)} = 0$ for all f by Remark 2.12. By the inequality (20) we obtain that there exist $f, g \in I$ such that

$$F^{(2)}(x_i^{(1)}), G^{(2)}(x_i^{(1)})$$

form a regular sequence in

$$k[x_1^{(i)}, x_2^{(j)}, x_3^{(k)}, x_4^{(l)} \mid i, j, k, l = 1, 2];$$

therefore these form a regular sequence in

$$k[x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}].$$

As $\mathrm{in}(f)(x_i^{(1)}) = F^{(2)}$, $\mathrm{in}(g)(x_i^{(1)}) = G^{(2)}$, it follows that $\mathrm{mult}_0 \, f = \mathrm{mult}_0 \, g = 2$

by Remark 2.12 and that $\mathrm{in}(f), \mathrm{in}(g)$ form a regular sequence in $k[x_1, x_2, x_3, x_4]$. This completes the proof of (i).

Now let A be a nonsingular variety of dimension 4 containing a neighborhood of the singularity $(X, 0)$ and let $A' \to A$ be the blow-up at 0 with the exceptional divisor $E \simeq \mathbb{P}^3$. Let $X' \subset A'$ be the strict transform of X in A'. Then, note that $E|_{X'} = E_X$ and we have

$$I_X \mathcal{O}_{A'} \subset I_{X'} \mathcal{O}_{A'}(-2E).$$

By taking a factorizing resolution $\Psi: \bar{A} \to A'$ of X' in A' with the strict transform $\overline{X}$ of X', we obtain

$$\hat{K}_{\overline{X}/X'} - J_{\overline{X}/X'} - \Psi^* E|_{\overline{X}} \geq \hat{K}_{\overline{X}/X} - J_{\overline{X}/X} \tag{21}$$

by Corollary 3.9. Now, by the assumption that X is MJ-log canonical at 0, it follows that (X', E_X) is MJ-log canonical, which implies $\mathrm{mld}_{\mathrm{MJ}}(y; X', E_X) \geq 0$ for every $y \in E_X$. Therefore we obtain

$$\mathrm{mld}_{\mathrm{MJ}}(y; X', \mathcal{O}_{X'}) \geq 1.$$

But such a two-dimensional singularity (X', y) is determined as either non-singular or a hypersurface singularity (see, for example [Ishii and Reguera 2013, Lemma 3.6]). Hence X' satisfies S_2 condition around E_X. Then, by Corollary 4.1, E_X is reduced and MJ-log canonical, which yields (ii). $\qquad \square$

Theorem 5.6. *Let $(X, 0)$ be a singularity on a two-dimensional reduced scheme with* $\mathrm{emb}(X, 0) = 4$.

 (i) *In case $(X, 0)$ is locally a complete intersection, X is MJ-log canonical at 0 if and only if $\widehat{\mathcal{O}_{X,,0}} \simeq k[\![x_1, x_2, x_3, x_4]\!]/(f, g)$, where f, g satisfy the conditions that $\mathrm{mult}_0 f = \mathrm{mult}_0 g = 2$ and $V(\mathrm{in}(f), \mathrm{in}(g)) \subset \mathbb{P}^3$ is a reduced curve with at worst ordinary nodes.*

 (ii) *In case $(X, 0)$ is not locally a complete intersection, X is MJ-log canonical at 0 if and only if X is a closed subscheme of a locally complete intersection surface M which is MJ-log canonical at 0.*

Proof. For the proof of (i), assume that $(X, 0)$ is locally a complete intersection and $\widehat{\mathcal{O}_{X,,0}} \simeq k[\![x_1, x_2, x_3, x_4]\!]/(f, g)$. Assume that $(X, 0)$ is MJ-log canonical. Then, by Lemma 5.5 it follows $\mathrm{mult}_0 f = \mathrm{mult}_0 g = 2$. Because in Lemma 5.5 it is proved that $E_X = V(\mathrm{in}(I))$ is a reduced curve with at worst ordinary nodes, it is sufficient to prove that $V(\mathrm{in}(f), \mathrm{in}(g)) = V(\mathrm{in}(I))$. In general for a complete intersection singularity defined by f, g the inequality

$$\mathrm{mult}(X, 0) \geq (\mathrm{mult}_0 f)(\mathrm{mult}_0 g)$$

holds. We have $\mathrm{mult}(X, 0) = \deg(V(\mathrm{in}(I)) \subset \mathbb{P}^3)$. Noting that $V(\mathrm{in}(I))$ is

contained in $V(\text{in}(f), \text{in}(g))$, we obtain $\deg V(\text{in}(I)) \leq \deg V(\text{in}(f), \text{in}(g))$, which implies

$$\text{mult}(X, 0) \leq (\text{mult}_0\, f_1)(\text{mult}_0\, f_2).$$

Therefore the equalities hold, in particular $V(\text{in}(I)) = V(\text{in}(f), \text{in}(g))$.

Conversely, if $\widehat{\mathcal{O}_{X,,0}} \simeq k[\![x_1, x_2, x_3, x_4]\!]/(f, g)$ and f, g satisfy the conditions in (i). The conditions claim that E_X is a MJ-log canonical curve. By Corollary 4.1, we have (X', E_X) is MJ-log canonical around E_X. On the other hand, in this case we have

$$I_X \mathcal{O}_{A'} = I_{X'} \mathcal{O}_{A'}(-2E).$$

Therefore by Corollary 3.9, we obtain the equality in (21)

$$\widehat{K}_{\overline{X}/X'} - J_{\overline{X}/X'} - \Psi^* E|_{\overline{X}} = \widehat{K}_{\overline{X}/X} - J_{\overline{X}/X},$$

which yields that X is MJ-log canonical at 0.

For the proof of (ii), first assume that X is a subscheme of an MJ-log canonical two-dimensional locally complete intersection scheme M. By adjunction formula in [Ishii 2013, Corollary 3.12] we have

$$\text{mld}_{\text{MJ}}(0; X, \mathcal{O}_X) \geq \text{mld}_{\text{MJ}}(0; M, \mathcal{O}_M).$$

As the right-hand side is nonnegative by the assumption, we obtain that X is MJ-log canonical at 0.

Conversely assume that X is MJ-log canonical at 0. Assume also that X is not locally a complete intersection at 0. Then, by Lemma 5.5, there are two elements $f, g \in I$ such that $\text{mult}_0\, f = \text{mult}_0\, g = 2$ and $\text{in}(f), \text{in}(g)$ define a curve in $\mathbb{P}^3$. Here I is the ideal as in the proof of Lemma 5.5. Let $E' = V(\text{in}(f), \text{in}(g)) \subset \mathbb{P}^3$. Let

$$\overline{A} \xrightarrow{\Psi} A' \to A, \quad \overline{X} \to X' \to X, \quad E \subset A', \quad E_X \subset X',$$

as in the proof of Lemma 5.5. Then, as $\text{in}(f), \text{in}(g) \in \text{in}(I)$, we have $E_X \subset E'$. Therefore $\deg E_X \leq \deg E' = 4$ in $\mathbb{P}^3$. By the assumption that X is not locally a complete intersection at 0, it follows that E_X is not a complete intersection; therefore

$$\deg E_X \leq 3. \tag{22}$$

On the other hand E_X is reduced and has at worst ordinary nodes by Lemma 5.5. By the result of (i), for the proof of the statement, it is sufficient to prove that there are two elements $f', g' \in I$ such that $V(\text{in}(f'), \text{in}(g'))$ is a reduced curve with at worst ordinary nodes. Therefore it is sufficient to prove that there exists in $\mathbb{P}^3$ a complete intersection reduced curve E'' which contains E_X such that E''

has at worst ordinary nodes. Here, we note that E_X is not a complete intersection, because if it is a complete intersection, then X is also a complete intersection.

An irreducible curve in $\mathbb{P}^3$ of degree ≤ 3 is classified as follows:

(a) $\deg C = 1 \Leftrightarrow C$ is a line.

(b) $\deg C = 2 \Leftrightarrow C$ is a conic in $\mathbb{P}^2$.

(c) $\deg C = 3 \Leftrightarrow C$ is either a plane cubic with genus 1 or a twisted cubic.

Case 1: The case $\deg E_X = 1$ does not happen. Because, if $\deg E_X = 1$, then E_X must be irreducible and by (a) it is a line; therefore E_X is a complete intersection, a contradiction.

Case 2: The case $\deg E_X = 2$. In this case, the possibilities for E_X are as follows:

(1) a plane conic;

(2) the union of two lines which intersect at one point;

(3) the disjoint union of two lines.

The cases (1), (2) do not happen as E_X, because in these cases the curve becomes a complete intersection. In case (3), E_X is the union of skew lines; therefore by a suitable coordinate system in $\mathbb{P}^3$, we can write $E_X = V(x_1, x_2) \cup V(x_3, x_4)$. Then E_X is contained in a complete intersection scheme $V(x_1 x_3, x_2 x_4)$. We can see that this scheme is a cycle of four $\mathbb{P}^1$'s with ordinary nodes. We can take this scheme $V(x_1 x_3, x_2 x_4)$ as E''.

Case 3: The case $\deg E_X = 3$. In this case, the possibilities for E_X are as follows:

(4) a plane cubic of genus 1;

(5) a twisted cubic;

(6) the union of a plane conic and a line;

(7) the union of three lines.

The case (4) does not happen as E_X, because in this case the curve is a complete intersection. If E_X is as in (5), then E_X is defined by

$$x_1 x_3 - x_2^2 = x_2 x_4 - x_3^2 = x_1 x_4 - x_2 x_3 = 0.$$

Then the complete intersection curve $V(x_1 x_3 - x_2^2 + x_2 x_4 - x_3^2, x_1 x_4 - x_2 x_3)$ contains E_X and it is reduced and has only ordinary nodes. So take this scheme as E''.

In case (6), first we show that the conic Q and the line l intersect. Let S be a surface defined by a general element in the vector space

$$\{a(\mathrm{in}(f)) + b(\mathrm{in}(g)) \mid a, b \in k\}.$$

Then S must be an irreducible surface, because otherwise S must be the union of two hyperplanes and E' becomes a line, a contradiction. Therefore S is a cone over a plane conic or nonsingular. If S is a cone, then a plane conic on S and a line on S intersect. If S is nonsingular, then $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and the lines on S are either of the type $C_p = \{p\} \times \mathbb{P}^1$ or of the type $D_q = \mathbb{P}^1 \times \{q\}$, where p, q are points in $\mathbb{P}^1$. A conic on S is linearly equivalent to $C_p + D_q$ which has a positive intersection number with C_p and D_q. Now we obtained $Q \cap l \neq \varnothing$.

Here, if the conic and the line lie on a plane, then the curve becomes a complete intersection. Therefore E_X is not of this type. Assume that the conic Q and the line l do not lie on a plane. We can take Q on a hyperplane $x_1 = 0$. By a suitable choice of the coordinate system, we may assume that $l = V(x_2, x_3)$. Let $g = g(x_2, x_3, x_4)$ be the defining equation of Q in the hyperplane and $\ell = ax_2 + bx_3$ a general linear combination of x_2 and x_3. Then the complete intersection scheme $V(g, x_1\ell)$ contains $Q \cup l$ and it is a reduced curve consisting of a plane conic and two lines l, l' intersecting normally at the point $(1, 0, 0, 0)$ with ordinary double intersection also at $Q \cap l'$. Therefore if $E_X = Q \cup l$, we can take $V(g, x_1\ell)$ as E''.

In case (7), take S as above. If S is a cone over a plane conic and if E_X consists of three lines, then by $E_X \subset S$ three lines must intersect at the vertex; therefore it is not ordinary double, which shows that E_X is not of this type. If S is nonsingular, then, as was stated above, a line on S is either of the form C_p or D_q. Because of the symmetry of C and D, we may assume that the union of three lines on S is either the union of three C_p's or the union of two C_p's and one D_q. The union of three C_p's is not possible for E_X. Because otherwise, $E_X \subset E'$ and $E' = S \cap H$, where H is a hypersurface of degree 2. Then

$$3 = (E_X \cdot D_q)_S \leq (E' \cdot D_q)_S = H \cdot D_q = 2,$$

which is a contradiction. Here, $(\cdot)_S$ is the intersection number of the divisors on S and $H \cdot D_q$ is the intersection number of the divisor H and a curve D_q in $\mathbb{P}^3$.

Now if E_X is the union of C_{p_1}, C_{p_2} and D_q, then it is a chain of lines and by a suitable choice of the coordinate system, these are represented as $C_{p_1} = V(x_1, x_2), C_{p_2} = V(x_3, x_4)$ and $D_q = V(x_2, x_3)$. Then the complete intersection $V(x_1x_3, x_2x_4)$ contains E_X and $V(x_1x_3, x_2x_4)$ is reduced and has at worst ordinary nodes. Thus every possible E_X is contained in a complete intersection curve which is reduced and has at worst ordinary nodes. $\square$

Example 5.7. Let $X \subset \mathbb{A}^4$ be defined by $f = x_1x_3, g = x_2x_4 \in k[x_1, x_2, x_3, x_4]$. Then $\mathrm{in}(f) = f$, $\mathrm{in}(g) = g$ and $V(f, g)$ is a cycle consisting of four $\mathbb{P}^1$'s such that the intersection of each two components is ordinary double. Then, by Theorem 5.6, X is MJ-log canonical at 0. Let C_i ($i = 1, 2, \ldots, 4$) be the

irreducible component of $V(f, g)$ such that $C_i \cdot C_{i+1} = 1$ for $i = 1, \ldots, 4$ and let $C_5 := C_1$. Note that X is the cone over the reduced projective scheme $\bigcup_{i=1}^{4} C_i \subset \mathbb{P}^3$.

Now take the cone X_1 over the reduced projective scheme $C_1 \cup C_2 \cup C_3 \subset \mathbb{P}^3$. By Theorem 5.6, X_1 is MJ-log canonical at 0. This example was proved to be non-semi-log canonical singularity by Kollár [2013, Example 5.16].

Next take the cone X_2 over the reduced projective scheme $C_1 \cup C_3 \subset \mathbb{P}^3$. By Theorem 5.6, $(X_2, 0)$ is also MJ-log canonical. This is an example of MJ-log canonical singularity but not S_2. Indeed X_2 is the union of two irreducible surfaces which intersect at a point 0; therefore X_2 does not satisfy S_2.

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