Adjoint associativity: an invitation to algebra in ∞-categories

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There appeared not long ago a reduction formula for derived Hochschild cohomology, that has been useful, for example, in the study of Gorenstein maps and of rigidity with respect to semidualizing complexes. The formula involves the relative dualizing complex of a ring homomorphism, so brings out a connection between Hochschild homology and Grothendieck duality. The proof, somewhat ad hoc, uses homotopical considerations via a number of noncanonical projective and injective resolutions of differential graded objects. Recent efforts aim at more intrinsic approaches, hopefully upgradable to "higher" contexts — like bimodules over algebras in ∞-categories. This would lead to wider applicability, for example to ring spectra; and the methods might be globalizable, revealing some homotopical generalizations of aspects of Grothendieck duality. (The original formula has a geometric version, proved by completely different methods coming from duality theory.) A first step is to extend Hom-Tensor adjunction — adjoint associativity — to the ∞-category setting.

Introduction

There are substantial overlaps between algebra and homotopy theory, making for mutual enrichment — better understanding of some topics, and wider applicability of results from both areas. In this vein, works of Quillen, Neeman, Avramov–Halperin, Schwede–Shipley, Dwyer–Iyengar–Greenlees, to mention just a few, come to mind. See also [Greenlees 2007]. In recent years, homotopy theorists like May, Toën, Joyal, Lurie (again to mention just a few) have been developing a huge theory of *algebra in ∞-categories,* dubbed by Lurie "higher algebra",[1] familiarity with which could be of significant benefit to (lower?) algebraists.

This little sales pitch will be illustrated here by one specific topic that arose algebraically, but can likely be illuminated by homotopical ideas.

1. Motivation: reduction of Hochschild (co)homology

Let R be a noetherian commutative ring, $\mathbf{D}(R)$ the derived category of the category of R-modules, and similarly for S. Let $\sigma\colon R \to S$ be a *flat* homomorphism essentially of finite type. Set $S^{\mathrm{e}} := S \otimes_R S$. Let $M, N \in \mathbf{D}(S)$, with M σ-*perfect*, that is, the cohomology modules $H^i(M)$ are finitely generated over S, and the natural image of M in $\mathbf{D}(R)$ is isomorphic to a bounded complex of flat R-modules.

Theorem 1.1 (reduction theorem [Avramov et al. 2010, Theorems 1 and 4.6]). *There exists a complex $D^\sigma \in \mathbf{D}(S)$ together with bifunctorial S-isomorphisms*

$$\mathbb{R}\operatorname{Hom}_{S^{\mathrm{e}}}(S, M \otimes_R^{\mathsf{L}} N) \xrightarrow{\sim} \mathbb{R}\operatorname{Hom}_S(\mathbb{R}\operatorname{Hom}_S(M, D^\sigma), N), \qquad (1.1.1)$$

$$S \otimes_{S^{\mathrm{e}}}^{\mathsf{L}} \mathbb{R}\operatorname{Hom}_R(M, N) \xrightarrow{\sim} \mathbb{R}\operatorname{Hom}_S(M, D^\sigma) \otimes_S^{\mathsf{L}} N. \qquad (1.1.2)$$

Remarks. (1) "Reduction" refers to the reduction, via (1.1.1) and (1.1.2), of constructions over S^{e} to constructions over S.

(2) The homology S-modules of the sources of (1.1.1) and (1.1.2) are the Hochschild cohomology modules of σ, with coefficients in $M \otimes_R^{\mathsf{L}} N$, and the Hochschild homology modules of σ, with coefficients in $\mathbb{R}\operatorname{Hom}_R(M, N)$.

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[1]Not to be confused with the contents of [Hall and Knight 1907].

(3) (Applications.) The isomorphism (1.1.1) is used to formulate a notion of rigidity with respect to a fixed semidualizing complex [Avramov et al. 2011, Section 3], leading to a broad generalization of the work of Yekutieli and Zhang summarized in [Yekutieli 2010].

The special case $M = N = S$ of (1.1.1) plays a crucial role in the proofs of [Avramov and Iyengar 2008, Theorems 3 and 4].

The special case $M = N = S$ of (1.1.2) is used in a particularly simple expression for the fundamental class of σ; see [Iyengar et al. 2015, Theorem 4.2.4].

(4) The complex D^σ is determined up to isomorphism by either (1.1.1) — which implies that D^σ corepresents the endofunctor $\mathbb{R}\operatorname{Hom}_{S^e}(S, S \otimes_R -)$ of $\mathbf{D}(S)$ — or, more directly, by (1.1.2), which yields an isomorphism

$$S \otimes_{S^e}^{\mathsf{L}} \mathbb{R}\operatorname{Hom}_R(S, S) \xrightarrow{\sim} D^\sigma.$$

In fact, if g is the map $\operatorname{Spec}(\sigma)$ from $V := \operatorname{Spec}(S)$ to $W := \operatorname{Spec}(R)$, then

$$D^\sigma \cong g^! \mathcal{O}_W,$$

that is, D^σ is a *relative dualizing complex for σ* [Avramov et al. 2010, Remark 6.2].

Thus we have a relation (one of several) between Hochschild homology and Grothendieck duality.

(5) For example, if $\operatorname{Spec}(S)$ is connected and σ is formally smooth, so that, with I the kernel of the multiplication map $S^e \to S$, the relative differential module $\Omega_\sigma := I/I^2$ is locally free of constant rank, say d, then

$$D^\sigma \cong \Omega_\sigma^d[d] := \left(\textstyle\bigwedge^d I/I^2\right)[d] \cong \operatorname{Tor}_d^{S^e}(S, S)[d].$$

Using local resolutions of S by Koszul complexes of S^e-regular sequences that generate I, one finds a chain of natural $\mathbf{D}(S)$-isomorphisms

$$\begin{aligned}
\mathbb{R}\operatorname{Hom}_{S^e}(S, S^e) \;&\xrightarrow{\sim}\; \left(H^d \mathbb{R}\operatorname{Hom}_{S^e}(S, S^e)\right)[-d] \\
&\xrightarrow{\sim}\; \left(H^d \mathbb{R}\operatorname{Hom}_{S^e}(S, S^e)\right)[-d] \otimes_{S^e} S \\
&\xrightarrow{\sim}\; \left(H^d \left(\mathbb{R}\operatorname{Hom}_{S^e}(S, S^e) \otimes_{S^e}^{\mathsf{L}} S\right)\right)[-d] \\
&\xrightarrow{\sim}\; \left(H^d \left(\mathbb{R}\operatorname{Hom}_S(S \otimes_{S^e}^{\mathsf{L}} S, S)\right)\right)[-d] \\
&\xrightarrow{\sim}\; \operatorname{Hom}_S \left(\operatorname{Tor}_d^{S^e}(S, S)[d], S\right) \\
&\xrightarrow{\sim}\; \operatorname{Hom}_S(D^\sigma, S) \\
&\xrightarrow{\sim}\; \mathbb{R}\operatorname{Hom}_S(D^\sigma, S).
\end{aligned}$$

The composition ϕ of this chain is (1.1.1) with $M = N = S$. (It is essentially the same as the isomorphism $H^d \mathbb{R} \operatorname{Hom}_{S^e}(S, S^e) \xrightarrow{\sim} \operatorname{Hom}_S\left(\bigwedge^d I/I^2, S\right)$ given by the "fundamental local isomorphism" of [Hartshorne 1966, Chapter III, §7].)

As σ is formally smooth, σ-perfection of M is equivalent to M being a perfect S^e-complex; and S too is a perfect S^e-complex. It is then straightforward to obtain (1.1.1) by applying to ϕ the functor

$$-\otimes_{S^e}^{\mathsf{L}} (M \otimes_R^{\mathsf{L}} N) = -\otimes_S^{\mathsf{L}} S \otimes_{S^e}^{\mathsf{L}} (M \otimes_R^{\mathsf{L}} N) \cong -\otimes_S^{\mathsf{L}} (M \otimes_S^{\mathsf{L}} N).$$

(Note that M and N may be assumed to be K-flat over S, hence over R.)

To prove (1.1.1) for arbitrary σ one uses a factorization

$$\sigma = (\text{surjection}) \circ (\text{formally smooth})$$

to reduce to the preceding formally smooth case. For this reduction (which is the main difficulty in the proof), as well as a scheme-theoretic version of Theorem 1.1, see [Avramov et al. 2010; Iyengar et al. 2015, Theorem 4.1.8].

2. Enter homotopy

So far no homotopical ideas have appeared. But they become necessary, via (graded-commutative) differential graded algebras (dgas), when the flatness assumption on σ is dropped. Then for Theorem 1.1 to hold, one must first define S^e to be a derived tensor product:

$$S^e := S \otimes_R^{\mathsf{L}} S := \bar{S} \otimes_R \bar{S},$$

where $\bar{S} \to S$ is a homomorphism of dg R-algebras that induces homology isomorphisms, with $\bar{S}$ *flat* over R. Such "flat dg algebra resolutions" of the R-algebra S exist; and any two are "dominated" by a third. (This is well-known; for more details, see [Avramov et al. 2010, Sections 2 and 3].) Thus S^e is not an R-algebra, but rather a class of quasi-isomorphic dg R-algebras.

By using suitable "semiprojective" dg $\bar{S}$-resolutions of the complexes M and N, one can make sense of the statements

$$M \otimes_R^{\mathsf{L}} N \in \mathbf{D}(S^e), \quad \mathbb{R}\operatorname{Hom}_R(M, N) \in \mathbf{D}(S^e);$$

and then, following Quillen, Mac Lane and Shukla, define complexes

$$\mathbb{R}\operatorname{Hom}_{S^e}(S, M \otimes_R^{\mathsf{L}} N), \quad S \otimes_{S^e}^{\mathsf{L}} \mathbb{R}\operatorname{Hom}_R(M, N),$$

whose homology modules are the *derived Hochschild cohomology* resp. *the derived Hochschild homology* modules of σ, with coefficients in $M \otimes_R^{\mathsf{L}} N$ and $\mathbb{R}\operatorname{Hom}_R(M, N)$, respectively. These complexes depend on a number of choices of resolution, so they are defined only up to a coherent family of isomorphisms,

indexed by the choices. This is analogous to what happens when one works with derived categories of modules over a ring.

In [Avramov et al. 2010], the reduction of Theorem 1.1 to the formally smooth case is done by the manipulation of a number of noncanonical dg resolutions, both semiprojective and semiinjective. Such an argument tends to obscure the conceptual structure. Furthermore, in section 6 of that paper a geometric version of Theorem 1.1 is proved by completely different methods associated with Grothendieck duality theory — but only for flat maps. A globalized theory of derived Hochschild (co)homology for analytic spaces or noetherian schemes of characteristic zero is given, for example, in [Buchweitz and Flenner 2008]; but there is as yet no extension of Theorem 1.1 to nonflat maps of such spaces or schemes.

The theory of algebra in ∞-categories, and its globalization "derived algebra geometry," encompass all of the above situations,[2] and numerous others, for instance "structured spectra" from homotopy theory. The (unrealized) underlying goal toward which this lecture is a first step is to prove a version of Theorem 1.1 — without flatness hypotheses — that is meaningful in this general context. The hope is that such a proof could unify the local and global versions in [Avramov et al. 2010], leading to better understanding and wider applicability; and perhaps most importantly, to new insights into, and generalizations of, Grothendieck duality.

3. Adjoint associativity

To begin with, such an upgraded version of Theorem 1.1 must involve some generalization of $\otimes$ and Hom; and any proof will most probably involve the basic relation between these functors, namely *adjoint associativity*.

For any two rings (not necessarily commutative) R, S, let $R\#S$ be the abelian category of R-S bimodules (R acting on the left and S on the right).

The classical version of adjoint associativity (compare [MacLane 1967, VI, (8.7)]) asserts that *for rings A, B, C, D, and $x \in A\#B$, $y \in B\#C$, $z \in D\#C$, there exists in $D\#A$ a functorial isomorphism*

$$a(x, y, z): \operatorname{Hom}_C(x \otimes_B y, z) \xrightarrow{\sim} \operatorname{Hom}_B(x, \operatorname{Hom}_C(y, z)), \tag{3.1}$$

such that for any fixed x and y, the corresponding isomorphism between the left adjoints of the target and source of a is the associativity isomorphism

$$- \otimes_A (x \otimes_B y) \xleftarrow{\sim} (- \otimes_A x) \otimes_B y.$$

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[2]to some extent, at least: see, e.g., [Shipley 2007]. But see also Example 10.3 and 12.3 below.

As hinted at above, to get an analogous statement for *derived categories,* where one needs flat resolutions to define (derived) tensor products, one has to work in the dg world; and that suggests going all the way to ∞-*categories*.

The remainder of this article will be an attempt to throw some light on how (3.1) can be formulated and proved in the ∞-context. There will be no possibility of getting into details, for which however liberal references will be given to the massive works [Lurie 2009; 2014], for those who might be prompted to explore the subject matter more thoroughly.[3]
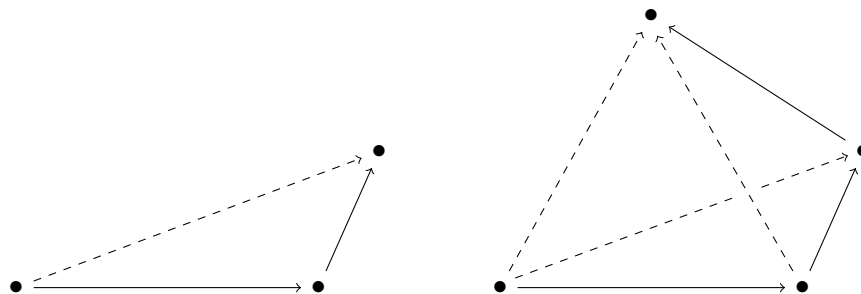
4. ∞-categories

It's time to say what an ∞-category is.

An ordinary small category $\mathcal{C}$ is, to begin with, a diagram

$$A_1 \underset{d_0}{\overset{d_1}{\underset{\longleftarrow}{\rightleftarrows}}} \overset{s_0}{\longleftarrow} A_0$$

where A_1 is the set of arrows in $\mathcal{C}$, A_0 is the set of objects, s_0 takes an object to its identity map, and d_0 (resp. d_1) takes an arrow to its target (resp. source). We can extend this picture by introducing sequences of composable arrows:

The first picture represents a sequence of two composable arrows, whose composition is represented by the dashed arrow; and the second picture represents a sequence of three composable arrows, with dotted arrows representing compositions of two or three of these. The pictures suggest calling a sequence of n composable arrows an n-*simplex*. (A 0-simplex is simply an object in $\mathcal{C}$.) The set of n-simplices is denoted A_n.

There are four *face maps* $d_i \colon A_3 \to A_2$ $(0 \le i \le 3)$, taking a sequence $\gamma \circ \beta \circ \alpha$ to the respective sequences $\gamma \circ \beta$, $\gamma \circ (\beta\alpha)$, $(\gamma\beta) \circ \alpha$ and $\beta \circ \alpha$. There are three *degeneracy maps* $s_j \colon A_2 \to A_3$ $(0 \le j \le 2)$ taking a sequence $\beta \circ \alpha$ to the

[3]The page numbers in the references to [Lurie 2014] refer to the preprint dated August, 2012.

respective "degenerate" (that is, containing an identity map) sequences $\beta \circ \alpha \circ \mathrm{id}$, $\beta \circ \mathrm{id} \circ \alpha$ and $\mathrm{id} \circ \beta \circ \alpha$.

Likewise, for any $n > 0$ there are face maps $d_i \colon A_n \to A_{n-1}$ $(0 \le i \le n)$ and degeneracy maps $s_j \colon A_{n-1} \to A_n$ $(0 \le j < n)$; and these maps satisfy the standard identities that define a *simplicial set* (see, e.g., [Goerss and Jardine 1999, p. 4, (1.3)]).

The simplicial set $\mathrm{N}(\mathcal{C})$ just defined is called the *nerve of* $\mathcal{C}$.

Example. For any $n \ge 0$, the totally ordered set of integers

$$0 < 1 < 2 < \cdots < n-1 < n$$

can be viewed as a category (as can any ordered set). The nerve of this category is the *standard n-simplex,* denoted Δ^n. Its m-simplices identify with the nondecreasing maps from the integer interval $[0, m]$ to $[0, n]$. In particular, there is a unique nondegenerate n-simplex ι_n, namely the identity map of $[1, n]$.

The collection of all the nondegenerate simplices of Δ^n, and their face maps, can be visualized by means of the usual picture of a geometric n-simplex and its subsimplices. (For $n = 2$ or 3, see the above pictures, with all dashed arrows made solid.)

The *horn* $\Lambda_i^n \subset \Delta^n$ is the simplicial subset whose m-simplices ($m \ge 0$) are the nondecreasing maps $s \colon [0, m] \to [0, n]$ with image not containing the set $([0, n] \setminus \{i\})$. For example, Λ_i^n has n nondegenerate $(n-1)$-simplices namely $d_j \iota_n$ $(0 \le j \le n,\ j \ne i)$.

Visually, the nondegenerate simplices of $\Lambda_i^n \subset \Delta^n$ are those subsimplices of a geometric n-simplex other than the n-simplex itself and its i-th face.

Small categories are the objects of a category $\mathcal{C}\mathrm{at}$ whose morphisms are functors; and simplicial sets form a category $\mathcal{S}\mathrm{et}_\Delta$ whose morphisms are *simplicial maps*, that is, maps taking m-simplices to m-simplices (for all $m \ge 0$) and commuting with all the face and degeneracy maps. The above map $\mathcal{C} \mapsto \mathrm{N}(\mathcal{C})$ extends in an obvious way to a *nerve functor* $\mathcal{C}\mathrm{at} \to \mathcal{S}\mathrm{et}_\Delta$.

Proposition 4.1 [Lurie 2009, p. 9, 1.1.2.2]. *The nerve functor* $\mathcal{C}\mathrm{at} \to \mathcal{S}\mathrm{et}_\Delta$ *is a fully faithful embedding. Its essential image is the full subcategory of* $\mathcal{S}\mathrm{et}_\Delta$ *spanned by the simplicial sets K with the following property*:

($*$) *For all $n > 0$ and $0 < i < n$, every simplicial map $\Lambda_i^n \to K$ extends* uniquely *to a simplicial map $\Delta^n \to K$.*

Remarks. By associating to each simplicial map $\Delta^n \to K$ the image of the nondegenerate n-simplex ι_n, one gets a bijective correspondence between such maps and n-simplices of K. (See, e.g., [Goerss and Jardine 1999, p. 6].)

By associating to each simplicial map $\Lambda_i^n \to K$ the image of the sequence $(d_j \iota_n)_{0 \le j \le n,\ j \ne i}$, one gets a bijective correspondence between such maps and

sequences $(y_j)_{0 \le j \le n,\, j \ne i}$ of $(n-1)$-simplices of K such that $d_j\, y_k = d_{k-1}\, y_j$ if $j < k$ and $j, k \ne i$. (See [Goerss and Jardine 1999, p. 10, Corollary 3.2].)

Thus $(*)$ means that *for all $n > 0$ and $0 < i < n$, if λ is the map from the set of n-simplices of K to the set of such sequences (y_j) that takes an n-simplex y to the sequence $(d_j\, y)_{0 \le j \le n,\, j \ne i}$, then λ is bijective.*

Definition 4.2. An ∞-*category* is a simplicial set K such that for all $n > 0$ and $0 < i < n$, every simplicial map $\Lambda_i^n \to K$ extends to a simplicial map $\Delta^n \to K$. Equivalently, it is a K for which the preceding map λ is *surjective*.

A *functor* from one ∞-category to another is a map of simplicial sets.

Thus ∞-categories and their functors form a full subcategory of $\mathcal{S}et_\Delta$, one that itself has a full subcategory canonically isomorphic to $\mathcal{C}at$.

Example 4.2.1. To any dg category $\mathcal{C}$ (one whose arrows between two fixed objects are complexes of abelian groups, composition being bilinear) one can assign the *dg-nerve* $\mathrm{N}_{dg}(\mathcal{C})$, an ∞-category whose construction is more complicated than that of the nerve $\mathrm{N}(\mathcal{C})$ because the dg structure has to be taken into account. (For details, see [Lurie 2014, Section 1.3.1].)

For instance, the complexes in an abelian category $\mathcal{A}$ can be made into a dg category $\mathcal{C}_{dg}(\mathcal{A})$ by defining $\mathrm{Hom}(E, F)$ for any complexes E and F to be the complex of abelian groups that is $\mathrm{Hom}\big(E, F[n]\big)$ in degree n, with the usual differential. When $\mathcal{A}$ is a Grothendieck abelian category, we will see below (Example 5.3) how one extracts from the ∞-category $N_{dg}(\mathcal{C}_{dg}(\mathcal{A}))$ the usual derived category $\mathbf{D}(\mathcal{A})$.

Example 4.2.2. To any topological category $\mathcal{C}$ — that is, one where the Hom sets are topological spaces and composition is continuous — one can assign a *topological nerve* $\mathrm{N}_{top}(\mathcal{C})$, again more complicated than the usual nerve $N(\mathcal{C})$ [Lurie 2009, p. 22, 1.1.5.5].

CW-complexes are the objects of a topological category $\mathcal{C}\mathcal{W}$. The topological nerve $\mathcal{S} := \mathrm{N}_{top}(\mathcal{C}\mathcal{W})$ is an ∞-category, the ∞-*category of spaces*. (See [loc. cit, p. 24, 1.1.5.12; p. 52, 1.2.16.3].) Its role in the theory of ∞-categories is analogous to the role of the category of sets in ordinary category theory.

Example 4.2.3. *Kan complexes* are simplicial sets such that the defining condition of ∞-categories holds for *all $i \in [0, n]$*. Examples are *the singular complex of a topological space* (a simplicial set that encodes the homotopy theory of the space), *the nerve of a groupoid* (= category in which all maps are isomorphisms), and *simplicial abelian groups*. (See [Goerss and Jardine 1999, Section I.3].)

Kan complexes span a full subcategory of the category of ∞-categories, the inclusion having a right adjoint [Lurie 2009, p. 36, 1.2.5.3]. The *simplicial nerve* of this subcategory [Lurie 2009, p. 22, 1.1.5.5] provides another model for the ∞-category of spaces [loc. cit, p. 51, 1.2.16].

Most of the basic notions from category theory can be extended to ∞-categories. Several examples will be given as we proceed. A first attempt at such an extension would be to express a property of categories in terms of their nerves, and then to see if this formulation makes sense for arbitrary ∞-categories. (This will not always be done explicitly; but as ∞-category notions are introduced, the reader might check that when restricted to nerves, these notions reduce to the corresponding classical ones.)

Example 4.2.4. An *object* in an ∞-category is a 0-simplex. A *map* f in an ∞-category is a 1-simplex. The *source* (resp. *target*) of f is the object $d_1 f$ (resp. $d_0 f$). The *identity map* id_x of an object x is the map $s_0 x$, whose source and target are both x.

Some history and motivation related to ∞-categories can be gleaned, for example, starting from ncatlab.org/nlab/show/quasi-category.

The notion of ∞-category as a generalization of that of category grew out of the study of operations in the homotopy category of topological spaces, for instance the composition of paths. Indeed, as will emerge, the basic effect of removing unicity from condition ($*$) above to get to ∞-categories (Definition 4.2) is to replace *equality* of maps in categories with a *homotopy* relation, with all that entails.

Topics of foundational importance in homotopy theory, such as *model categories,* or *spectra* and their products, are closely related to, or can be treated via, ∞-categories [Lurie 2009, p. 803; 2014, Sections 1.4, 6.3.2]. Our concern here will mainly be with relations to algebra.
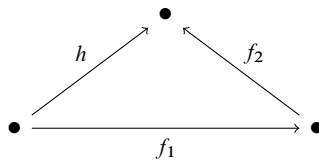
5. The homotopy category of an ∞-category

5.1. The nerve functor of Proposition 4.1 has a left adjoint h: $\mathrm{Set}_\Delta \to \mathrm{Cat}$, the *homotopy functor*; see [loc. cit., p. 28, 1.2.3.1].

If the simplicial set $\mathcal{C}$ is an ∞-category, the homotopy category $h\mathcal{C}$ can be constructed as follows. For maps f and g in $\mathcal{C}$, write $f \sim g$ (and say that "f is homotopic to g") if there is a 2-simplex σ in $\mathcal{C}$ such that $d_2\sigma = f$, $d_1\sigma = g$ and $d_0\sigma = \mathrm{id}_{d_0 g} = \mathrm{id}_{d_0 f}$:
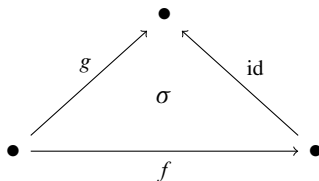
(This can be intuited as the skeleton of a deformation of f to g through a "continuous family" of maps with fixed source and target.) Using the defining

property of ∞-categories, one shows that this homotopy relation is an equivalence relation. Denoting the class of f by $\bar{f}$, one defines the composition $\bar{f}_2 \circ \bar{f}_1$ to be $\bar{h}$ for any h such that there exists a 2-simplex

$$
\begin{array}{ccc}
 & \bullet & \\
h \nearrow & & \nwarrow f_2 \\
\bullet & \xrightarrow{ f_1 } & \bullet
\end{array}
$$

One shows that this composition operation is well-defined, and associative. There results a category whose objects are those of $\mathcal{C}$, and whose maps are the homotopy equivalence classes of maps in $\mathcal{C}$, with composition as just described. (For details, see [Lurie 2009, Section 1.2.3].) This is the homotopy category $h\mathcal{C}$.

Example 5.2. Let $\mathcal{S}$ be the ∞-category of spaces (Example 4.2.2). Its homotopy category $\mathcal{H} := h\mathcal{S}$ is called the *homotopy category of spaces*. The objects of $\mathcal{H}$ are CW-complexes, and the maps are homotopy-equivalence classes of continuous maps. (See [loc. cit., p. 16].)

Example 5.3 (extending Example 4.2.1). In the category of complexes in a Grothendieck abelian category $\mathcal{A}$, the (injectively) *fibrant* objects are those complexes I such that for any $\mathcal{A}$-diagram of complexes $X \xleftarrow{s} Y \xrightarrow{f} I$ with s both a (degreewise) monomorphism and a quasi-isomorphism, there exists $g \colon X \to I$ such that $gs = f$. (See [Lurie 2014, p. 93, 1.3.5.3].)

The *q-injective*, or *K-injective*, objects are those I such that for any diagram of complexes $X \xleftarrow{s} Y \xrightarrow{f} I$ with s a quasi-isomorphism, there exists $g \colon X \to I$ such that gs is homotopic to f.

(Recall that, following Spaltenstein, right-derived functors are defined via q-injective resolutions.)

Lemma 5.3.1. *An $\mathcal{A}$-complex I is q-injective if and only if I is homotopy-equivalent to a fibrant complex.*

Proof. Fix a fibrant Q. By [loc. cit., p. 97, 1.3.5.11], if the complex M is exact then so is the complex $\mathrm{Hom}^{\bullet}(M, Q)$; and so by [Lipman 2009, 2.3.8(iv) and (2.3.8.1)], Q is q-injective, whence so is any complex homotopy-equivalent to Q.

If I is q-injective, then factoring $I \to 0$ as fibration$\circ$ (trivial cofibration) ([Lurie 2014, p. 93, 1.3.5.3]) one gets a monomorphic quasi-isomorphism $j \colon I \hookrightarrow Q$ with Q fibrant, hence q-injective; so j is a homotopy equivalence [Lipman 2009, 2.3.2.2]. $\qquad\square$

Remarks. (1) If the complex Q is bounded below and injective in each degree, then Q is fibrant, hence q-injective, see [Lurie 2014, p. 96, 1.3.5.6].

(2) Any split short exact sequence, extended infinitely both ways by zeros, is a complex homotopically equivalent to the fibrant complex 0, but not necessarily itself fibrant, since fibrant complexes are term-wise injective; see again [Lurie 2014, p. 96, 1.3.5.6].

5.3.2. For any additive category $\mathcal{A}$, two maps in the dg-nerve $N_{dg}(Ch(\mathcal{A}))$ are homotopic if and only if they are so as chain maps, see [loc. cit., p. 64, 1.3.1.8]. Thus the homotopy category $hN_{dg}(Ch(\mathcal{A}))$ is just the category with objects the $\mathcal{A}$-complexes and arrows the homotopy-equivalence classes of chain maps.

Similarly, when $\mathcal{A}$ is a Grothendieck abelian category and $Ch(\mathcal{A})^0$ is the full subcategory of $Ch(\mathcal{A})$) spanned by the fibrant complexes, the homotopy category of the derived ∞-category $\mathcal{D}(\mathcal{A}) := N_{dg}(Ch(\mathcal{A})^0)$ [loc. cit., p. 96, 1.3.5.8] is the quotient of $Ch(\mathcal{A})^0$ by the homotopy relation on chain maps, and thus is equivalent to the similar category whose objects are the fibrant complexes, which by Lemma 5.3.1 is equivalent to the usual derived category $\mathbf{D}(\mathcal{A})$.

In summary: $h\mathcal{D}(\mathcal{A})$ *is equivalent to* $\mathbf{D}(\mathcal{A})$.

(A more general result for any dg category is in [loc. cit., p. 64, 1.3.1.11].)

5.4. The homotopy category of a *stable* ∞-category is triangulated. (See Introduction to [loc. cit., Section 1.1].) For instance, the ∞-category $\mathcal{D}(\mathcal{A})$ (just above) is stable [loc. cit., p. 96, 1.3.5.9]. So is the ∞-category of *spectra* — whose homotopy category underlies stable homotopy theory [loc. cit., p. 16, 1.1.1.11].

Example 5.5. A localization $\mathcal{D} \to \mathcal{D}_V$ of an ordinary category $\mathcal{D}$ with respect to a set V of maps in $\mathcal{D}$ is an initial object in the category of functors with source D that take the maps in V to isomorphisms.

A localization $\mathcal{C} \to \mathcal{C}[W^{-1}]$ of an ∞-category $\mathcal{C}$ w.r.t a set W of maps (i.e., 1-simplices) in $\mathcal{C}$ is similarly universal *up to homotopy* for those ∞-functors out of $\mathcal{C}$ that take the maps in W to equivalences. (For more precision, see [loc. cit., p. 83, 1.3.4.1].) Such a localization exists, and is determined uniquely up to equivalence by $\mathcal{C}$ and W [loc. cit., p. 83, 1.3.4.2].

For functors of the form $\mathcal{C} \to N(\mathcal{D})$ with $\mathcal{D}$ an ordinary category, the words "up to homotopy" in the preceding paragraph can be omitted. (This follows from the precise definition of localization, because in ∞-categories of the form $Fun(\mathcal{C}, N(\mathcal{D}))$ — see 8.2 — the only equivalences are identity maps.)

So composition with the localization map (see (12.4.1)) gives a natural bijection from the set of ∞-functors $\mathcal{C}[W^{-1}] \to N(\mathcal{D})$ to the set of those ∞-functors $\mathcal{C} \to N(\mathcal{D})$ that take the maps in W to equivalences, that is, from the set of functors $h(\mathcal{C}[W^{-1}]) \to \mathcal{D}$ to the set of those functors $h\mathcal{C} \to \mathcal{D}$ that take the maps in the image $\overline{W}$ of W to isomorphisms. Hence there is a natural isomorphism

$$h(\mathcal{C}[W^{-1}]) \xrightarrow{\sim} (h\mathcal{C})_{\overline{W}} \qquad (5.5.1)$$

giving commutativity of the homotopy functor with localization.

For instance, to every *model category* **A** one can associate naturally an "underlying ∞-category" $\mathbf{A}_\infty$. Under mild assumptions, $\mathbf{A}_\infty$ can be taken to be the localization $(\mathrm{N}(\mathbf{A}))[W^{-1}]$, with W the set of weak equivalences in **A**. Without these assumptions, one can just replace **A** by its full subcategory spanned by the cofibrant objects, see [Lurie 2014, p. 89, 1.3.4.16].

Equation (5.5.1), with $\mathcal{C} = \mathrm{N}(\mathbf{A})$, shows that the homotopy category $\mathrm{h}\mathbf{A}_\infty$ is canonically isomorphic to $\mathbf{A}_{\overline{W}}$, the classical homotopy category of **A** [Goerss and Jardine 1999, p. 75, Theorem 1.11].

Example 5.3, with $\mathcal{A}$ the category of right modules over a fixed ring R, is essentially the case where **A** is the category of complexes in $\mathcal{A}$, with "injective" model structure as in [Lurie 2009, p. 93, 1.3.5.3]. Indeed, $\mathbf{A}_\infty$ can be identified with $\mathcal{D}(\mathcal{A})$ [Lurie 2014, p. 98, 1.3.5.15], and the classical homotopy category of **A**, obtained by inverting weak equivalences (= quasi-isomorphisms), with $\mathbf{D}(\mathcal{A})$.

Example 5.6. Let $\mathcal{D}$ be a dg category. For any two objects $x, y \in \mathcal{D}$ replace the mapping complex $\mathrm{Map}_\mathcal{D}(x, y)$ by the simplicial abelian group associated by the Dold–Kan correspondence to the truncated complex $\tau_{\leq 0} \mathrm{Map}_\mathcal{D}(x, y)$, to produce a simplicial category $\mathcal{D}_\Delta$. (See [loc. cit., p. 65, 1.3.1.13], except that indexing here is cohomological rather than homological.)

For example, with notation as in Example 5.3, $\mathbf{D}(\mathcal{A})$ is also the homotopy category of the simplicial nerve of the simplicial category thus associated to $\mathrm{Ch}(\mathcal{A})^0$ [loc. cit., p. 66, 1.3.1.17].

For the category $\mathcal{A}$ of abelian groups, and $\mathcal{D} := \mathrm{Ch}(\mathcal{A})^0$, [loc. cit., p. 47, Remark 1.2.3.14] (in light of [loc. cit., p. 46, 1.2.3.13]) points to an agreeable interpretation of the homotopy groups of the Kan complex $\mathrm{Map}_{\mathcal{D}_\Delta}(x, y)$ with base point 0 (or of its geometric realization, see [Goerss and Jardine 1999, bottom, p. 60]):

$$\pi_n(\mathrm{Map}_{\mathcal{D}_\Delta}(x, y)) \cong \mathrm{H}^{-n} \mathrm{Map}_\mathcal{D}(x, y) =: \mathrm{Ext}^{-n}(x, y) \quad (n \geq 0).$$

(See also [Lurie 2014, p. 32, 1.2.1.13] and [loc. cit., p. 29; Section 1.2, second paragraph].)

6. Mapping spaces; equivalences

6.1. An important feature of ∞-categories is that any two objects determine not just the set of maps from one to the other, but also a topological *mapping space*. In fact, with $\mathcal{H}$ as in Example 5.2, the homotopy category $\mathrm{h}\mathcal{C}$ of an ∞-category $\mathcal{C}$ can be upgraded to an $\mathcal{H}$-*enriched* category, as follows:

For any objects x and y in $\mathcal{C}$, one considers not only maps with source x and target y, but all "arcs" of n-simplexes ($n \geq 0$) that go from the trivial n-simplex $\Delta^n \to \Delta^0 \xrightarrow{x} \mathcal{C}$ to the trivial n-simplex $\Delta^n \to \Delta^0 \xrightarrow{y} \mathcal{C}$ — more precisely, maps $\theta \colon \Delta^1 \times \Delta^n \to \mathcal{C}$ such that the compositions

$$\Delta^n = \Delta^0 \times \Delta^n \xrightarrow{j \times \mathrm{id}} \Delta^1 \times \Delta^n \xrightarrow{\theta} \mathcal{C} \quad (j = 0, 1)$$

are the unique n-simplices in the constant simplicial sets $\{x\}$ and $\{y\}$ respectively. Such θ are the n-simplices of a Kan subcomplex $M_{x,y}$ of the "function complex" $\mathbf{Hom}(\Delta^1, \mathcal{C})$ [Goerss and Jardine 1999, Section I.5]. The *mapping space* $\mathrm{Map}_{\mathcal{C}}(x, y)$ is the geometric realization of $M_{x,y}$. It is a CW-complex [loc. cit., Section I.2]. For objects $x, y, z \in \mathcal{C}$, there is in $\mathcal{H}$ a composition map

$$\mathrm{Map}_{\mathcal{C}}(y, z) \times \mathrm{Map}_{\mathcal{C}}(x, y) \longrightarrow \mathrm{Map}_{\mathcal{C}}(x, z),$$

that, unfortunately, is not readily describable (see [Lurie 2009, pages 27–28, 1.2.2.4, 1.2.2.5]); and this composition satisfies associativity.

The unenriched homotopy category is the underlying ordinary category, obtained by replacing each $\mathrm{Map}_{\mathcal{C}}(x, y)$ by the set $\pi_0 \mathrm{Map}_{\mathcal{C}}(x, y)$ of its connected components.

All page and section numbers in the next example refer to [Lurie 2009].

Example 6.2. When $\mathcal{C} = \mathrm{N}(C)$ for an ordinary category C, the preceding discussion is pointless: the spaces $\mathrm{Map}_{\mathcal{C}}(x, y)$ are isomorphic in $\mathcal{H}$ to discrete topological spaces (see, e.g., p. 22, 1.1.5.8; p. 25, 1.1.5.13), so that the $\mathcal{H}$-enhancement of $h\mathrm{N}(C)$ is trivial; and one checks that the counit map is an isomorphism of ordinary categories $h\mathrm{N}(C) \xrightarrow{\sim} C$.

More generally — and much deeper, for any topological category $\mathcal{D}$ and any simplicial category $\mathcal{C}$ that is *fibrant* — that is, all its mapping complexes are Kan complexes, one has natural $\mathcal{H}$-enriched isomorphisms

$$h\mathrm{N}_{\mathrm{top}}(\mathcal{D}) \xrightarrow{\sim} h\mathcal{D} \quad \text{and} \quad h\mathrm{N}_{\Delta}(\mathcal{C}) \xrightarrow{\sim} h\mathcal{C},$$

where $\mathrm{N}_{\mathrm{top}}(\mathcal{D})$ is the topological nerve of $\mathcal{D}$ (an ∞-category: p. 24, 1.1.5.12) and $\mathrm{N}_{\Delta}(\mathcal{C})$ is the simplicial nerve of $\mathcal{C}$ (an ∞-category: p. 23, 1.1.5.10), where the topological homotopy category $h\mathcal{D}$ is obtained from $\mathcal{D}$ by replacing each topological space $\mathrm{Map}_{\mathcal{D}}(x, y)$ by a weakly homotopically equivalent CW complex considered as an object of $\mathcal{H}$ (p. 16, 1.1.3.4), and the simplicial homotopy category $h\mathcal{C}$ is obtained from $\mathcal{C}$ by replacing each simplicial set $\mathrm{Map}_{\mathcal{C}}(x, y)$ by its geometric realization considered as an object of $\mathcal{H}$ (see p. 19). Using the description of the homotopy category of a simplicial set given in p. 25, 1.1.5.14, one finds that the first isomorphism is essentially that of p. 25, 1.1.5.13; and likewise, the second is essentially that of p. 72, 2.2.0.1.

6.3. A functor $F \colon \mathcal{C}_1 \to \mathcal{C}_2$ between two ∞-categories induces a functor

$$\mathrm{h}F \colon \mathrm{h}\mathcal{C}_1 \to \mathrm{h}\mathcal{C}_2$$

of $\mathcal{H}$-enriched categories: the functor $\mathrm{h}F$ has the same effect on objects as F does, and there is a natural family of $\mathcal{H}$-maps

$$\mathrm{h}F_{x,y} \colon \mathrm{Map}_{\mathcal{C}_1}(x, y) \to \mathrm{Map}_{\mathcal{C}_2}(Fx, Fy) \quad (x, y \text{ objects in } \mathcal{C}_1)$$

that respects composition (for whose existence see [Lurie 2009, p. 25, 1.1.5.14 and p. 27, 1.2.2.4].)

The functor F is called a *categorical equivalence* if for all x and y, $\mathrm{h}F_{x,y}$ is a homotopy equivalence ($=$ isomorphism in $\mathcal{H}$), and for every object $z \in \mathcal{C}_2$, there exists an object $x \in \mathcal{C}_1$ and a map $f \colon z \to Fx$ whose image in $\mathrm{h}\mathcal{C}_2$ is an isomorphism.

Example 6.4. For any ∞-category $\mathcal{C}$, the unit map $\mathcal{C} \to \mathrm{N}(\mathrm{h}\mathcal{C})$ induces an isomorphism of ordinary homotopy categories (because N and h are adjoint, so that $\mathrm{h}N\mathrm{h} = \mathrm{h}$); but it is a categorical equivalence only when the mapping spaces of $\mathcal{C}$ are isomorphic in $\mathcal{H}$ to discrete topological spaces, that is, their connected components are all contractible (because this holds for the mapping spaces of $\mathrm{N}(\mathrm{h}\mathcal{C})$).

The "interesting" properties of ∞-categories are those which are invariant under categorical equivalence. In other words, the $\mathcal{H}$-enriched homotopy category is the fundamental invariant of an ∞-category $\mathcal{C}$; the role of $\mathcal{C}$ itself is to generate information about this invariant.

For this purpose, $\mathcal{C}$ can be replaced by any equivalent ∞-category, that is, an ∞-category that can be joined to $\mathcal{C}$ by a chain of equivalences (or even by equivalent topological or simplicial categories, as explained in [loc. cit., Section 1.1], and illustrated by Example 6.2 above). Analogously, one can think of a single homology theory in topology or algebra being constructed in various different ways.

Along these lines, a *map* in $\mathcal{C}$ is called an equivalence if the induced map in $\mathrm{h}\mathcal{C}$ is an isomorphism; and the interesting properties of objects in $\mathcal{C}$ are those which are invariant under equivalence.

Example 6.5. An ∞-category $\mathcal{C}$ is a Kan complex (Example 4.2.3) if and only if every map in $\mathcal{C}$ is an equivalence, that is, $\mathrm{h}\mathcal{C}$ is a groupoid [loc. cit., Section 1.2.5]. For a Kan complex $\mathcal{C}$, $\mathrm{h}\mathcal{C}$ is the *fundamental groupoid* of $\mathcal{C}$ (or of its geometric realization), see [loc. cit., p. 3, 1.1.1.4].

7. Colimits

To motivate the definition of colimits in ∞-categories, recall that a colimit of a functor $\tilde{p}\colon K \to C$ of ordinary categories is an initial object in the category $C_{\tilde{p}/}$ whose objects are the extensions of $\tilde{p}$ to the right cone $K^{\triangleright}$ — that is, the disjoint union of K and the trivial category $*$ (the category with just one map) together with one arrow from each object of K to the unique object of $*$ — and whose maps are the obvious ones.

Let us now reformulate this remark in the language of ∞-categories. (A fuller discussion appears in [Lurie 2009, Sections 1.2.8, 1.2.12 and 1.2.13].)

First, an *initial object* in an ∞-category $\mathcal{C}$ is an object $x \in \mathcal{C}$ such that for every object $y \in \mathcal{C}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(x, y)$ is contractible. It is equivalent to say that x is an initial object in the $\mathcal{H}$-enriched homotopy category $h\mathcal{C}$. Thus any two initial objects in $\mathcal{C}$ are equivalent. (In fact, if nonempty, the set of initial objects in $\mathcal{C}$ spans a contractible Kan subcomplex of $\mathcal{C}$ [loc. cit., p. 46, 1.2.12.9].)

Next, calculation of the nerve of the above right cone $K^{\triangleright}$ suggests the following definition. For *any* simplicial set K, the *right cone* $K^{\triangleright}$ is the simplicial set whose set of n-simplices $K_n^{\triangleright}$ is the disjoint union of all the sets K_m ($m \leq n$) and Δ_n^0 (the latter having a single member $*_n$), with the face maps d_j — when $n > 0$ — (resp. degeneracy maps s_j) restricting on K_m to the usual face (resp. degeneracy) maps for $0 \leq j \leq m$ (except that d_0 maps all of K_0 to $*_{n-1}$), and to identity maps for $m < j \leq n$, and taking $*_n$ to $*_{n-1}$ (resp. $*_{n+1}$). It may help here to observe that for $n > 0$, the nondegenerate n-simplices in $K^{\triangleright}$ are just the nondegenerate n-simplices in K_n together with the nondegenerate $(n-1)$ simplices in K_{n-1}, the latter visualized as being joined to the "vertex" $*_n$.

This is a special case of the construction (which we'll not need) of the *join* of two simplicial sets [loc. cit., Section 1.2.8]. The join of two ∞-categories is an ∞-category [loc. cit., p. 41, 1.2.8.3]; thus if K is an ∞-category then so is $K^{\triangleright}$.

Define $K^{\triangleright^n}$ inductively by $K^{\triangleright^1} := K^{\triangleright}$ and (for $n > 1$) $K^{\triangleright^n} := (K^{\triangleright^{n-1}})^{\triangleright}$. There is an obvious embedding of K into $K^{\triangleright}$, and hence into $K^{\triangleright^n}$. For a map $p\colon K \to \mathcal{C}$ of ∞-categories, the corresponding *undercategory* $\mathcal{C}_{p/}$ is a simplicial set whose $(n-1)$-simplices ($n > 0$) are the extensions of p to maps $K^{\triangleright^n} \to \mathcal{C}$, see [loc. cit., p. 43, 1.2.9.5]. This undercategory is an ∞-category [loc. cit., p. 61, 2.1.2.2].

Definition 7.1. A *colimit* of a map $p\colon K \to \mathcal{C}$ of ∞-categories is an initial object in the ∞-category $\mathcal{C}_{p/}$.

Being an object (= 0-simplex) in $\mathcal{C}_{p/}$, any colimit of p is an extension of p to a map $\bar{p}\colon K^{\triangleright} \to \mathcal{C}$. Often one refers loosely to the image under $\bar{p}$ of $*_0 \in K^{\triangleright}$ as the colimit of p.

Some instances of colimits are the ∞-categorical versions of *coproducts* (where K is the nerve of a category whose only maps are identity maps), *pushouts* (where K is the horn Λ_0^2), and *coequalizers* (where K is the nerve of a category with exactly two objects x_1 and x_2, and such that $\mathrm{Map}(x_i, x_j)$ has cardinality $j - i + 1$); see [Lurie 2009, Section 4.4].

Example 7.2. Suppose $\mathcal{C}$ is the nerve $\mathrm{N}(C)$ of an ordinary category C. A functor $p\colon K \to \mathcal{C}$ corresponds under the adjunction $\mathrm{h} \dashv \mathrm{N}$ to a functor $\tilde{p}\colon \mathrm{h}K \to C$. There is a natural isomorphism of ordinary categories

$$\mathrm{h}(K^{\triangleright}) \cong (\mathrm{h}K)^{\triangleright},$$

whence an extension of p to $K^{\triangleright}$ corresponds under $\mathrm{h} \dashv \mathrm{N}$ to an extension of $\tilde{p}$ to $(\mathrm{h}K)^{\triangleright}$. More generally, one checks that there is a natural isomorphism

$$\mathcal{C}_{p/} = \mathrm{N}(C)_{p/} \cong \mathrm{N}(C_{\tilde{p}/}).$$

Any colimit of p is an initial object in $\mathrm{h}\mathcal{C}_{p/} \cong \mathrm{h}\mathrm{N}(C_{\tilde{p}/}) \cong C_{\tilde{p}/}$; that is, *the homotopy functor takes a colimit of p to a colimit of $\tilde{p}$*.

For more general $\mathcal{C}$, and most p, the homotopy functor does not preserve colimits. For example, in any *stable* ∞-category, like the derived ∞-category of a Grothendieck abelian category [Lurie 2014, p. 96, 1.3.5.9], the pushout of 0 with itself over an object X is the suspension $X[1]$ (see [loc. cit., p. 19, bottom paragraph]), but the pushout in the homotopy category is 0.

8. Adjoint functors

For a pair of functors (= simplicial maps) $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{g} \mathcal{C}$ of ∞-categories one says that *f is a left adjoint of g*, or that *g is a right adjoint of f*, if there exists a homotopy u from the identity functor $\mathrm{id}_{\mathcal{C}}$ to gf (that is, a simplicial map $u\colon \mathcal{C} \times \Delta^1 \to \mathcal{C}$ whose compositions with the maps

$$\mathcal{C} = \mathcal{C} \times \Delta^0 \xrightarrow{\{i\}} \mathcal{C} \times \Delta^1 \quad (i = 0, 1)$$

corresponding to the 0-simplices $\{0\}$ and $\{1\}$ of Δ^1 are $\mathrm{id}_{\mathcal{C}}$ and gf, respectively) such that, for all objects $C \in \mathcal{C}$ and $D \in \mathcal{D}$, the natural composition

$$\mathrm{Map}_{\mathcal{D}}(f(C), D) \to \mathrm{Map}_{\mathcal{C}}(gf(C), g(D)) \xrightarrow{u(C)} \mathrm{Map}_{\mathcal{C}}(C, g(D))$$

is an isomorphism in $\mathcal{H}$.

(For an extensive discussion of adjunction, see [Lurie 2009, Section 5.2]. The foregoing definition comes from [loc. cit., p. 340, 5.2.2.8].)

Such adjoint functors f and g induce adjoint functors $\mathrm{h}\mathcal{C} \xrightarrow{\mathrm{h}f} \mathrm{h}\mathcal{D} \xrightarrow{\mathrm{h}g} \mathrm{h}\mathcal{C}$ between the respective $\mathcal{H}$-enriched homotopy categories.

As a partial converse, it holds that if the functor $\mathrm{h}f$ induced by a functor $f\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories has an $\mathcal{H}$-enriched right adjoint, then f itself has a right adjoint [Lurie 2009, p. 342, 5.2.2.12].

The following *adjoint functor theorem* gives a powerful criterion (to be used subsequently) for $f\colon \mathcal{C} \to \mathcal{D}$ to have a right adjoint. It requires a restriction — *accessibility* — on the sizes of the ∞-categories $\mathcal{C}$ and $\mathcal{D}$. This means roughly that $\mathcal{C}$ is generated under filtered colimits by a small ∞-subcategory, and similarly for $\mathcal{D}$, see [loc. cit., Chap. 5]. (If necessary, see also [loc. cit., p. 51] for the explication of "small" in the context of Grothendieck universes.) Also, $\mathcal{C}$ and $\mathcal{D}$ need to admit colimits of all maps they receive from small simplicial sets K. The conjunction of these properties is called *presentability* [loc. cit., Section 5.5].

For example, the ∞-category $\mathcal{S}$ of spaces (see Example 4.2.2) is presentable [loc. cit., p. 460, 5.5.1.8]. It follows that the ∞-category of spectra $\mathrm{Sp} := \mathrm{Sp}(\mathcal{S}_*)$ (see [Lurie 2014, p. 116, 1.4.2.5 and p. 122, 1.4.3.1]) is presentable. Indeed, presentability is an equivalence-invariant property (see, e.g., [Lurie 2009, p. 457, 5.5.1.1(4)]), hence by the presentability of $\mathcal{S}$ and by [loc. cit., p. 719, 7.2.2.8; p. 242, 4.2.1.5; p. 468, 5.5.3.11], $\mathcal{S}_*$ is presentable, whence, by [Lurie 2014, p. 127, 1.4.4.4], so is Sp.

Theorem 8.1 [Lurie 2009, p. 465, 5.5.2.9]. *A functor $f\colon \mathcal{C} \to \mathcal{D}$ between presentable ∞-categories has a right adjoint if and only if it preserves small colimits.*

8.2. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. The simplicial set $\mathbf{Hom}(\mathcal{C}, \mathcal{D})$ [Goerss and Jardine 1999, Section I.5] is an ∞-category, denoted $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ [Lurie 2009, p. 39, 1.2.7.3]. Its 0-simplices are functors (= simplicial maps). Its 1-simplices are *homotopies* between functors, that is, simplicial maps $\phi\colon \mathcal{C} \times \Delta^1$ to $\mathcal{D}$ such that the following functor is f when $i = 0$ and g when $i = 1$:

$$\mathcal{C} = \mathcal{C} \times \Delta^0 \xrightarrow{\mathrm{id} \times i} \mathcal{C} \times \Delta^1 \xrightarrow{\phi} \mathcal{D}.$$

Let $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ (resp. $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}, \mathcal{D})$) be the full ∞-subcategories spanned by the functors which are left (resp. right) adjoints, that is, the ∞-categories whose simplices are all those in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ whose vertices are such functors.

The *opposite* $\mathcal{E}^{\mathrm{op}}$ of an ∞-category $\mathcal{E}$ [loc. cit., Section 1.2.1] is the simplicial set having the same set $\mathcal{E}_n$ of n-simplices as $\mathcal{E}$ for all $n \geq 0$, but with face and degeneracy operators

$$(d_i \colon \mathcal{E}_n^{\mathrm{op}} \to \mathcal{E}_{n-1}^{\mathrm{op}}) := (d_{n-i} \colon \mathcal{E}_n \to \mathcal{E}_{n-1}),$$
$$(s_i \colon \mathcal{E}_n^{\mathrm{op}} \to \mathcal{E}_{n+1}^{\mathrm{op}}) := (s_{n-i} \colon \mathcal{E}_n \to \mathcal{E}_{n+1}).$$

It is immediate that $\mathcal{E}^{\mathrm{op}}$ is also an ∞-category.

The next result, when restricted to ordinary categories, underlies the notion of *conjugate functors* (see, e.g., [Lipman 2009, 3.3.5–3.3.7].) It plays a role in Theorem 12.1 below.

Proposition 8.3. *There is a canonical* (*up to homotopy*) *equivalence*

$$\varphi \colon \mathrm{Fun}^R(\mathcal{C}, \mathcal{D}) \xrightarrow{\approx} \mathrm{Fun}^L(\mathcal{D}, \mathcal{C})^{\mathrm{op}}. \qquad (8.3.1)$$

that takes any object $g \colon \mathcal{C} \to \mathcal{D}$ *in* $\mathrm{Fun}^R(\mathcal{C}, \mathcal{D})$ *to a left-adjoint functor* g'.

9. Algebra objects in monoidal ∞-categories

A *monoidal category* $\mathcal{M}$ is a category together with a monoidal structure, that is, a *product functor* $\otimes \colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ that is associative up to isomorphism, plus a *unit object* $\mathcal{O}$ and isomorphisms (unit maps)

$$\mathcal{O} \otimes M \ \xrightarrow{\sim}\ M \ \xrightarrow{\sim}\ M \otimes \mathcal{O} \quad (M \in \mathcal{M})$$

compatible with the associativity isomorphisms.

An *associative algebra* $A \in \mathcal{M}$ ($\mathcal{M}$-algebra for short) is an object equipped with maps $A \otimes A \to A$ (multiplication) and $\mathcal{O} \to A$ (unit) satisfying associativity etc. *up to isomorphism*, such isomorphisms having the usual relations, expressed by commutative diagrams.

(No additive structure appears here, so one might be tempted to call algebras "monoids". However, that term is reserved in [Lurie 2014, Section 2.4.2] for a related, but different, construct.)

Examples 9.1. (a) $\mathcal{M} := \{\mathrm{Sets}\}$, $\otimes$ is the usual direct product, and $\mathcal{M}$-algebras are monoids.

(b) $\mathcal{M} := $ modules over a fixed commutative ring $\mathcal{O}$, $\otimes$ is the usual tensor product over $\mathcal{O}$, and $\mathcal{M}$-algebras are the usual $\mathcal{O}$-algebras.

(c) $\mathcal{M} := $ dg modules over a fixed commutative dg ring $\mathcal{O}$, $\otimes$ is the usual tensor product of dg $\mathcal{O}$-modules, and an $\mathcal{M}$-algebra is a dg $\mathcal{O}$-algebra (i.e., a dg ring A plus a homomorphism of dg rings from $\mathcal{O}$ to the center of A).

(d) $\mathcal{M} := $ the derived category $\mathbf{D}(X)$ of $\mathcal{O}$-modules over a (commutative) ringed space $(X, \mathcal{O})$, $\otimes$ is the derived tensor product of $\mathcal{O}$-complexes. Any dg $\mathcal{O}$-algebra gives rise to an $\mathcal{M}$-algebra; but there might be $\mathcal{M}$-algebras not of this kind, as the defining diagrams may now involve quasi-isomorphisms and homotopies, not just equalities.

The foregoing notions can be extended to ∞-categories. The key is to formulate how algebraic structures in categories arise from *operads,* in a way that can be upgraded to ∞-categories and ∞-operads. Details of the actual implementation are not effortless to absorb. (See [loc. cit., Section 4.1].)

The effect is to replace isomorphism by "coherent homotopy". Whatever this means (see [Lurie 2009, Section 1.2.6]), it turns out that any monoidal structure on an ∞-category $\mathcal{C}$ induces a monoidal structure on the ordinary category $h\mathcal{C}$, and any $\mathcal{C}$-algebra (very roughly: an object with multiplication associative up to coherent homotopy) is taken by the homotopy functor to an $h\mathcal{C}$-algebra [Lurie 2014, p. 332, 4.1.1.12; 4.1.1.13].

The $\mathcal{C}$-algebras are the objects of an ∞-category $\mathrm{Alg}(\mathcal{C})$ [loc. cit., p. 331, 4.1.1.6]. The point is that the homotopy-coherence of the associativity and unit maps are captured by an ∞-category superstructure.

Similar remarks apply to *commutative* $\mathcal{C}$-algebras, that is, $\mathcal{C}$-algebras whose multiplication is commutative up to coherent homotopy.

Example 9.2. In the monoidal ∞-category of spectra [loc. cit., Sections 1.4.3, 6.3.2], algebras are called A_∞-rings, or A_∞-ring spectra; and commutative algebras are called $\mathbb{E}_\infty$-rings, or $\mathbb{E}_\infty$-ring spectra. The *discrete* A_∞-(resp. $\mathbb{E}_\infty$-) rings — those algebras S whose homotopy groups $\pi_i S$ vanish for $i \neq 0$ — span an ∞-category that is equivalent to (the nerve of) the category of associative (resp. commutative) rings [loc. cit., p. 806, 8.1.0.3].

In general, for any commutative ring R, there is a close relation between dg R-algebras and A_∞-R-algebras; see [Lurie 2014, p. 824, 8.1.4.6; Shipley 2007, Theorem 1.1]; and when R contains the rational field $\mathbb{Q}$, between graded-commutative dg R-algebras and $\mathbb{E}_\infty$-R-algebras [Lurie 2014, p. 825, 8.1.4.11]; in such situations, every A_∞-(resp. $\mathbb{E}_\infty$-)R-algebra is equivalent to a dg R-algebra.

See also [loc. cit., Section 4.1.4] for more examples of ∞-category-algebras that have concrete representatives.

10. Bimodules, tensor product

10.1. For algebra objects A and B in a monoidal ∞-category $\mathcal{C}$, there is a notion of A-B-*bimodule* — an object in $\mathcal{C}$ on which, via $\otimes$ product in $\mathcal{C}$, A acts on the left, B on the right and the actions commute up to coherent homotopy. (No additive structure is required.) The bimodules in $\mathcal{C}$ are the objects of an ∞-category $_A\mathrm{BMod}_B(\mathcal{C})$, to be denoted here, once the ∞-category $\mathcal{C}$ is fixed, as $A\#B$. (See [Lurie 2014, Section 4.3].)

10.2. Let A, B, C be algebras in a monoidal ∞-category $\mathcal{C}$ that admits small colimits, and in which product functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable. There is a tensor-product functor

$$(A\#B) \times (B\#C) \xrightarrow{\otimes} A\#C, \tag{10.2.1}$$

defined to be the *geometric realization* — a kind of colimit (see [Lurie 2009, p. 542, 6.1.2.12]) of a two-sided bar-construction. (See [Lurie 2014, p. 409, 4.3.5.11]; and for some more motivation, [Lurie 2007, pp.145–146].)

Tensor product is associative up to canonical homotopy [Lurie 2014, p. 416, 4.3.6.14]. It is unital on the left in the sense that, roughly, the endofunctor of $B \# C$ given by tensoring on the left with the B-B-bimodule B is canonically homotopic to the identity; and similarly on the right [loc. cit., p. 417, 4.3.6.16]. Also, it preserves colimits separately in each variable [loc. cit., p. 411, 4.3.5.15].

Example 10.3 (Musings). It is natural to ask about direct connections between (10.2.1) and the usual tensor product of bimodules over rings. If there is an explicit answer in the literature I haven't found it, except when $A = B = C$ is an ordinary commutative ring regarded as a discrete $\mathbb{E}_\infty$-ring, a case addressed by [loc. cit., p. 817, 8.1.2.13] (whose proof might possibly be adaptable to a more general situation).

What follows are some related remarks, in the language of model categories, which in the present context can presumably be translated into the language of ∞-categories. (Compare, e.g., [loc. cit., p. 90, 1.3.4.21, and p. 824, 8.1.4.6].)

Let $\mathcal{M}$ be the (ordinary) symmetric monoidal category of abelian groups, and let A, B, C be $\mathcal{M}$-algebras, that is, ordinary rings (set $\mathcal{O} := \mathbb{Z}$ in Example 9.1(b)). The tensor product over B of an $A \otimes B^{\mathrm{op}}$-complex (i.e., a left A- right B-complex, or A-B-bicomplex) and a $B \otimes C^{\mathrm{op}}$-complex is an $A \otimes C^{\mathrm{op}}$-complex. *Can this bifunctor be extended to a derived functor*

$$\mathbf{D}(A \otimes B^{\mathrm{op}}) \times \mathbf{D}(B \otimes C^{\mathrm{op}}) \to \mathbf{D}(A \otimes C^{\mathrm{op}})?$$

The problem is to construct, in $\mathbf{D}(A \otimes C^{\mathrm{op}})$, a derived tensor product, over B, of an A-B-bicomplex X and a B-C-bicomplex Y. As in the classical situation where $C = \mathbb{Z}$, this requires building something like a quasi-isomorphism $f \colon Y \to Y'$ of B-complexes with Y' flat over B; but now *f must be compatible with the right C-action.* How can this be done?

To deal with the question it seems necessary to move out into the dg world. Enlarge $\mathcal{M}$ to the category of complexes of abelian groups, made into a symmetric monoidal model category by the usual tensor product and the "projective" model structure (weak equivalences being quasi-isomorphisms, and fibrations being surjections), see [loc. cit., p. 816, 8.1.2.11]. It results from [Schwede and Shipley 2000, Theorem 4.1(1)] (for whose hypotheses see [Shipley 2007, p. 356, Section 2.2 and p. 359, Proposition 2.9]) that:

(1) For any $\mathcal{M}$-algebra (i.e., dg ring) S, the category $\mathcal{M}_S \subset \mathcal{M}$ of left dg S-modules has a model structure for which maps are weak equivalences (resp. fibrations) if and only if they are so in $\mathcal{M}$.

(2) For any commutative $\mathcal{M}$-algebra (i.e., graded-commutative dg ring) R, the category of R-algebras in $\mathcal{M}$ has a model structure for which maps are weak equivalences (resp. fibrations) if and only if they are so in $\mathcal{M}$.

In either case (1) or (2), the *cofibrant* objects are those I such that for any diagram $I \xrightarrow{f} Y \xleftarrow{s} X$ with s a surjective quasi-isomorphism, there exists (in the category in play) $g \colon I \to X$ such that $sg = f$. Any object Z in these model categories is the target of a quasi-isomorphism $\widetilde{Z} \to Z$ with cofibrant $\widetilde{Z}$; such a quasi-isomorphism (or its source) is called a *cofibrant replacement* of Z.

Note that the derived category $\mathbf{D}(S)$ — obtained by adjoining to $\mathcal{M}_S$ formal inverses of its quasi-isomorphisms — is the homotopy category of the model category $\mathcal{M}_S$.

Now fix a graded-commutative dg ring R. The *derived tensor product* $S \otimes_R^{\mathsf{L}} T$ of two dg R-algebras S, T is the tensor product $\widetilde{S} \otimes_R \widetilde{T}$.[4] This construction depends, up to quasi-isomorphism, on the choice of the cofibrant replacements. However, two such derived tensor products have canonically isomorphic derived categories [Schwede and Shipley 2000, Theorem 4.3]. Any such derived category will be denoted $\mathbf{D}(S \otimes_R^{\mathsf{L}} T)$.

If either S or T is *flat* over R then the natural map $S \otimes_R^{\mathsf{L}} T \to S \otimes_R T$ is a quasi-isomorphism; in this case one need not distinguish between the derived and the ordinary tensor product.

More generally, let S and T be dg R-algebras, let M be a dg S-module and N a dg T-module. Let $\widetilde{S} \to S$ and $\widetilde{T} \to T$ be cofibrant replacements. Let $\widetilde{M} \to M$ (resp. $\widetilde{N} \to N$) be a cofibrant replacement in the category of dg $\widetilde{S}$-(resp. $\widetilde{T}$-)modules. Then $\widetilde{M} \otimes_R \widetilde{N}$ is a dg module over $\widetilde{S} \otimes_R \widetilde{T}$. Using "functorial factorizations" [Hovey 1999, Definition 1.1.3], one finds that this association of $\widetilde{M} \otimes_R \widetilde{N}$ to (M, N) gives rise to a functor

$$\mathbf{D}(S) \times \mathbf{D}(T) \to \mathbf{D}(S \otimes_R^{\mathsf{L}} T),$$

and that different choices of cofibrant replacements lead canonically to isomorphic functors.

If A, B and C are dg R-algebras, with B commutative, then setting $S := A \otimes_R B$ and $T := B \otimes_R C^{\mathrm{op}}$, one gets, as above, a functor

$$\mathbf{D}(A \otimes_R B) \times \mathbf{D}(B \otimes_R C^{\mathrm{op}}) \to \mathbf{D}\big((A \otimes_R B) \otimes_B^{\mathsf{L}} (B \otimes_R C^{\mathrm{op}})\big).$$

Then, via restriction of scalars through the natural map

$$\mathbf{D}(A \otimes_R^{\mathsf{L}} C^{\mathrm{op}}) \to \mathbf{D}\big((A \otimes_R B) \otimes_B^{\mathsf{L}} (B \otimes_R C^{\mathrm{op}})\big),$$

---

[4]This is an instance of the passage from a monoidal structure on a model category $\mathcal{M}$ to one on the homotopy category of $\mathcal{M}$ [Hovey 1999, Section 4.3] — a precursor of the passage from a monoidal structure on an ∞-category to one on its homotopy category.

one gets a version of the desired functor, of the form

$$\mathbf{D}(A \otimes_R B) \times \mathbf{D}(B \otimes_R C^{\mathrm{op}}) \to \mathbf{D}(A \otimes_R^{\mathsf{L}} C^{\mathrm{op}}).$$

What does this functor have to do with the tensor product of 10.2?

Here is an approach that should lead to an answer; but details need to be worked out.

Restrict R to be an ordinary commutative ring, and again, B to be graded-commutative. By [Shipley 2007, 2.15] there is a zigzag $\mathbb{H}$ of three "weak monoidal Quillen equivalences" between the model category of dg R-modules (i.e., R-complexes) and the model category of symmetric module spectra over the Eilenberg–Mac Lane symmetric spectrum HR (see, e.g., [Greenlees 2007, 4.16]), that induces a monoidal equivalence between the respective homotopy categories. (The monoidal structures on the model categories are given by the tensor and smash products, respectively; see, e.g., [Schwede 2012, Chapter 1, Theorem 5.10].) For A-B-bimodules M and B-C-bimodules N, the tensor product $M \otimes_B N$ coequalizes the natural maps $M \otimes_R B \otimes_R N \rightrightarrows M \otimes_R N$, and likewise for the smash product of HM and HN over HB; so it should follow that when M and N are cofibrant, these products also correspond, up to homotopy, under $\mathbb{H}$. This would reduce the problem, modulo homotopy, to a comparison of the smash product and the relative tensor product in the associated ∞-category of the latter category. But it results from [Shipley 2000, 4.9.1] that these bifunctors become naturally isomorphic in the homotopy category of symmetric spectra, that is, the classical stable homotopy category.

For a parallel approach, based on "sphere-spectrum-modules" rather than symmetric spectra, see [Elmendorf et al. 1997, Section IV.2 and Proposition IX.2.3].

Roughly speaking, then, any homotopical — that is, equivalence-invariant — property of relative tensor products in the ∞-category of spectra (whose homotopy category is the stable homotopy category) should entail a property of derived tensor products of dg modules or bimodules over appropriately commutative (or not) dg R-algebras.

11. The ∞-functor $\mathscr{H}om$

One shows, utilizing [Lurie 2014, p. 391, 4.3.3.10], that if $\mathcal{C}$ is presentable then so is $A\#B$. Then one can apply the adjoint functor Theorem 8.1 to prove:

Proposition 11.1. *Let A, B, C be algebras in a fixed presentable monoidal ∞-category. There exists a functor*

$$\mathscr{H}om_C \colon (B\#C)^{\mathrm{op}} \times A\#C \to A\#B,$$

such that for every fixed $y \in B\#C$, *the functor*

$$z \mapsto \mathscr{H}om_C(y, z) \colon A\#C \to A\#B$$

is right-adjoint to the functor $x \mapsto x \otimes_B y \colon A\#B \to A\#C$.

As adjoint functors between ∞-categories induce adjoint functors between the respective homotopy categories, and by unitality of tensor product, when $x = A = B$ one gets:

Corollary 11.2 ("global sections" of $\mathscr{H}om$ = mapping space). *There exists an* $\mathcal{H}$-*isomorphism of functors* (*going from* $\mathrm{h}(A\#C)^{\mathrm{op}} \times \mathrm{h}(A\#C)$ *to* $\mathcal{H}$):

$$\mathrm{Map}_{A\#A}(A, \mathscr{H}om_C(y, z)) \xrightarrow{\sim} \mathrm{Map}_{A\#C}(A \otimes_A y, z) \cong \mathrm{Map}_{A\#C}(y, z).$$

12. Adjoint associativity in ∞-categories

We are finally in a position to make sense of adjoint associativity for ∞-categories. The result and proof are similar in spirit to, if not implied by, those in [Lurie 2014, p. 358, 4.2.1.31 and 4.2.1.33(2)] about "morphism objects".

Theorem 12.1. *There is in* $\mathrm{Fun}((A\#B)^{\mathrm{op}} \times (B\#C)^{\mathrm{op}} \times D\#C, \ D\#A)$ *a functorial equivalence* (*canonically defined*, *up to homotopy*)

$$\alpha(x, y, z) \colon \mathscr{H}om_C(x \otimes_B y, z) \to \mathscr{H}om_B(x, \mathscr{H}om_C(y, z)),$$

such that for any objects $x \in A\#B$ *and* $y \in B\#C$, *the map* $\alpha(x, y, -)$ *in* $\mathrm{Fun}^{\mathrm{R}}(D\#C, D\#A)$ *is taken by the equivalence* (8.3.1) *to the associativity equivalence*, *in* $\mathrm{Fun}^{\mathrm{L}}(D\#A, D\#C)^{\mathrm{op}}$,

$$- \otimes_A (x \otimes_B y) \leftarrow (- \otimes_A x) \otimes_B y.$$

Using Corollary 11.2, one deduces:

Corollary 12.2. *In the homotopy category of spaces there is a trifunctorial isomorphism*

$$\mathrm{Map}_{A\#C}(x \otimes_B y, z) \xrightarrow{\sim} \mathrm{Map}_{A\#B}(x, \mathscr{H}om_C(y, z))$$

$$(x \in A\#B, \ y \in B\#C, \ z \in A\#C).$$

12.3. What conclusions about ordinary algebra can we draw?

Let us confine attention to spectra, and try to understand the homotopy invariants of the mapping spaces in the preceding corollary, in particular the maps in the corresponding unenriched homotopy categories (see last paragraph in 6.1).

As in Example 10.3, the following remarks outline a possible approach, whose details I have not completely verified.

Let R be an ordinary commutative ring. Let S and T be dg R-algebras, and U a derived tensor product $U := S \otimes_R^{\mathsf{L}} T^{\mathsf{op}}$ (see Example 10.3). For any dg R-module V, let $\widehat{V}$ be the canonical image of $\mathbb{H}V$ ($\mathbb{H}$ as in Example 10.3) in the associated ∞-category $\mathcal{D}$ of the model category $\mathbf{A}$ of HR-modules. The ∞-category $\mathcal{D}$ is monoidal via a suitable extension, denoted $\wedge$, of the smash product; see [Lurie 2014, p. 619, (S1)]. Recall from the second-last paragraph in Example 5.5 that the homotopy category $h\mathcal{D}$ is equivalent to the homotopy category of $\mathbf{A}$.

As indicated toward the end of Example 10.3, *there should be,* in $\mathcal{D}$, an equivalence

$$\widehat{U} \xrightarrow{\approx} \widehat{S} \wedge \widehat{T^{\mathsf{op}}}.$$

whence, by [loc. cit., p. 650, 6.3.6.12], an equivalence

$$\widehat{S} \# \widehat{T} \simeq \mathrm{LMod}_{\widehat{U}},$$

whence, for any dg S-T bimodules a and b, and $i \in \mathbb{Z}$, isomorphisms in the homotopy category $\mathcal{H}$ of spaces

$$\mathrm{Map}_{\widehat{S}\#\widehat{T}}(\widehat{a}, \widehat{b[i]}) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{LMod}_{\widehat{U}}}(\widehat{a}, \widehat{b[i]}). \qquad (12.3.1)$$

(For the hypotheses of [loc. cit., p. 650, 6.3.6.12], note that the ∞-category Sp of spectra, being presentable, has small colimits — see the remarks preceding Theorem 8.1; and these colimits are preserved by smash product [loc. cit., p. 623, 6.3.2.19].)

By [loc. cit., p. 393, 4.3.3.17] and again, [loc. cit., p. 650, 6.3.6.12], the stable ∞-category $\mathrm{LMod}_{\widehat{U}}$ is equivalent to the associated ∞-category of the model category of HU-module spectra, and hence to the associated ∞-category $\mathcal{U}$ of the equivalent model category of dg U-modules. There results an $\mathcal{H}$-isomorphism

$$\mathrm{Map}_{\mathrm{LMod}_{\widehat{U}}}(\widehat{a}, \widehat{b[i]}) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{U}}(a, b[i]). \qquad (12.3.2)$$

Since the homotopy category $h\mathrm{LMod}_{\widehat{U}}$ is equivalent to $h\mathcal{U} := \mathbf{D}(U)$, (12.3.1) and (12.3.2) give isomorphisms, with $\mathrm{Ext}^i_{\widehat{S}\#\widehat{T}}$ as in [loc. cit., p. 24, 1.1.2.17]:

$$\mathrm{Ext}^i_{\widehat{S}\#\widehat{T}}(\widehat{a}, \widehat{b}) := \pi_0 \, \mathrm{Map}_{\widehat{S}\#\widehat{T}}(\widehat{a}, \widehat{b[i]}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(U)}(a, b[i]) = \mathrm{Ext}^i_U(a, b).$$

(In particular, when $S = T = a$, one gets the derived Hochschild cohomology of S/R, with coefficients in b.)

Thus, Corollary 12.2 implies a derived version, involving Exts of adjoint associativity for dg bimodules.

12.4. *Proof of Theorem 12.1* (*Sketch*). The associativity of tensor product gives a canonical equivalence, in

$$\mathrm{Fun}((D\#A)\times(A\#B)\times(B\#C),(D\#C)),$$

between the composed functors

$$(D\#A)\times(A\#B)\times(B\#C)\xrightarrow{\otimes\times\mathrm{id}}(D\#B)\times(B\#C)\xrightarrow{\otimes}(D\#C),$$

$$(D\#A)\times(A\#B)\times(B\#C)\xrightarrow{\mathrm{id}\times\otimes}(D\#A)\times(A\#C)\xrightarrow{\otimes}(D\#C).$$

The standard isomorphism $\mathrm{Fun}(X\times Y,Z)\xrightarrow{\sim}\mathrm{Fun}(X,\mathrm{Fun}(Y,Z))$ (see [Goerss and Jardine 1999, p. 20, Proposition 5.1]) turns this into an equivalence ξ between the corresponding functors from $(A\#B)\times(B\#C)$ to $\mathrm{Fun}(D\#A,D\#C)$. These functors factor through the full subcategory $\mathrm{Fun}^{\mathrm{L}}(D\#A,D\#C)$: this need only be checked at the level of objects $(x,y)\in(A\#B)\times(B\#C)$, whose image functors are, by Proposition 11.1, left-adjoint, respectively, to $\mathcal{H}om_B(x,\mathcal{H}om_C(y,-))$ and to $\mathcal{H}om_C(x\otimes y,-)$. Composition with $(\varphi^{-1})^{\mathrm{op}}$ (φ as in (8.3.1)) takes ξ into an equivalence in

$$\mathrm{Fun}((A\#B)\times(B\#C),\mathrm{Fun}^{\mathrm{R}}(D\#C,D\#A)^{\mathrm{op}})$$

$$=\mathrm{Fun}((A\#B)^{\mathrm{op}}\times(B\#C)^{\mathrm{op}},\mathrm{Fun}^{\mathrm{R}}(D\#C,D\#A)),$$

to which α corresponds. (More explicitly, note that for any ∞-categories X, Y and Z, there is a *composition functor*

$$\mathrm{Fun}(Y,Z)\times\mathrm{Fun}(X,Y)\to\mathrm{Fun}(X,Z),\qquad(12.4.1)$$

corresponding to the natural composed functor

$$\mathrm{Fun}(Y,Z)\times\mathrm{Fun}(X,Y)\times X\to\mathrm{Fun}(Y,Z)\times Y\to Z;$$

and then set

$$X:=(A\#B)^{\mathrm{op}}\times(B\#C)^{\mathrm{op}},$$
$$Y:=\mathrm{Fun}^{\mathrm{L}}(D\#A,D\#C)^{\mathrm{op}},$$
$$Z:=\mathrm{Fun}^{\mathrm{R}}(D\#C,D\#A)\dots).$$

The rest follows in a straightforward manner from Proposition 8.3. $\square$

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