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On the subadditivity problem for maximal shifts in free resolutions

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We present some partial results regarding subadditivity of maximal shifts in finite graded free resolutions.

Let K be field, $S = K[x_1, \dots, x_n]$ the polynomial ring over K in the indeterminates $x_1, \dots, x_n$ and $I \subset S$ a graded ideal. Let $(\mathbb{F}, \partial)$ be a graded free S-resolution of $R = S/I$. Each free module $\mathbb{F}_a$ in the resolution is of the form $\mathbb{F}_a = \bigoplus_j S(-j)^{b_{aj}}$. We set

$$t_a(\mathbb{F}) = \max\{j : b_{aj} \neq 0\}.$$

In the case that $\mathbb{F}$ is the graded minimal free resolution of I we write $t_a(I)$ instead of $t_a(\mathbb{F})$.

We say $\mathbb{F}$ satisfies the *subadditivity condition*, if $t_{a+b}(\mathbb{F}) \le t_a(\mathbb{F}) + t_b(\mathbb{F})$.

Remark 1. The Taylor complex and the Scarf complex satisfy the subadditivity condition. Indeed, both complexes are cellular resolutions supported on a simplicial complex. From this fact the assertion follows immediately.

The minimal resolution of a graded algebra S/I does not always satisfy the subadditivity condition as pointed out in [Avramov et al. 2015]. Additional assumptions on the ideal I are required. Somewhat weaker inequalities can be shown in certain ranges of a and b, and in particular the inequality $t_{a+1}(I) \le t_a(I) + t_1(I)$ if $R = S/I$ is Koszul and $a \le \operatorname{height} I$; see [Avramov et al. 2015, Theorem 4.1]. Another case of interest for which the subadditivity condition holds is when $\dim S/I \le 1$ and $a + b = n$ as shown by David Eisenbud, Craig Huneke and Bernd Ulrich in [Eisenbud et al. 2006, Theorem 4.1]. No counterexample is known for monomial ideals.

For a general graded ideal I we have the following result.

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Proposition 2. *Let $I \subset S$ be a graded ideal, $\mathbb{F}$ the graded minimal free resolution of S/I. Suppose there exists a homogeneous basis $f_1, \dots, f_r$ of F_a such that*

$$\partial(\mathbb{F}_{a+1}) \subset \bigoplus_{i=1}^{r-1} S f_i.$$

Then $\deg f_r \le t_{a-1} + t_1$.

Proof. We denote by $(\mathbb{F}^*, \partial^*)$ the complex $\operatorname{Hom}_S(\mathbb{F}, S)$ which is dual to $\mathbb{F}$. For any basis $h_1, \dots, h_l$ of $\mathbb{F}_b$ we denote by h_i^* the basis element of $\mathbb{F}_b^*$ with $h_i^*(h_j) = 1$ if $j = i$ and $h_i^*(h_j) = 0$, otherwise. Then $h_1^*, \dots, h_l^*$ is a basis of $\mathbb{F}_b^*$, the so-called dual basis of $h_1, \dots, h_l$.

Our assumption implies that $\partial^*(f_r^*) = 0$. This implies that f_r^* is a generator of $H^a(\mathbb{F}^*) = \operatorname{Ext}_S^a(S/I, S)$, and hence $I f_r^* = 0$ in $H^a(\mathbb{F}^*)$, since $\operatorname{Ext}^a(S/I, S)$ is an S/I-module. On the other hand, if $g_1, \dots, g_m$ is a basis of $\mathbb{F}_{a-1}$ and $\partial(f_r) = c_1 g_1 + \cdots + c_n g_m$, then $\partial^*(g_i^*) = c_i f^* + m_i$ where each m_i is a linear combination of the remaining basis elements of $\mathbb{F}_a^*$. Let $c \in I$ be a generator of maximal degree. Then by definition, $\deg c = t_1(I)$. Since $I f_r^* = 0$ in $H^a(\mathbb{F}^*)$, there exist homogeneous elements $s_i \in S$ such that $c f_r^* = \sum_{i=1}^m s_i(c_i f_r^* + m_i)$. This is only possible if $t_1(I) = \deg c_i + \deg s_i$ for some i. In particular, $\deg c_i \le t_1(I)$. It follows that $\deg f_r = \deg c_i + \deg g_i \le t_1(I) + t_{a-1}(I)$, as desired. □

Jason McCullough [2012, Theorem 4.4] shows $t_p(I) \le \max_a\{t_a(I) + t_{p-a}(I)\}$, where $p = \operatorname{proj\,dim} S/I$. As an immediate consequence of Proposition 2 we obtain the following improvement of McCullough's inequality:

Corollary 3. *Let $I \subset S$ be a graded ideal of projective dimension p. Then*

$$t_p(I) \le t_{p-1}(I) + t_1(I).$$

For monomial ideals one even has the following corollary.

Corollary 4. *Let I be a monomial ideal. Then $t_a(I) \le t_{a-1}(I) + t_1(I)$ for all* $a \ge 1$.

For the proof of this and the following results we will use the restriction lemma as given in [Herzog et al. 2004, Lemma 4.4]: let I be a monomial ideal with multigraded (minimal) free resolution $\mathbb{F}$ and let $\alpha \in \mathbb{N}^n$. Then the restricted complex $\mathbb{F}^{\le\alpha}$ which is the subcomplex of $\mathbb{F}$ for which $(\mathbb{F}^{\le\alpha})_i$ is spanned by those basis elements of $\mathbb{F}_i$ whose multidegree is componentwise less than or equal to α, is a (minimal) multigraded free resolution of the monomial ideal $I^{\le\alpha}$ which is generated by all monomials $\mathbf{x}^{\mathbf{b}} \in I$ with $\mathbf{b} \le \alpha$, componentwise.

Proof of Corollary 4. Let $\mathbb{F}$ the minimal multigraded free S-resolution of S/I, and let $f \in F_a$ be a homogeneous element of multidegree $\alpha \in \mathbb{N}^n$ whose total degree is $t_a(I)$. We apply the restriction lemma and consider the restricted complex $\mathbb{F}^{\le\alpha}$.

Let $f_1, \dots, f_r$ be a homogeneous basis of $(\mathbb{F}^{\leq\alpha})_a$ with $f_r = f$. Since there is no basis element of $(\mathbb{F}^{\leq\alpha})_{a+1}$ of a multidegree which is coefficient bigger than α, and since the resolution $\mathbb{F}^{\leq\alpha}$ is minimal, it follows that $\partial((\mathbb{F}^{\leq\alpha})_{a+1}) \subset \bigoplus_{i=1}^{r-1} Sf_i$. Thus we may apply Proposition 2 and deduce that $t_a(I^{\leq\alpha}) \leq t_{a-1}(I^{\leq\alpha}) + t_1(I^{\leq\alpha})$. Since $t_a(I) = t_a(I^{\leq\alpha})$, $t_{a-1}(I^{\leq\alpha}) \leq t_{a-1}(I)$ and $t_1(I^{\leq\alpha}) \leq t_1(I)$, the assertion follows. □

The preceding corollary generalizes a result by Oscar Fernández-Ramos and Philippe Gimenez [2014, Corollary 1.9] who showed that $t_a \leq t_{a-1} + 2$ for any monomial ideal generated in degree 2.

Let $I \subset S$ be a monomial ideal, and $\alpha, \beta \in \mathbb{N}^n$ be two integer vectors. We say that (α, β) is a *covering pair* for I, if

$$I = I^{\leq\alpha} + I^{\leq\beta}.$$

Theorem 5. *Let (α, β) be a covering pair for the monomial ideal I, and suppose that $p = \operatorname{proj\,dim} S/I^{\leq\alpha}$ and $q = \operatorname{proj\,dim} S/I^{\leq\beta}$. Then $\operatorname{proj\,dim} S/I \leq p + q$, and for all integers $a \leq \operatorname{proj\,dim} S/I$ we have*

$$t_a(I) \leq \max\{t_i(I) + t_j(I) : i + j = a, i \leq p, j \leq q\}.$$

Proof. We consider the complex $\mathbb{G} = \mathbb{F}^{\leq\alpha} * \mathbb{F}^{\leq\beta}$ defined in [Herzog 2007]. Then $\mathbb{G}$ is a multigraded free resolution of $I^{\leq\alpha} + I^{\leq\beta}$ of length $p + q$, and hence a multigraded free resolution of I. In particular, it follows that $\operatorname{proj\,dim} S/I \leq p+q$.

By construction,

$$\mathbb{G}_a = \bigoplus_{i+j=a} (\mathbb{F}^{\leq\alpha})_i * (\mathbb{F}^{\leq\beta})_j,$$

where each direct summand $(\mathbb{F}^{\leq\alpha})_i * (\mathbb{F}^{\leq\beta})_j$ is a free multigraded S-module. If $f_1, \dots, f_s$ is a multihomogeneous basis of $(\mathbb{F}^{\leq\alpha})_i$ and $g_1, \dots, g_r$ a multihomogeneous basis of $(\mathbb{F}^{\leq\beta})_j$, then the symbols $f_k * g_l$ with $k = 1, \dots, s$ and $l = 1, \dots, r$ establish a multihomogeneous basis of $(\mathbb{F}^{\leq\alpha})_i * (\mathbb{F}^{\leq\beta})_j$, and if σ_k is the multidegree of f_k and τ_l is the multidegree of g_l, then $\sigma_k \vee \tau_l$ is the multidegree of $f_k * g_l$, where for two integer vectors $\gamma, \delta \in \mathbb{N}^n$ we denote by $\gamma \vee \delta$ the integer vector which is obtained from γ and δ by taking componentwise the maximum. It follows that the element of maximal (total) degree in $(\mathbb{F}^{\leq\alpha})_i * (\mathbb{F}^{\leq\beta})_j$ has degree less than or equal to $t_i(\mathbb{F}^{\leq\alpha}) + t_j(\mathbb{F}^{\leq\beta})$. Consequently we obtain

$$\begin{aligned} t_a(I) = t_a(\mathbb{F}) \leq t_a(\mathbb{G}) &\leq \max\{t_i(\mathbb{F}^{\leq\alpha}) + t_j(\mathbb{F}^{\leq\beta}) : i + j = a, i \leq p, j \leq q\} \\ &\leq \max\{t_i(I) + t_j(I) : i + j = a, i \leq p, j \leq q\}. \end{aligned}$$ □

The following example illustrates that Theorem 5 leads to inequalities which are not implied by Corollary 3.

Example 6. Set $S = k[x, y, z, u, v, w, a]$ and let $I \subset S$ be given by

$$I = (x^2w^2v^2, a^2x^3y^2u^2w^2, a^2z^2u^2, u^2y^2z^3, x^3y^2z^2, x^5, y^5, z^5, u^5, w^5, v^6, a^6).$$

We choose $\alpha = (5, 5, 5, 5, 0, 0, 0)$ and $\beta = (3, 3, 2, 2, 6, 5, 6)$. Then

$$I^{\leq\alpha} = (x^5, y^5, z^5, u^5, x^3y^3z^2, u^2y^2z^3),$$
$$I^{\leq\beta} = (w^5, v^6, a^6, x^2w^2v^2, a^2x^3y^2u^2w^2, a^2z^2u^2).$$

Here, $p = 4$, $q = 5$ and $\operatorname{proj\,dim} S/I = 7$. Thus by Theorem 5,

$$t_7(I) \leq \max\{t_2(I) + t_5(I), t_3(I) + t_4(I)\}.$$

Corollary 7. *Let $s = p + q - a$. Then with the notation and assumptions of Theorem 5 we have*

$$t_a(I) \leq \max\{t_i(I) + t_{a-i}(I) : p - s \leq i \leq p\}.$$

As a special case one obtains:

Corollary 8. *Let $I \subset S = K[x_1, \dots, x_n]$ be a monomial ideal with $\dim S/I = 0$ which is minimally generated by $m \leq 2n - 6$ monomials, and let a be an integer with $(m + 4)/2 \leq a \leq n$. Then*

$$t_a(I) \leq \min\bigl\{t_1(I)+t_{a-1}(I), \max\bigl\{t_i(I)+t_{a-i}(I) : p-(m-a) \leq i \leq \min\{p, a/2\}\bigr\}\bigr\}$$

for all $p = m - a + 2, \dots, a - 2$.

Proof. Due to Corollary 3 we only need to show that

$$t_a(I) \leq \max\bigl\{t_i(I) + t_{a-i}(I) : p + a - m \leq i \leq \min\{p, a/2\}\bigr\}.$$

Since $\dim S/I = 0$, among the minimal set of generators $G(I)$ of I are the pure powers $x_1^{a_1}, \dots, x_n^{a_n}$ for suitable $a_i > 0$. We let $\alpha = (a_1, \dots, a_p, 0, \dots, 0)$. Then $I^{\leq\alpha}$ has all its generators in $K[x_1, \dots, x_p]$ so that $\operatorname{proj\,dim} S/I = p$. Let J be the ideal which is generated by the set of monomials $G(I) \setminus \{x_1^{a_1}, \dots, x_p^{a_p}\}$, and let x^β be the least common multiple of the generators of J. Then $J = I^{\leq\beta}$ and (α, β) is a covering pair for I. Since J is generated by $m - p$ elements it follows that $q = \operatorname{proj\,dim} S/J \leq m - p$. Hence the desired inequality follows from Corollary 7. The conditions on the integers a, m and p only make sure that $i \geq 2$ and $a - i \geq 2$ for all i with $p + a - m \leq i \leq p$, and that $m - a + 2 \leq a - 2$. $\square$

The bound in Corollary 8 is a partial improvement of the results in [Eisenbud et al. 2006] and [McCullough 2012] since the bound is also valid for certain $a < n$. For $a = n$, it is weaker than the one in [Eisenbud et al. 2006] for zero dimensional rings and is stronger than the one in [McCullough 2012]. For example, if $n = 7$ and $m = 8$ one has $t_6 \leq t_1 + t_2 + t_3$, and if $6 \leq n \leq 20$ and $m \leq 2n - 6$, then one has $t_7 \leq t_1 + t_2 + t_4$.

Remark 9. With the same methods as applied in the proof of Theorem 5 one can show the following statement: let $I \subset S$ be a monomial ideal with graded minimal free resolution $\mathbb{F}$, and $f_i \in F_{a_i}$ multihomogeneous basis elements of multidegree α_i for $i = 1, \ldots, r$. Assume that $I = \sum_{i=1}^{r} I^{\leq \alpha_i}$. Then

$$t_{a_1+a_2+\cdots+a_r}(I) \leq t_{a_1}(I) + t_{a_2}(I) + \cdots + t_{a_r}(I).$$

To satisfy the condition $I = \sum_{i=1}^{r} I^{\leq \alpha_i}$ requires in general that either r is big enough or that the α_i are large enough (with respect to the partial order given by componentwise comparison). Here is an example with $r = 2$ to which Remark 9 applies: let

$$I = (x^2w^2v^2, a^2x^3y^2u^2w^2, a^2z^2u^2, u^2y^2z^3, x^3y^2z^2) \subset k[x, y, z, w, u, v, a].$$

The Betti numbers of R/I are 1, 5, 8, 5, 1. Even though the Betti sequence is symmmetric, the ideal I is not Gorenstein, since it is of height 2 and projective dimension 4. The two multidegrees in F_2 which form a covering pair for I are $(3, 2, 2, 2, 2, 0, 2)$ and $(2, 2, 3, 2, 2, 2, 0)$. In this example we have $t_1 = 11$, $t_2 = 13$, $t_3 = 15$, $t_4 = 16$ and we clearly have $t_i \leq t_2 + t_2$.

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