

Scoring games: the state of play

URBAN LARSSON, RICHARD J. NOWAKOWSKI
AND CARLOS PEREIRA DOS SANTOS

We survey scoring-play combinatorial game theory, and reflect upon similarities and differences with normal- and misère-play. We illustrate the theory by using new and old scoring rulesets, and we conclude with a survey of scoring games that originate from graph theory.

1. Introduction

Recent progress in scoring-play combinatorial game theory motivates a survey on the subject. There are similarities with classical settings in normal- and misère-play, but the subject is richer than both those combined. This survey has a three-fold purpose: first to survey the combinatorial game theory (CGT) work in the area (such as disjunctive sum, game comparison, game reduction and game values), secondly to point at some important ideas about scoring rulesets (in relation with normal- and misère-play), and at last we show that existing literature includes many scoring combinatorial games which have not yet been studied in the broader CGT context. Although CGT was first developed in positional (scoring) games by Milnor (inspired by game decomposition in the game of Go), the field took off only with the advances in normal-play during the 1970-80s, and recently via successes in understanding misère games.

1.1. Normal- and misère-play. The family of combinatorial games consists of two-player games with perfect information (no hidden information as in some card games), no chance moves (no dice), and where the two players move alternately. We primarily consider games in which the positions decompose into independent subpositions. This class of games has been called *additive* to distinguish it from maker-maker and maker-breaker positional games [5], such as HEX.

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When a player has no more moves (in *any* game component, which is called the “long” disjunctive sum) the game ends and some convention is required to be able to determine the outcome. There are two natural conventions that tie together the last move and the outcome:

- (1) *Normal-play convention*: last player wins;
- (2) *Misère-play convention*: last player loses.

The obtained body of results considering these conventions is what we call *classical combinatorial game theory*. See [16; 7; 1; 47] for background, [25] for a survey, and [39] for a list of open problems.

The theory of normal-play was developed first. The analysis of NIM, by Charles Bouton, was published in the early twentieth century [10]. After this, the Sprague–Grundy theory for impartial combinatorial games¹ [49; 27] appeared, played with normal-play convention. This was a fundamental step for the subsequent establishment of combinatorial game theory; the important concept of *disjunctive sum* of games was established, and this idea was already present in the game of NIM, where, of course, the most central concept is that of “nim-sum”. The next big step was in the 1950s, by Milnor, when the notion of game comparison first appeared [38] in so-called “positional games”, and this theory was inspired by the famous eastern game of GO. The game appears to decompose into components during play; the longer play continues, the more independent the regions become. By the end of play, the game board has decomposed into a finite number of regions, and only the scoring part remains, where the difference of the total numbers of captured pieces together with the “captured territories” determines the final result. We will return to Milnor’s class of games in Section 2. Later, in the 1970s, Conway [16] was able to expand the theory to include partizan game theory² followed up in the 1980s with Conway, Berlekamp and Guy’s “Winning Ways” [7].

It is important to note that *normal-play convention* is a very special case, with many nice properties. (The notation for a normal-play game is $G = \{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}$, where $G^{\mathcal{L}}$ is the set of G ’s Left options, and similar for Right. Also, $G^{\mathcal{L}}$ and $G^{\mathcal{R}}$ are elements of $G^{\mathcal{L}}$, $G^{\mathcal{R}}$, respectively.) A “ten commandments list” is given next. The properties of all the other conventions are compared to this list.

Properties of normal-play. N1 Combinatorial games played with normal-play convention, together with the disjunctive sum, are an ordered, abelian group.

N2 The inverse of G is obtained recursively by $-G = \{-G^{\mathcal{R}} \mid -G^{\mathcal{L}}\}$.

¹A impartial combinatorial ruleset is a ruleset in which the allowable moves depend only on the positions and not on which of the two players is currently moving. In other words, from any subposition, the options are exactly the same for both players.

²There are two players, Left and Right, with usually different move options.

N3 To check if $G \succcurlyeq H$, it is only necessary to see if Left wins $G - H$ playing second.

N4 The empty set is the worst possible set of options.

N5 There are two fundamental reductions: domination and reversibility. The second happens when there is a Right response to a Left play in G such that $G^{LR} \preccurlyeq G$ (or the mirror image, from Right's point of view). In that case, G^L may be replaced by $G^{LR\mathcal{L}}$. This can be done even if $G^{LR\mathcal{L}} = \emptyset$ (atomic reversibility) and in this case, G^L is erased. In other words, G^L is replaced by the empty set!

N6 Given a game G , there is a unique simplest game equivalent to G . That game is obtained by making all the possible reductions in G and its followers.

N7 There is a bijection between the outcome classes and the order:

$$G > 0 \Leftrightarrow G \in \mathcal{L}; \quad G = 0 \Leftrightarrow G \in \mathcal{P}; \quad G \parallel 0 \Leftrightarrow G \in \mathcal{N}; \quad G < 0 \Leftrightarrow G \in \mathcal{R}.$$

N8 The disjunctive sum table of the outcome classes is not particularly complex:

+	\mathcal{P}	\mathcal{N}	\mathcal{R}	\mathcal{L}
\mathcal{P}	\mathcal{P}	\mathcal{N}	\mathcal{R}	\mathcal{L}
\mathcal{N}	\mathcal{N}	all	$\mathcal{N} \cup \mathcal{R}$	$\mathcal{N} \cup \mathcal{L}$
\mathcal{R}	\mathcal{R}	$\mathcal{N} \cup \mathcal{R}$	\mathcal{R}	all
\mathcal{L}	\mathcal{L}	$\mathcal{N} \cup \mathcal{L}$	all	\mathcal{L}

N9 If $LS(G) < RS(G)$ (“the number of move advantage is worse for Left if Left plays first”) then G is a *zugzwang*. In the normal-play convention, there is no canonical zugzwangs because this inequality always corresponds to a number.

N10 If $LS(G) > RS(G)$ then G is *hot*. If a game is hot then $G^{\mathcal{L}} \neq \emptyset$ and $G^{\mathcal{R}} \neq \emptyset$, because otherwise G would be a number, and numbers are cold.

All these properties fail in misère universes: for **N1**, misère games have only monoid structures; for instance, considering impartial misère games, $*2$ has no inverse. Also, it is possible to have invertible elements that don't satisfy **N2** [37; 40]. In misère, $\{ | * \} > 0$ and Left does not win going second in $\{ | * \}$, therefore **N3** fails. About **N4**, in misère, an empty set of options may be a good thing; in $\{ | * \}$, Left wins going first (the worst thing is an infinite string of moves, which only exists in “loopy” games). In [21], the authors proposed nice reductions for dicot misère games; for reversibility, in some situations, if $G^{LR\mathcal{L}} = \emptyset$ then G^L should be replaced by $*$. Hence, **N5** also fails. The failure of **N6** will be discussed in the next section. **N7** fails because, as we have seen, $\{ | * \} > 0$ and $\{ | * \} \in \mathcal{N}$. For **N8**, the algebraic table of the outcomes is

+	\mathcal{P}	\mathcal{N}	\mathcal{R}	\mathcal{L}
\mathcal{P}	all	all	all	all
\mathcal{N}	all	all	all	all
\mathcal{R}	all	all	all	all
\mathcal{L}	all	all	all	all

Finally, in misère, **N9** and **N10** fail because we don't have numbers to define stops.

2. Scoring combinatorial game theory

In scoring-play there is no direct correspondence between the last player and the outcome. When a player has no more moves, the game ends, but an evaluation about the outcome is still needed. In common language, the player wants to get as many points as possible irrespective of who moves last. Therefore, the information that the set of options is empty ($G^{\mathcal{L}} = \emptyset$ or $G^{\mathcal{R}} = \emptyset$) is not enough. It reveals the end of the game, but not the outcome of the game. It is necessary to include that information in the description of the game forms.

In [50], Fraser Stewart introduced a notation based in triples, instead of pairs. Considering a totally ordered group of results \mathbb{B} , in Stewart's notation, a game is a triple $\{G^{\mathcal{L}} \mid b \in \mathbb{B} \mid G^{\mathcal{R}}\}$ where, as usual, $G^{\mathcal{L}}$ and $G^{\mathcal{R}}$ are sets of games. For instance, the game of day 0,

$$\{\mid 3 \mid\} \underbrace{=}_{\text{definition}} 3$$

is a game where the players have no options, but the last player gets 3 points (which is usually bad for Right, who wants a negative number of points). Another example is the game $\{\mid 2 \mid -3\}$; if it is Left's turn, she has no options, the game ends, and she gets 2 points; if it is Right's turn, he has an option, that is $-3 = \{\mid -3 \mid\}$. After that, the game ends and the final result is -3 .

Later, in [34], the authors proposed an alternative notation with pairs instead of triples. Basically, the different empty sets are adorned with elements of \mathbb{B} . This allows a representation of the result when a player has no more moves. Using the new notation, $\{\mid 3 \mid\}$ becomes $\langle \emptyset^3 \mid \emptyset^3 \rangle$ and $\{\mid 2 \mid -3\}$ is represented by $\langle \emptyset^2 \mid -3 \rangle$. The notation with adorns has two fundamental advantages: first, it is more suitable for asymmetrical forms like $\langle \emptyset^3 \mid \emptyset^4 \rangle$, which is particularly interesting when we discuss reductions to simplest form! Second, and the most important, it has practical advantages when we want to relate classical combinatorial game theory to scoring combinatorial game theory. On the other hand, Stewart's notation may be useful when analyzing operators different than the disjunctive sum. A different type of bracket is used to distinguish scoring-play from the classical context. See Figure 1 for how this relates to game trees.

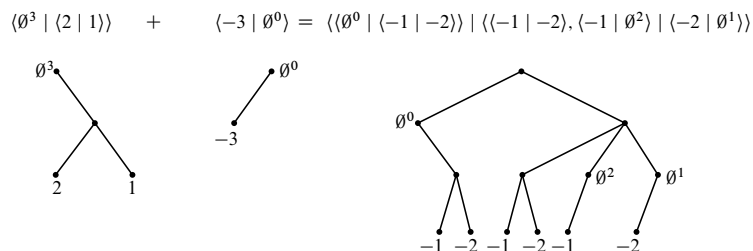


Figure 1. The disjunctive sum of two game trees.

There is a problem with inverses, so we define the *conjugate* as $\sim G = \langle \sim G^{\mathcal{R}} \mid \sim G^{\mathcal{L}} \rangle$. The game $G = \langle \langle 1 \mid -1 \rangle \mid \langle 1 \mid -1 \rangle \rangle$ is a zugzwang; if Left starts, the final result is -1 , if Right starts, the final result is 1 . The conjugate of that form is exactly the same, and so, $G + (\sim G) = \langle \langle 1 \mid -1 \rangle \mid \langle 1 \mid -1 \rangle \rangle + \langle \langle 1 \mid -1 \rangle \mid \langle 1 \mid -1 \rangle \rangle$. We may observe that $G + (\sim G)$ is not zero, but a new zugzwang. In conclusion, $\sim G$ is not the inverse of G . In [23], Mark Ettinger proved that G is not invertible. Therefore, as in misère-play, scoring-play often leads to monoid structures, instead of group structures.

The research done on scoring combinatorial game theory always considered short games, requiring that the sets $G^{\mathcal{L}}$ and $G^{\mathcal{R}}$ be finite. Also, in some works, only *dicot* forms were considered — from any subposition, a player can move if and only if the opponent also can.

The first mathematical approach to scoring-play was done by John Milnor and Olof Hanner with the publication of the papers [38; 29]. In the first, Milnor restricted the studied universe to nonzugzwang dicot forms. That means that every Milnor dicot form G satisfies $Ls(G) \geq Rs(G)$ so these games still satisfy **N9** and **N10**. (We write Ls, Left score, in scoring-play instead of LS, which is the Left stop, in normal-play.) In fact, Milnor’s universe of scoring games is very well behaved, constituting a group structure. Following Milnor’s approach, Olof Hanner proved the existence of a mean value of a game; a kind of “average value” for G [29]. This result was very important, inspiring the modern temperature theory [16; 7; 1; 47]. Because of that, in a game theorist’s mind, this work is fundamentally important for the establishment of CGT, more than a first contribution for scoring-play analysis³.

The next stride forward is in the 1990s when Mark Ettinger published his visionary PhD thesis ([23]; see also [22]). Ettinger studied dicot scoring games

³In their terminology they used a pay-off function instead of outcome function/Left stop/Left score and so on, probably inspired by the novelties in economic game theory regarding Nash equilibrium, mixed strategies, and so on. However, since then the fields ECG and CGT seem to have developed in completely different directions. A big open question is if there is a merging theory, but the authors are not yet aware of any.

without the restriction $Ls(G) \geq Rs(G)$, allowing zugzwang positions as $G = \langle \langle 1 \mid -1 \rangle \mid \langle 1 \mid -1 \rangle \rangle$. Even with the dicot restriction, Ettinger was able to prove useful reductions in forms and the conjugation property **N2**. Also, he proved a very useful theorem for checking if a player should prefer a game G or some number of points r ([23], Ettinger’s theorem, page 20). We used the adjective “visionary” because his ideas, already in the 1990s, are very similar to those used 15 years later in misère theory. In particular, with different words, he proved the “downlink concept” ([23], page 22), used later by Aaron Siegel [48] and a “carousel” procedure for proving the conjugate property that is very useful in other contexts [33]. Finally, in Ettinger’s universe, there are reversibility situations with $G^{LR\mathcal{L}} = \emptyset$. In some of them, the reversible option *cannot be erased*, but replaced by a well-chosen representative (atomic reversibility). Ettinger was the first who understood this particular situation of reversibility, filling in the cases that do not occur in the normal-play convention.

Fraser Stewart [50] analyzed scoring games almost without restrictions (only asymmetrical empty trees like $\langle \emptyset^3 \mid \emptyset^4 \rangle$ were not allowed). The absence of restrictions brought the possibility of non-well-behaved forms like $\langle \emptyset^1 \mid -1 \rangle$; the problematic forms are those hot games without options (*hot atomic games*). Left wants to have the right to move, even knowing that there are no available moves (seemingly resembling local misère-play)! To understand the interesting mathematical implications of these forms, we need to clarify first the different nature of the concepts r -score and r moves. For instance, 1-score is the game $1 = \langle \emptyset^1 \mid \emptyset^1 \rangle$ whose tree is empty; on the other hand, one move is the game $\langle \langle \emptyset^0 \mid \emptyset^0 \rangle \mid \emptyset^0 \rangle$, whose tree is analogous to the tree of 1 in classical normal-play (a hat is used for moves, trees adorned only with zeros; $\hat{1} = \langle \langle \emptyset^0 \mid \emptyset^0 \rangle \mid \emptyset^0 \rangle$). In the second, Left does not win points, but the extra move may be very useful in presence of zugzwang components. In the symbiosis between scores and moves lies the soul of scoring combinatorial game theory.

2.1. Normal-play behavior in scoring games. In misère- or scoring-play, if we consider the game forms just like trees without scores, we can think about their behavior under normal-play convention. For example, in scoring, if we replace all the adorns by zero, we essentially obtain moves. Only the shape of the trees is relevant. The relation between a specific convention (sc) and the normal-play convention (nc) is very interesting. Often, we have an order preserving relation, that is,

$$G \succ_{sc} H \Rightarrow G \succ_{nc} H.$$

To give some intuition, consider $G \succ H$ in misère-play and suppose that $G \not\succeq H$ in normal-play. For some game X , in normal-play, we have *Left loses playing first $G + X$ and Left wins playing first $H + X$ or Left loses playing second $G + X$ and*

Left wins playing second $H + X$. Without loss of generality, assume the first. Now, consider the form $\{\hat{n} \mid -\hat{n}\}$ where n is arbitrary large, \hat{n} is a string of moves for Left, and $-\hat{n}$ a string of moves for Right. It is an easy check that, with misère convention, Left loses going first in $G + X + \{\hat{n} \mid -\hat{n}\}$ and Left wins going first in $H + X + \{\hat{n} \mid -\hat{n}\}$. This happens because $\{\hat{n} \mid -\hat{n}\}$ works like a zugzwang and zugzwangs promote normal-play (players want to make the last move in the disjunctive sum of the remaining components). Hence, this situation contradicts $G \succ H$ in misère-play and we have the mentioned order-preserving. Of course, in scoring context, we can build an analogous argument with a “large zugzwang” $\langle -r \mid r \rangle$.

The one million dollar question is the opposite question: are extra moves always good? That is, is it true that $\hat{1} = \langle \langle \emptyset^0 \mid \emptyset^0 \rangle \mid \emptyset^0 \rangle \succ 0$? In other words, under what circumstances do we have an order embedding? In Stewart’s universe we do not have it. Consider $X = \langle \emptyset^1 \mid -1 \rangle$: if Left plays first in $\hat{1} + X$, the final result is -1 , if Left plays first in $0 + X$, the final result is 1 . Therefore, $\hat{1} \not\succeq 0$, and that happens because of the existence of hot atomic games.

It is also possible to prove that if $G = 0$ in Stewart’s universe then the game tree of G is empty. Without loss of generality, consider $G^{\mathcal{L}} \neq \emptyset$ and r a negative real number less than all the atoms of G . Playing first, Left loses $G + \langle \emptyset^1 \mid r \rangle$ because she has a move in G . On the other hand, Left wins $0 + \langle \emptyset^1 \mid r \rangle$ with the final result 1 . Hence G with options cannot be equal to 0 . Again, that happens because of the existence of hot atomic games.

Finally, consider an hypothetical *atomic reversibility* such that $G^{LR\mathcal{L}} = \emptyset$. It cannot exist! If so, $G \succ \langle \emptyset^a \mid G^{LR\mathcal{R}} \rangle$, for some real number a , and this inequality is contradictory with a distinguishing game like $X = \langle \emptyset^1 \mid r \rangle$, where r is an arbitrary small negative real number. Namely, $G + X \not\succeq \langle \emptyset^a \mid G^{LR\mathcal{R}} \rangle + X$. The importance of Stewart’s contribution was to bring this whole range of unusual situations to scoring theory research.

In the 2010s, the authors and João Pedro Neto proposed the analysis of *guaranteed scoring games*, $\mathbb{G}\mathbb{S}$, in which all the atoms in $G^{\mathcal{R}}$ of a game of the form $\langle \emptyset^r \mid G^{\mathcal{R}} \rangle$, have scores larger than or equal to r and analogously for $\langle G^{\mathcal{L}} \mid \emptyset^r \rangle$ [34; 33]. In the guaranteed scoring universe, in each game component, it is always good to continue playing instead of being without moves. More importantly, in that universe *there are no hot atomic games* and, because of that, an order-embedding of normal-play was proved [34]. Also, Ettinger’s ideas were adapted in order to obtain comparison techniques, the conjugate property, reductions (Figure 2), and to propose useful canonical forms [33].

As in Ettinger’s universe, the guaranteed universe allows for atomic reversibility. To better understand this concept, we make some preliminary observations. A *game* is a directed acyclic graph $(P, (M^{\mathcal{L}}, M^{\mathcal{R}}))$ where the set of nodes P contains the *game positions*, and the edges are partitioned into the distinguished

sets M^L and M^R , which are the *Left-* and *Right-moves* respectively. On the other hand, with a recursive construction and the definitions of disjunctive sum and order, the structure of *forms* $(\mathbb{G}\mathbb{S}, +, \succcurlyeq)$ is obtained (game positions are described by forms) where the equivalence classes of the quotient $(\mathbb{G}\mathbb{S}, +, \succcurlyeq)/\equiv$ are what we call *game values*. In normal-play, because there is a unique simplest form in each equivalence class $[G]$, that is the natural choice for a representative — it is called the *canonical form*. However, in scoring context, we lose the uniqueness: there is more than one simplest form in an equivalence class $[G]$. Even so, the problem of canonical forms has an interesting approach [34]: an atomic reversible option G^L is replaced by $r - n + 1$ where n is minimal such that $G \geq r - \hat{n}$ (and r is unique); this procedure constitutes a *choice*. With that choice, considering a form G , it is possible to prove that, after all reductions, a unique and simplest form is achieved, representing $[G]$. In conclusion, there is not a unique simplest form in the class $[G]$ but, *after a choice in the atomic reversibility procedure*, we get a unique simplest form (canonical form). The reversibility is illustrated in Figure 2.

	Milnor	Ettinger	Larsson et al.	Stewart
structure	group	monoid	monoid	monoid
existence of atomic reversible options	yes	yes	yes	no
existence of zugzwangs	no	yes	yes	yes
existence of hot empty games	no	no	no	yes
large equivalence class of zero	yes	yes	yes	no

This section would be incomplete without mentioning Will Johnson. His work imposes a big restriction on scoring games (which also restricts its relation to surrounding work). Johnson’s universe consists of dicot games in which, for a given game G , the length of any play (distance to any leaf on the game tree) has the same parity [31]; see Section 4 for some graph games in this universe. The games are called *odd-* or *even-tempered* if all the lengths are odd or even, respectively. A game G is *inversive* if $Ls(G + X) \geq Rs(G + X)$ for every even-tempered game X . Johnson proved that the universe of inversive games is well behaved, having a group structure. Every inversive game has a canonical form and an additive inverse which is equal to its conjugate; moreover, $G \succcurlyeq H$ if G and H have the same “temper” and $R_s(G - H) \geq 0$.

3. Scoring rulesets

Many rulesets, usually played in normal (or *misère*) convention adapt easily to some scoring convention, and we may even mix rulesets in one and the same game; the scoring aspects may vary to create various effects such as zugzwangs or change the class of games for consideration to obtain more structure and nicer

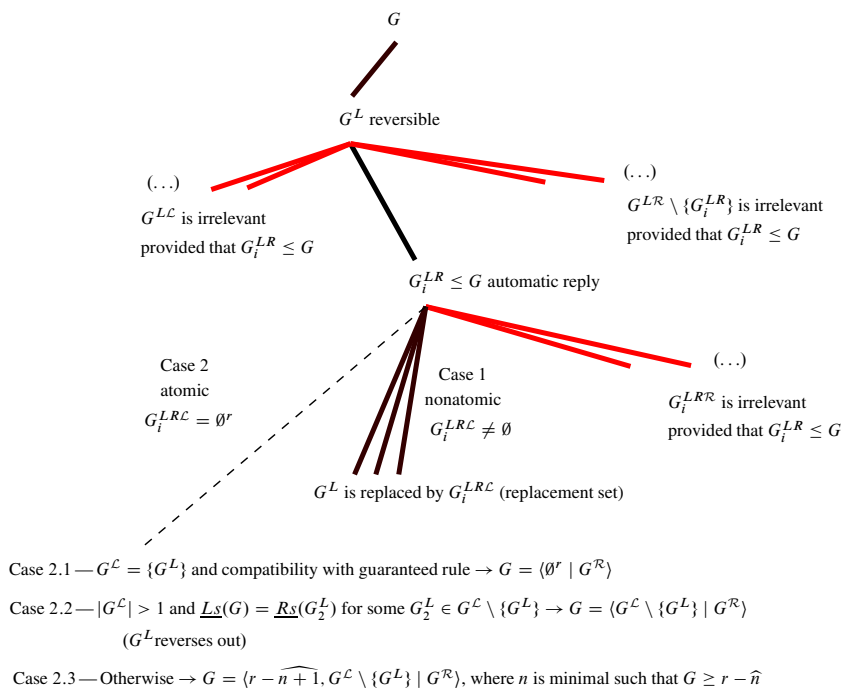


Figure 2. Reversibility of guaranteed games.

game reductions. We consider some games and properties in detail in this section and look briefly at many others in the next.

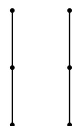
In scoring-play, as in normal- and misère-play, there are impartial rulesets, and there are partizan rulesets, and between them, there are dicot rulesets (the latter class is called “all-small” in normal-play). A scoring ruleset is *impartial* if all subpositions can be expressed as $r + G$, where the game tree of G is symmetric both in terms of moves and scores. A scoring ruleset is *partizan* if it is not impartial. A scoring ruleset is *dicot* if, from any nonterminal position, *both* players have moves; otherwise, the ruleset is *nondicot*.

In this section we also introduce another interesting classification for scoring rulesets:

- (1) In an *all-normal* ruleset, making the last move is *never worse* than allowing the opponent to make the last move.
- (2) In an *all-misère* ruleset, allowing the opponent to make the last move is *never worse* than making the last move.
- (3) A *hybrid* ruleset is a ruleset that does not satisfy any of the above items.

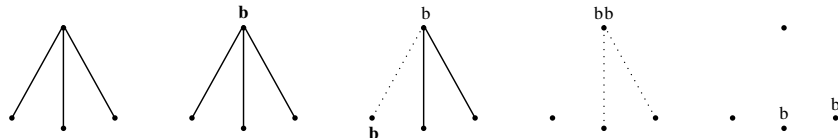
We will return to this classification scheme, when we have defined some rulesets, but we mention that being a hybrid ruleset is richer than either of the other two.

3.1. DOTS-AND-BOXES. A classical impartial scoring ruleset is DOTS-AND-BOXES, a pencil and paper ruleset carefully studied in [7] and [6]. In DOTS-AND-BOXES, zugzwang positions are very common. The players alternate to connect nodes in an $n \times n$ grid. If you connect all four edges surrounding a cell, you get another move, until you are not able to capture any more cells. By the end of game, the captured number of cells is your score. Because of the nature of the game, near the end it decomposes very nicely into many nearly filled strips of cells, and the strategy depends only on parity. Therefore the concept of a “double-box” is one of the most interesting features of this game. An open-ended double-box, shown below, is a (scoring) zugzwang. Note however that in this example, the parity problem can be avoided (in case parity is already in your favor), by connecting the middle nodes.



If we for the moment ignore the “extra moves”, this position is the zugzwang $\langle\langle 2 \mid -2 \rangle\rangle, \langle 2, \langle 2 \mid -2 \rangle \mid -2, \langle 2 \mid -2 \rangle \rangle \mid \langle 2 \mid -2 \rangle, \langle 2, \langle 2 \mid -2 \rangle \mid -2, \langle 2 \mid -2 \rangle \rangle$. Because of the special rule in DOTS-AND-BOXES where you get an extra move (in another component) when you finish off a game component, disjunctive (long) sum is not exactly the right concept.

3.2. BRUSH. The game of BRUSH is an impartial scoring ruleset where the players alternate to place a brush on any node of a given finite graph. A vertex is “primed” if it has at least as many brushes as its degree. When a primed vertex v is “fired”, a brush from v is placed on each adjacent vertex and v and all incident edges are erased—the edges have been “cleaned”. If other vertices are primed then chose one and fire it and continue until there are no more primed vertices. (The order of firing is irrelevant.) For each move, the current player adds a number of points according to the number of “cleaned” edges. For example:



The left most picture represents the starting position, and the second picture represents the first move. When the second player plays the brush on the lower left vertex, as in the third picture, then a sequence of automatic cleaning finishes

the game. As this happens, the second player obtains three points (before this move no point was awarded). If the player instead would have played on the upper vertex, then one more move would have been required to finish the game.

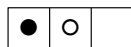
More generally, let $S(l, d)$ be the star with l leaves and needing d brushes for the center vertex to fire. It is easy to show

$$S(l, 1) = \langle l \mid -l \rangle \quad \text{and} \quad S(l, 2) = \langle l, \langle l \mid -l \rangle \mid -l, \langle l \mid -l \rangle \rangle.$$

When $d = 3$, the game is a zugzwang—the first play can take one edge but, regardless, the second player will take all the remaining edges—and $S(l, 3) = \langle \emptyset^{-l+1} \mid \emptyset^{l-1} \rangle$. The canonical forms become increasingly more complicated as d increases.

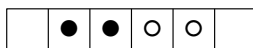
We describe some other variants under CLEANING GAMES in Section 4.

3.3. Variations of KONANE. Konane is a traditional Hawaiian normal-play game, played with black and white stones on an $n \times n$ grid. This game is convenient to adapt to scoring-play. In KONANE, the players alternate to capture the other player's pieces by jumping neighboring pieces, one at the time, and not changing direction in a single move. Played on the same game-board, the ruleset for normal-play CLOBBER is even simpler; by moving, a player captures one single neighboring piece by replacement. One interesting dicot scoring ruleset is KOBBER, mixing the rulesets KONANE and CLOBBER. In KOBBER, a player may choose between a capture as in KONANE or a capture as in CLOBBER. Each captured piece as in KONANE provides one point. Pieces captured by a CLOBBER move do not add to the score. KOBBER is dicot, but it is not impartial. (The convention is that Left plays black.)



$$\langle 0, 1 \mid 0 \rangle = \langle 1 \mid 0 \rangle$$

In KOBBER, as in DOTS-AND-BOXES, there are zugzwang positions. This is one example:



3.4. Bonus/penalty rules. For many (nondicot) rulesets, one can adjoin a natural bonus/penalty at the end of play. When a player has no moves, he takes a final penalty equal to the number of own dead stones over the board. Consider TERMINAL KONANE: the rules are exactly as in KONANE except that each captured piece corresponds to one gained point. When a player has no moves he takes a final penalty equal to the number of his own stones still on the board. Next, a zugzwang position of TERMINAL KONANE:

●			○
	●	○	
○			●

Another KONANE variant is DISKONNECT. In this ruleset a piece is *insecure* if it can be captured by the opponent with a well-chosen sequence of moves (ignoring the alternating-move condition). Otherwise, the piece is *safe*. When a player has no more moves he takes a final penalty equal to the number of own *insecure* stones remaining on the board (this is a GO-type rule).

3.5. Examples of all-normal and hybrid games. These examples of partizan scoring games fit well into our all-normal, all-misère, or hybrid classification.

In TERMINAL KONANE, if a player makes the last move, then he takes any remaining opponent's pieces. Therefore, to make the last move is *never worse* than to allow the opponent to make the last move; it is all-normal. Played in a disjunctive sum, the strategy is identical to that of normal-play.

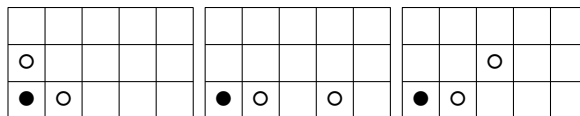
DISKONNECT is a more interesting ruleset. In the following example, Left can make the last move. However, in the only winning line for Left, Right makes the last move. Therefore DISKONNECT is not all-normal. It is also easy to check that it is not all-misère, and so it is hybrid.

○			
	○		
	○		
	○		
	●	○	

actual score = -1

Observe that classical normal-play theory is enough to analyze good TERMINAL KONANE strategies. Although the last move is not guaranteed to win, if you cannot win by making the last move, then you cannot win at all. On the other hand, because DISKONNECT is hybrid (in a sense, it has both normal-play and misère-play behavior), for that game we need scoring combinatorial game theory. Note that in DISKONNECT, as in TERMINAL KONANE, players always prefer to keep playing rather than having no moves. These rulesets satisfy the *guaranteed property*, with the accompanying theory [34]. If a player, say Left, runs out of moves in a component game, then the score of this component cannot decrease

(in case of Right opening the component for more play). For instance, consider the following DISCONNECT positions:



It is known, and not difficult to see, that the canonical forms of these positions are $\langle 1 \mid \emptyset^2 \rangle$, $\langle 2 \mid \emptyset^2 \rangle = 2 + \hat{1}$, and $\langle \langle 2 \mid \emptyset^2 \rangle \mid \emptyset^2 \rangle = 2 + \hat{2}$, respectively. (The middle picture needs some theory, but intuitively it is clear that Left will never jump just one stone: if she wants to play first in some other component, then she will do this directly, otherwise she captures two stones, and lets Right play first elsewhere. The penalty rule guarantees her 2 points in this component either way.) We can also compare

$$\langle 1 \mid \emptyset^2 \rangle < 2 + \hat{1} < 2 + \hat{2}.$$

Online software [33] is available for computations of reduced guaranteed forms, where more complicated positions may be analyzed. This module also contains the games TAKE-SMALL and TAKE-TALL. It turns out that the former is more interesting (because it is hybrid).

3.6. TAKE-SMALL. A nondicot scoring ruleset is TAKE-SMALL. It is played with a strip of a finite number of sticks of different (positive) integer length, possibly with empty spots between some of the sticks. Left chooses two adjacent sticks of lengths a and b where $a \geq b$ (the a -stick to the left). She captures the b -stick by removing it and putting the a -stick in its position. She is not allowed to move the stick over an empty spot. Right plays similarly in the opposite direction. If Right cannot move, Left will be rewarded a bonus: to complete all her possible moves. The reverse bonus is applied to Right if Left cannot move. The winner is the player who has accumulated the greater total length of sticks.

We'll represent a TAKE-SMALL position as a string of numbers where empty spots are indicated by \cdot (dot). For example, in $(3, 2, \cdot, 1, 4)$ Left can move to the position $(\cdot, 3, \cdot, 4, 1)$ plus a capture of 2; Right can move to $(3, 2, \cdot, 4, \cdot)$ plus a capture of 1. Note that $(3, 2, \cdot, 1, 4)$ is actually the disjunctive sum $(3, 2) + (1, 4)$.

TAKE-SMALL is a good example of a partizan scoring ruleset with zugzwang positions that is not all-normal. A position like $(1, 0, 1) = \langle (1, 1) \mid (1, 1) \rangle = \langle \langle 1 \mid -1 \rangle \mid \langle 1 \mid -1 \rangle \rangle$ is a zugzwang. The canonical form for this position is $\langle \emptyset^{-1} \mid \emptyset^1 \rangle$, therefore the position behaves like a *purely atomic game*, the empty tree with a Left-score and a Right-score.

Consider now the position $(1, 2, 2, 9, 8, 1) - 2$. It has an interesting feature. Left's options are

- $(1, \cdot, 2, 9, 8, 1)$;
- $(1, 2, 2, \cdot, 9, 1) + 6$;
- $(1, 2, 2, 9, \cdot, 8) - 1$.

Right's options are

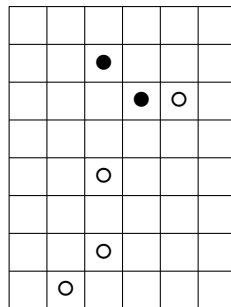
- $(2, \cdot, 2, 9, 8, 1) - 3$;
- $(1, 2, \cdot, 9, 8, 1) - 4$;
- $(1, 2, 9, \cdot, 8, 1) - 4$.

The canonical form of this position is

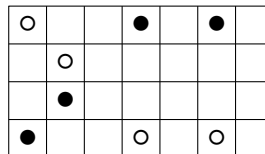
$$\langle\langle(11 \mid \emptyset^{11}) \mid 6\rangle, \langle(11 \mid \emptyset^{11}) \mid (1 \mid \emptyset^1)\rangle \mid \langle\emptyset^{-4} \mid -4\rangle\rangle - 2.$$

Playing *without points*, as if it were normal-play, the option $\langle(11 \mid \emptyset^{11}) \mid 6\rangle$ reverses out. However, regarded as a scoring game (which it is), the left option $\langle(11 \mid \emptyset^{11}) \mid 6\rangle - 2$ is the only winning move. Left does not make the last move, but it does not matter. The existence of such positions in TAKE-SMALL explains why it is a hybrid ruleset.

3.7. Order in scoring-play and outcomes. As mentioned before, scoring order does not directly relate to outcomes. Consider the following DISKONNECT position G :

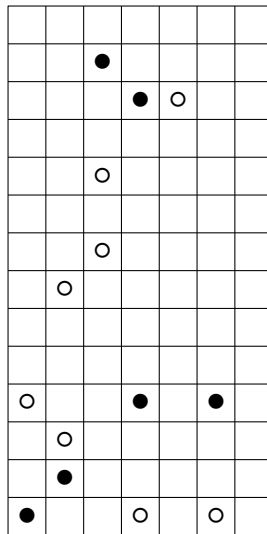


After reductions, the canonical form of G is $\langle 1 \mid 1 \rangle$. Left always wins, playing first or second. However, $G \not\approx 0$. In order to distinguish G from 0, consider the following zugzwang distinguishing game X :

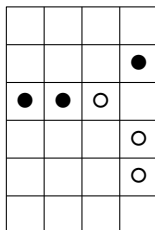


In fact, X reduces a lot! Its canonical form is $\langle \emptyset^{-2} \mid \emptyset^2 \rangle$; it behaves like an empty tree. Hence, $G + X = \langle 1 \mid 1 \rangle + \langle \emptyset^{-2} \mid \emptyset^2 \rangle$ whose canonical form is $\langle\langle \emptyset^{-1} \mid \emptyset^3 \rangle \mid \langle \emptyset^{-1} \mid \emptyset^3 \rangle\rangle$.

Therefore, in X , Right loses going first and, in $G + X$, Right wins going first. Here we display the game $G + X$, as it might appear in a play-situation:



3.8. Order in different contexts. We recall that the behavior of a position may change depending on the context. Consider the following KOBBER position G :



Without reductions, the literal form of G is

$$\langle 0, \langle \langle 2 \mid -1, \langle 1 \mid -1 \rangle \rangle \mid \langle 3, \langle 3 \mid 1 \rangle \mid 0 \rangle \rangle \mid \langle 1 \mid 0 \rangle \rangle,$$

where the only Right option is $\langle 1 \mid 0 \rangle$ (the left-most white stone can capture a black one via a COBBER move). Left has two options, and in the dicot context, $G \succcurlyeq 1$, that is, in all situations Left *should prefer to have this component rather than add one point to her score* [23]. On the other hand, in the partizan context, there are situations where one point is better than having that form. Consider the position $G - \hat{2}$ (Right has two free moves); Left cannot finish with a final score better than 0.

3.9. A scoring games calculator. The algebra of games (disjunctive sums, canonical forms, and so on) is not a trivial task and cannot be done manually except for very simple positions. A computer program is required for more complex

situations. The *scoring games calculator* [33] is such a program, suitable for the universe of guaranteed scoring games. It was implemented as a set of Haskell modules that run on an interpreter available in any Haskell distribution or embedded in a program that imports these modules.

4. Scoring games on graphs

Despite not having a theory of scoring-play, several scoring games have been studied in the literature. A typical approach, first identified in [43], is to approximate a graph parameter by having the players choose, for example, vertices or edges. When the game is over, an instance of the structure or set has been created. Left wants the cardinality of the structure or set as large as possible, i.e., Left is the *maximizer*, and Right wants it as small as possible, i.e., he is the *minimizer*. Left always wins because Right never scores points, therefore “this is boring for Right” is a valid comment. To convert to a game to be potentially interesting for both players, prelabel the connected components as either Left subgraphs and Right subgraphs. Left’s score is then the sum of the cardinalities in the Left components, Right’s score is the sum in the Right components and the score in the game is the difference. In all of the games *M2*, *O1*, *O3*, *O4*, and *O8*, the game finishes after a fixed number of moves, which is either the number of vertices or the number of edges. The techniques of Will Johnson [31] will be useful in analyzing these games.

We consider the maximizer/minimizer games first and then report on some others.

4.1. *Maximizer-minimizer games.*

M1: CLEANING GAMES [9]. The toppling number of a graph can be defined in the context of a zero-sum two-player game played on the graph as follows. The maximizer (who moves first in the original game of [9]) and the minimizer alternately place chips on the vertices of the graph. If the number of chips at a vertex is equal to its degree, it sends one chip to each neighbor vertex. The game ends if there exist an infinite sequence of vertices that send chips to their neighbors. Player 1 wants to maximize the number of chips played during the game, while player 2 tries to minimize this.

M2: COMPETITION-REACHABILITY OF A GRAPH [44]. On a turn, players choose an undirected edge and give it an orientation. After all edges have been chosen in a subgraph H , the score is

$$|\{(x, y) \mid x, y \in V(H) \text{ and there is a directed path from } x \text{ to } y\}|.$$

If H is a Left subgraph then the score is added to Left’s total, otherwise it is added to Right’s total.

Let P_n be the path on n vertices. In [44], they give $Rs(P_n)$ and $Ls(P_n)$ and, in particular, show that if n is even the game is a zugzwang. This is an impartial game but not all-normal.

M3: DOMINATION GAME. Given a graph G , start with $S = \emptyset$. Players choose a vertex and add it to S but a new vertex of G must be dominated by S . The players are called Staller (Left) and Dominator (Right). This is an impartial game but not all-normal.

This is considered in several papers. In [11] it is shown that $|Ls(G) - Rs(G)|$ is at most 1 and that $\gamma(G) \leq Ls(G) \leq 2\gamma - 1$, where γ is the cardinality of the least dominating set. In [30] they conjecture that $Ls(G) \leq \frac{3}{4}|V(G)|$.

M4: GAME CHROMATIC NUMBER and GRUNDY NUMBERS. In the CHROMATIC NUMBER GAME, there is a graph G and a fixed number of colors k . On a turn a player chooses an uncolored vertex and colors it with one of the allowed colors, maintaining a proper coloring. Right wins if G can be colored and Left wins otherwise. Note that for a given k this is a *short disjunctive sum* since Left only has to win in one component. The game chromatic number of G is the least k such that Right can win going first. Despite this concept not coming directly from a scoring game there are several scoring games that have been used to approximate it. See [35] for a follow up and [4] for a survey.

GAME COLORING NUMBER: There are two variants.

- (1) **MARKING GAME:** Given a graph G , a move consists of a player marking an unmarked vertex of G . The game ends when all vertices are marked. For $v \in V(G)$, let $b(x)$ be the number of neighbors of v that are marked before v is marked. The score of the game is $s = 1 + \max\{b(v), v \in V(G)\}$. By convention of this game, Right (the minimizer) starts. The game coloring number $\text{col}_g(G)$ of G is $Rs(G)$. Some known results include

$$\text{col}_g(G) \leq \begin{cases} 4 & \text{if } G \text{ is a forest, [24];} \\ 17 & \text{if } G \text{ is a planar, [55];} \\ 7 & \text{if } G \text{ is a outerplanar, [28; 32].} \end{cases}$$

- (2) **GAME GRUNDY NUMBER:** The colors are the positive integers. Players choose uncolored vertices and it is colored the least integer not in its neighborhood, i.e., let $c(x)$ be the color of vertex x , so that $c(v) = \text{mex}\{c(x) : x \sim v\}$.

M5: The GAME TRANSVERSAL NUMBER. Let $H = (V, E)$ be a hypergraph with vertex set V and edge set E of order $n(H) = |V|$ and size $m(H) = |E|$. A transversal in H is a subset of vertices in H that has a nonempty intersection with every edge of H . A vertex hits an edge if it belongs to that edge. In the

transversal game played on H , Left and Right choose vertices where each vertex chosen must hit at least one edge not hit by the vertices previously chosen. The game ends when the set of vertices chosen becomes a transversal in H . This game was introduced in [12]. The game is impartial but not all-normal.

M6: GRAPH SATURATION GAME. Given a family of graphs $\{H_i\}$ and an integer n , build a graph by adding edges such that no copy of any H_i is built. The score is the number of edges.

The original game, a misère-play game was proposed in [13] and rediscovered by Hajnal. In both, the first player to create a triangle K_3 lost.

In [8] they show that $Ls(K_3, n) \leq (n \log n)/2 - 2n \log \log n + O(n)$ and report that Erdős claimed $n^2/5$ as the correct value. See also [45; 35].

The most general formulation of the game is given by Doug West [52].

Also related is the GRAPH MATCHING GAME [53]. Players choose independent edges, and the score is the number of edges. In [17] it is shown that $|Ls(G) - Rs(G)| \leq 1$ and that $Ls(G) \geq \frac{2}{3}m(G)$, where $m(G)$ is the cardinality of the largest matching of G .

4.2. Other games.

O1: EDGE-BALANCE INDEX GAME. Given a Graph $G = (V, E)$, Left colors edges blue and Right colors edges by red. Vertex v is colored as blue if it is incident with more blue than red edges and is colored red if it is incident with more red than blue edges and remains uncolored otherwise. The score is the difference between the blue and red vertices. The game is impartial but not all-normal.

The edge balance index is defined in [26] where the indications are that it will be difficult to find this number. In [15] the game is introduced. In addition, the edge-balance index with terminals is also introduced. Here, there are two specified vertices s and t . The play is the same but Left wins if there is a completely blue path (both edges and vertices) from s to t ; Right wins if there is a completely red path and there is a tie if there is one of each. This is not a scoring game but a normal-play partizan game, except a tie is possible. This becomes an impartial, not all-normal game by making the Left score is the number of blue paths and the Right score the number of red paths.

O2: BRIDGE BUILDING. Suppose a directed graph G , where each arc has an associated cost, and some $s, t \in V(G)$. The goal is to build a path from s to t . The first choice is an edge out of s and then, from each vertex, the next player selects the next vertex to be visited among all neighboring vertices of the current vertex — a path from s to t must still be constructible. The player has to pay the associated cost of the arc. The score is the difference between what the players pay. This is an impartial game but not all-normal, particularly if negative costs (a profit can be made) are allowed.

The game was introduced in [19] where the authors consider the “economic” game where each player minimizes its costs taking into account that also the other player acts in a selfish and rational way. They show that the decision problem associated with such a path is PSPACE-complete even for bipartite graphs. They also note that forcing the play around an even cycle is not a rational play. If we play the difference-of-scores then playing around an even cycle could be a good strategy.

In [20], the authors introduce a variant, SHORTEST CONNECTION GAME, where the two players start at different vertices, say s and t , and build their own paths. The game ends when the two paths first intersect say at vertex m . There are now two paths: L , which is Left’s from s to m , and R , Right’s from t to m . Left’s score S_L is the sum of the edges of L and Right’s score R_S is the sum of the edges of R and the game score is the $R_S - L_S$. They prove some complexity results; for example, SHORTEST CONNECTION GAME is PSPACE-complete for directed bipartite graphs even if all costs are bounded by a constant. This is not impartial nor all-normal.

O3: GRAPH GRABBING GAME. The vertices of the graph have coins, of possibly different denominations, at each vertex. On a turn, a player removes a noncut vertex and pockets the coins on the vertex. The score is the difference; that is, the winner is the player with the most coins. This is an impartial, not all-normal game.

This is a generalization of the original game played on a path [54]. The next version was played on trees [41] and generalized to the graphs in [36]. In [42], it is shown that if played on a tree T with an even number of vertices then $Ls(T)$ is at least half the total amount of money.

O4: MEDIAN GRAPH GAME. For a set of vertices X , set $d(v, X) = \sum_{x \in X} d(x, v)$ and $\text{med}(X) = \min\{d(v, X) : v \in X\}$. The players choose vertices until all are chosen. Let L be the set chosen by Left and R that by Right. The Left and Right scores are $-\text{med}(L)$ and $-\text{med}(R)$ respectively and the alternating play score, assuming Left plays first, is $\mu(G) = \text{med}(R) - \text{med}(L)$. Compare this with the Weiner game *O8*.

The game was introduced in [14], where the authors give $\mu(G)$ for various G including a subset of trees and for complete bipartite graphs $K_{m,n}$, $m \geq n$, where

$$\mu(K_{m,n}) = \begin{cases} 1 & \text{if } m \neq n \text{ and both are odd;} \\ 0 & \text{if } m = n \text{ and both are even;} \\ -1 & \text{if } m \text{ is odd and } n \text{ is even;} \\ -2 & \text{if } m \text{ is even and } n \text{ is odd.} \end{cases}$$

O5: GRAPH OCCUPATION GAME. Let G be a connected graph and let L and R be empty sets. On a turn, Left chooses a vertex and adds it to L , and Right adds a vertex to R . The extra condition is that both sets must be connected and

$L \cap R = \emptyset$. If a player cannot move the other player is allowed to take the rest of the (legal) vertices. The Left score is $|L|/|V(G)|$ and Right's is $|R|/|V(G)|$.

The game is introduced in [46]. The author notes a similarity to GO and shows that for each $\epsilon > 0$ there exists graphs where $Ls(G) \geq 1 - \epsilon$ but also that there exists graphs with $Ls(G) \leq \epsilon$.

O6: PIRATES AND TREASURE. Suppose a graph G with weights on some vertices (treasure) and some blue and red tokens on some vertices — there is no standard starting position. On a turn, Left must move a blue token to an adjacent vertex that contains treasure that they take and which is added to the player's score. The game is over when the player to move cannot and the player with the greatest amount of treasure wins.

The game is introduced in [51] where it is shown to be NP-hard to determine the final scores. In [2] it is shown to be PSPACE complete to determine if Right can win.

O7: SUBSET SUM GAME. Suppose an integer n and a set of weights $\{w_i\}$. The sum S starts at 0 and is not allowed to exceed n . On a turn, a player chooses a weight, removes from the set and adds it to the sum. A player's score is the sum of the weights they added to S .

The game, coming from the *knapsack* problem, is introduced in [18], where they consider two strategies: the greedy strategy and the one move look-ahead.

O8: WEINER INDEX GAME. Given a graph G , players choose vertices. The Left score is the sum of all the distances between the vertices Left chose. Similarly for Right. Since the winner is the player with the least sum, the score is defined as $R_s - L_s$. This is introduced in [3], where the authors require that $|V(G)|$ be even. It arises out of the study of the Wiener index from mathematical chemistry. They report that $|Ls(K_{1,2n-1}) - Rs(K_{1,2n-1})| = n - 1$.

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urban031@gmail.com *Department of Industrial Engineering and Management,
Technion - Israel Institute of Technology, Haifa, Israel*

rjn@mathstat.dal.ca *Department of Mathematics and Statistics,
Dalhousie University, Halifax, NS, Canada*

cmfsantos@fc.ul.pt *Center for Functional Analysis, Linear Structures and
Applications, University of Lisbon, Lisboa, Portugal*

