Unsolved problems in combinatorial games

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During the recent development of combinatorial game theory many more problems have been suggested than solved. Here is a collection of problems that many people have found interesting. Included is a discussion of recent developments in these areas. This collection was started by Richard Guy in 1991, and has been updated in each *Games of No Chance* volume.

The problems are divided into five sections:

A. Taking and breaking

B. Pushing and placing pieces

C. Playing with pencil and paper

D. Disturbing and destroying

E. Theory of games

In the sections, the problems are identified by letters and numbers. Any number in parenthesis is the old number used in each of the lists of unsolved problems given on pp. 183–189 of AMS *Proc. Sympos. Appl. Math.* **43** (1991), called PSAM **43** below; on pp. 475–491 of *Games of No Chance*, hereafter referred to as GONC.

Previous versions of the unsolved problems can also be found on pp. 457–473 of *More Games of No Chance* (MGONC); pp. 475–500 of *Games of No Chance* 3 (GONC3); and pp. 279–308 of *Games of No Chance* 4 (GONC4).

Some problems have little more than the statement of the problem if there is nothing new to be added. References [#] are at the end of this article. Useful references for the rules and an introduction to many of the specific games mentioned below are:

- M. Albert, R. J. Nowakowski, and D. Wolfe, *Lessons in Play*: *An Introduction to the Combinatorial Theory of Games*, A K Peters, 2007 (*LIP*);

- Berlekamp, Conway, and Guy, *Winning Ways for your Mathematical Plays*, vol. 1–4, A K Peters, 2000–2004 (*WW*);

- Siegel, *Combinatorial Game Theory*, American Math. Society, 2013 (*CGT*).

The term "game" is ambiguous and needs to be clarified. If a board is not specified then the question pertains to all possible boards. Berlekamp noted that many games have a standard opening position, an empty board for example. However, some results include positions, such as Garden-of-Eden positions, that could never occur by any sequence of moves from a standard opening position, and ones which would never occur in decent play. Indeed, Plambeck has asked at several meetings whether there is any difference in the complexity results for all positions and those that would occur in decent play.

Common usage suggests the term "game-name" as meaning the ruleset without a specified opening board. For example, Drummond-Cole's *2 in DOMINEERING has "holes" which could not be formed starting from the empty board. The term *restricted* will be used if the positions are (or must be) obtainable from a standard opening board.

Acknowledgement. I have had help in compiling this collection from all those mentioned, and from others. I would especially like to mention Elwyn Berlekamp, Gabriel Drummond-Cole, Alex Fink, Aviezri Fraenkel, Urban Larsson, Richard Low, Neil McKay, Carlos Santos, Simon Rubinstein-Salzedo, Alex Sierra Ballarin, and Eric Sopena.

A. Taking and breaking games

A1. (1) Subtraction games with finite subtraction sets are known to have periodic nim-sequences. Investigate the relationship between the subtraction set and the length and structure of the period. The same question can be asked about *partizan* subtraction games, in which each player is assigned an individual subtraction set. See Fraenkel and Kotzig [36].

(A move in the game $S(s_1, s_2, s_3, \ldots)$ is to take a number of beans from a heap, provided that number is a member of the *subtraction-set*, $\{s_1, s_2, s_3, \ldots\}$. Analysis of such a game and of many other heap games is conveniently recorded by a *nim-sequence*,

$$n_0 n_1 n_2 n_3 \ldots,$$

meaning that the nim-value of a heap of h beans is n_h; i.e., that the value of a heap of h beans in this particular game is the *nimber* $*n_h$.)

For examples, see Table 2 in Section 4 on p. 67 of "Impartial games" in GONC.

It would now seem feasible to give the complete analysis for games whose subtraction sets have just three members, though this has so far eluded us. Several people, including Mark Paulhus and Alex Fink, have given a complete analysis

for all sets $\{1, b, c\}$ and for sets $\{a, b, c\}$ with $a < b < c < 32$. Mark Daniel Ward has pushed this to $a < b < c < 4096$ and has conjectured [127]:

Case 1: Suppose $c = a + b$. Let $j \equiv b - a \bmod 2a$, $0 \leq j < 2a$. Then

$$p = \begin{cases} 2b + a - j & \text{if } 0 \leq j < a, \\ a(2b - a + j)/\gcd(a, 2a - j) & \text{if } a \leq j < 2a. \end{cases}$$

Case 2: Suppose $c \neq a + b$; then p is a divisor of at least one of $a + b$, $a + c$, and $b + c$. Moreover, if G is the subset of these terms which p divides then $p = \gcd_{(x+y) \in G} (x + y)$.

In general, period lengths can be surprisingly long, and it has been suggested that they could be superpolynomial in terms of the size of the subtraction set. However, Guy conjectures that they are bounded by polynomials of degree at most $\binom{n}{2}$ in s_n, the largest member of a subtraction set of cardinality n. It would also be of interest to characterize the subtraction sets which yield a purely periodic nim-sequence, i.e., for which there is no preperiod. Carlos Santos [101] reduced this upper bound slightly by using a dynamical system approach.

Angela Siegel [109] (and *LIP*) considered infinite subtraction sets which are the complement of finite ones and showed that the nim-sequences are always arithmetic periodic. That is, the nim-values belong to a finite set of arithmetic progressions with the same common difference. The number of progressions is the period and their common difference is called the *saltus*. For instance, the game $S\{\hat{4}, \hat{9}, \hat{26}, \hat{30}\}$ (in which a player may take any number of beans except 4, 9, 26, or 30) has a preperiod of length 243, period-length of 13014, and saltus of 4702. Marla Clusky and Danny Sleator have proved her conjecture that the class of games $S\{\hat{a}, \hat{b}, \widehat{a+b}\}$ is purely periodic with period length $3(a + b)$.

For infinite subtraction games in general there are corresponding questions about the length and purity of the period.

Suppose the elements of the finite subtraction sets are not constants. It was shown by Fraenkel [35] that the game with subtraction set

$$\{1, 2, \ldots, t - 1, \lfloor n/t \rfloor\},$$

where $t \geq 2$ is a fixed integer and n is the pile size, has an *aperiodic nim-sequence*. See also Fraenkel [35], Guo [47]. Sopena [112] has a variant, I-MARK(S, D), where S contains the subtraction, and D the division elements so that a move from n is one of $n - s$ for $s \in S$, and $\lfloor n/d \rfloor$ for $d \in D$. He shows that I-MARK$(\{1\}, \{2\})$ has an aperiodic nim-sequence but a periodic outcome sequence. The nim-sequence has a "period" except for one element which is nonperiodic and the values are taken from $\{0, 1, 2\}$. Larsson and Fox [70] show that the subtraction game with $S = \{F_{2n+1} - 1\}$ has a nonperiodic nim-sequence with values from $\{0, 1, 2\}$.

It is possible to get a subtraction-division game with nonperiodic nim-sequence with values from {0, 1}. Write n in binary and a move is to remove a leading 1, but not if n is a power of 2, or remove a trailing 0. The number of moves is fixed so this is a version of SHE-LOVES-ME-SHE-LOVES-ME-NOT. Can this be done with a subtraction game set?

An old, voiced but not written, question asks for an infinite subtraction set where the associated outcome-sequence is periodic but the nim-sequence is aperiodic. These are easy to find in octal games.

A2. (2) Are all finite *octal games* ultimately periodic? (If the binary expansion of the k-th code digit in the game with code $\mathbf{d}_0 \cdot \mathbf{d}_1 \mathbf{d}_2 \mathbf{d}_3 \ldots$ is

$$\mathbf{d}_k = 2^{a_k} + 2^{b_k} + 2^{c_k} + \cdots,$$

where $0 \le a_k < b_k < c_k < \ldots$, then it is legal to remove k beans from a heap, provided that the rest of the heap is left in exactly a_k or b_k or c_k or $\ldots$ nonempty heaps. See [*WW*, pp. 81–115]. Some specimen games are exhibited in Table 3 of Section 5 of "Impartial games" in GONC.)

Resolve any number of outstanding particular cases, e.g., ·**6** (OFFICERS), ·**04**, ·**06**, ·**14**, ·**36**, ·**37**, ·**64**, ·**74**, ·**76**, ·**004**, ·**005**, ·**006**, ·**007**, ·**014**, ·**015**, ·**016**, ·**024**, ·**026**, ·**034**, ·**064**, ·**114**, ·**125**, ·**126**, ·**135**, ·**136**, ·**142**, ·**143**, ·**146**, ·**162**, ·**163**, ·**164**, ·**166**, ·**167**, ·**172**, ·**174**, ·**204**, ·**205**, ·**206**, ·**207**, ·**224**, ·**244**, ·**245**, ·**264**, ·**324**, ·**334**, ·**336**, ·**342**, ·**344**, ·**346**, ·**362**, ·**364**, ·**366**, ·**371**, ·**374**, ·**404**, ·**414**, ·**416**, ·**444**, ·**564**, ·**604**, ·**606**, ·**744**, ·**764**, ·**774**, ·**776**, and GRUNDY'S GAME (split a heap into two unequal heaps [*WW*, pp. 96–97, 111–112; *LIP*, p. 142]), which has been analyzed, first by Dan Hoey, and later by Achim Flammenkamp, as far as heaps of 2^{35} beans.

In this volume, J. P. Grossman, using a new approach based on the sparse space phenomenon, has analyzed ·**6** up to heaps of size 2^{47} and has found no periodicity.

Perhaps the most notorious and deserving of attention is the game ·**007**, one-dimensional TIC-TAC-TOE, or TREBLECROSS, which Flammenkamp has pushed to 2^{25}. Figure 1 shows the first 11 million nim-values, a small proportion of which are ≥ 1024; the largest, $\mathcal{G}(6193903) = 1401$ is shown circled. Will 2048 ever be reached?

Flammenkamp has settled ·**106**: it has the remarkable period and preperiod lengths of 328226140474 and 465384263797. For information on the current status of each of these games, we refer the reader to Flammenkamp's web page at uni-bielefeld.de/~achim/octal.html.

A game similar to GRUNDY'S GAME, and which is also unsolved, is John Conway's COUPLES-ARE-FOREVER [*LIP*, p. 142], where a move is to split any
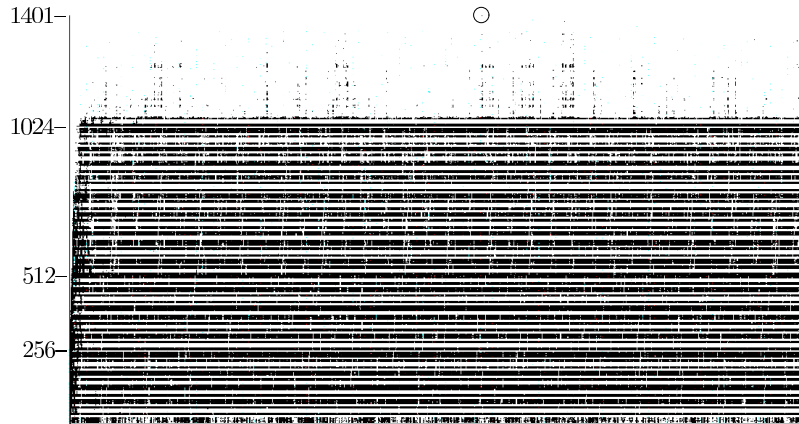
Figure 1. Plot of 11000000 nim-values of the octal game ·**007**.

heap except a heap of two. The first 50 million nim-values haven't displayed any periodicity. See Caines et al. [14]. More generally, Bill Pulleyblank suggests looking at splitting games in which you may only split heaps of size $> h$, so that $h = 1$ is SHE-LOVES-ME-SHE-LOVES-ME-NOT and $h = 2$ is COUPLES-ARE-FOREVER. David Singmaster suggested a similar generalization — you may split a heap provided that the resulting two heaps each contain at least k beans: $k = 1$ is the same as $h = 1$, while $k = 2$ is the third cousin of DAWSON'S CHESS.

Explain the structure of the periods of games known to be periodic.

In *Discrete Math.*, 44 (1983), pp. 331–334, Problem 38, Fraenkel raised questions concerning the computational complexity (see Section E1 below) of octal games. In Problem 39, he and Kotzig define *partizan octal games* in which distinct octals are assigned to the two players. Mesdal [GONC3, pp. 447–461] shows that in many cases, if the game is "all-small" [*WW*, pp. 229–262; *LIP*, pp. 183–207], then the atomic weights are arithmetic periodic. (See Section E13 for an explanation of the new name "dicot" games.) In Problem 40, Fraenkel introduces *poset games*, played on a partially ordered set of heaps, each player in turn selecting a heap and then removing a nonnegative number of beans from this heap and from each heap above it in the ordering, with at least one heap being reduced in size. For posets of height one, new regularities in the nim-sequence can occur. Tanya Khovanova and the author propose the following simple version. Given n nim-heaps of sizes $a_1, a_2, \ldots, a_n$, a move is to choose some heap i and some number $1 \leq x \leq a_i$ and for each $j \geq i$ replace a_j by $\max\{a_j - x, 0\}$. The solution for three heaps involves the continued fraction expansion of $\sqrt{2}$.

Note that this includes, as particular cases, SUBSET TAKEAWAY, CHOMP or DIVISORS, and GREEN HACKENBUSH forests. Compare Problems A3, A13, D1, and D2 below.

A3. (3) Hexadecimal games have code digits $\mathbf{d}_k$ in the interval from 0 to f ($= 15$), so that there are options splitting a heap into three heaps. See [*WW*, pp. 116–117].

Such games may be arithmetically periodic. Nowakowski has calculated the first 100000 nim-values for each of the 1-, 2-, and 3-digit games. Richard Austin's Theorem 6.8 in his thesis [4] and the generalization by Howse and Nowakowski [59] suffice to confirm the arithmetic periodicity of several of these games.

Some interesting specimens are ·**28** = ·**29**, which have period 53 and saltus 16, the only exceptional value being $\mathcal{G}(0) = 0$; ·**9c**, which has period 36, preperiod 28, and saltus 16; and ·**f6** with period 43 and saltus 32, but its apparent preperiod of 604 and failure to satisfy one of the conditions of the theorem prevent us from verifying the ultimate periodicity. The game ·**205200c** is arithmetic periodic with preperiod length 4, period length 40, and saltus 16, except that $40k + 19$ has nim-value 6 and $40k + 39$ has nim-value 14. This regularity (which also seems to be exhibited by ·**660060008** with a period length of approximately 300000) was first reported in Horrocks and Nowakowski [57] (see Problem A2). Grossman and Nowakowski [GONC4] have shown that the nim-sequences for ·**200…0048**, with an odd number of zero code digits, exhibit "ruler function" patterns. The game ·**9** has not so far yielded its complete analysis, but, as far as analyzed (to heaps of size 100000), exhibits a remarkable fractal-like set of nim-values. See Howse and Nowakowski [59]. Also of special interest are ·**e**; ·**7f** (which has a strong tendency to period 8, saltus 4, but, for $n \leq 100000$, has 14 exceptional values, the largest being $\mathcal{G}(94156) = 26614$); ·**b6** (which "looks octal"); ·**b33b** (where a heap of size n has nim-value n except for 27 heap sizes which appear to be random).

Other unsolved hexadecimal games are

·**1x**,	$x \in \{8, 9, c, d, e, f\}$,	·**2x**,	$a \leq x \leq f$,
·**3x**,	$8 \leq x \leq e$,	·**4x**,	$x \in \{9, b, d, f\}$,
·**5x**,	$8 \leq x \leq f$,	·**6x**,	$8 \leq x \leq f$,
·**7x**,	$8 \leq x \leq f$,	·**9x**,	$1 \leq x \leq a$,
·**9d**,		·**bx**,	$x \in \{6, 9, d\}$,
·**dx**,	$1 \leq x \leq f$,	·**fx**,	$x \in \{4, 6, 7\}$.

A4. (53) N-HEAP WYTHOFF GAME. We are given $N \geq 2$ heaps of finitely many tokens, whose sizes are $p_1, \ldots, p_N$ with $p_1 \leq \cdots \leq p_N$. Players take turns removing any positive number of tokens from a *single* heap or removing $(a_1, \ldots, a_N)$ from *all* the heaps — a_i from the i-th heap — subject to the conditions:

(i) $0 \leq a_i \leq p_i$ for each i,

(ii) $\sum_{i=1}^{N} a_i > 0$,

(iii) $a_1 \oplus \cdots \oplus a_N = 0$, where $\oplus$ is nim-addition.

The player making the last move wins and the opponent loses. Note that the classical Wythoff game is the case $N = 2$.

For $N \geq 3$, Fraenkel makes the following conjectures.

Conjecture 1. For every fixed set $K := (A^1, \ldots, A^{N-2})$ there exists an integer $m = m(K)$ (i.e., m depends only on K) such that

$$(A^1, \ldots, A^{N-2}, A_n^{N-1}, A_n^N), \quad A^{N-2} \leq A_n^{N-1} \leq A_n^N,$$

with $A_n^{N-1} < A_{n+1}^{N-1}$ for all $n \geq 1$, is the n-th $\mathcal{P}$-position, and

$$A_n^{N-1} = \text{mex}(\{A_i^{N-1}, A_i^N : 0 \leq i < n\} \cup T), \quad A_n^N = A_n^{N-1} + n,$$

for all $n \geq m$, where $T = T(K)$ is a (small) set of integers.

That is, if you fix $N - 2$ of the heaps, the $\mathcal{P}$-positions resemble those for the classical Wythoff game. For example, for $N = 3$ and $A^1 = 1$, we have $T = \{2, 17, 22\}$ and $m = 23$.

Conjecture 2. For every fixed K there exist integers $a = a(K)$ and $M = M(K)$ such that
$$A_n^{N-1} = \lfloor n\phi \rfloor + \varepsilon_n + a \quad \text{and} \quad A_n^N = A_n^{N-1} + n,$$

for all $n \geq M$, where $\phi = \frac{1}{2}(1 + \sqrt{5})$ is the golden section, and $\varepsilon_n \in \{-1, 0, 1\}$.

In Fraenkel and Krieger [37] the following was shown, inter alia: Let $t \in \mathbb{Z}_{\geq 1}$, $\alpha = \frac{1}{2}(2 - t + \sqrt{t^2 + 4})$, where $\alpha = \phi$ for $t = 1$, $T \subset \mathbb{Z}_{\geq 0}$ a finite set, and $A_n = \text{mex}(\{A_i, B_i : 0 \leq i < n\} \cup T)$, where $B_n = A_n + nt$. Let $s_n := \lfloor n\alpha \rfloor - A_n$. Then there exist $a \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 1}$ such that for all $n \geq m$, either $s_n = a$, or $s_n = a + \varepsilon_n$, $\varepsilon_n \in \{-1, 0, 1\}$. If $\varepsilon_n \neq 0$, then $\varepsilon_{n-1} = \varepsilon_{n+1} = 0$. Also the general structure of the ε_n was characterized succinctly.

This result was then applied to the N-heap WYTHOFF GAME. In particular, for $N = 3$ (such that $K = A^1$) it was proved that $A_n^2 = \text{mex}(\{A_i^2, A_i^3 : 0 \leq i < n\} \cup T)$, where

$$T = \{x \geq K : \text{there exists } 0 \leq k < K \text{ such that } (k, K, x) \text{ is a } \mathcal{P}\text{-position}\}$$

$$\cup \{0, \ldots, K - 1\}.$$

The following upper bound for A_n^3 was established: $A_n^3 \leq (K + 3)A_n^2 + 2K + 2$. It was also proved that Conjecture 1 implies Conjecture 2.

In Sun and Zeilberger [117], a sufficient condition for the conjectures to hold was given. It was then proved that the conjectures are true for the case $N = 3$,

where the first heap has up to ten tokens. For those ten cases, the parameter values m, M, a, T were listed in a table.

Sun [116] obtained results similar to those in Fraenkel and Krieger [37], but the proofs are different. In that work, it was also proved that Conjecture 1 implies Conjecture 2. A method was given to compute a in terms of certain indexes of the A_i and B_j.

A5. (23) BURNING-THE-CANDLE-AT-BOTH-ENDS. Conway and Fraenkel ask us to analyze NIM played with a row of heaps. A move may only be made in the leftmost or in the rightmost heap. When a heap becomes empty, then its neighbor becomes the end heap.

Albert and Nowakowski [2] have determined the outcome classes in impartial and partizan versions (called END-NIM [*LIP*, pp. 210, 263]) with finite heaps, and Duffy, Kolpin, and Wolfe [GONC3, pp. 419–425] extend the partizan case to infinite ordinal heaps. Wolfe asks for the actual values.

Nowakowski suggested to analyze impartial and partizan END-WYTHOFF: take from either end-pile, or the *same* number from both ends. The impartial game is solved by Fraenkel and Reisner [GONC3, pp. 329–347]. Fraenkel [32] asks a similar question about a generalized Wythoff game: take from either end-pile or take $k > 0$ from one end-pile and $\ell > 0$ from the other, subject to $|k - \ell| < a$, where a is a fixed integer parameter ($a = 1$ is END-WYTHOFF).

There is also HUB-AND-SPOKE NIM, proposed by Fraenkel. One heap is the hub and the others are arranged in rows forming spokes radiating from the hub. Albert notes that this game can be generalized to playing on a forest, i.e., a graph where each of whose components is a tree. The most natural variant is that beans may only be taken from a leaf (valence 1) or isolated vertex (valence 0).

The partizan game of RED-BLUE CHERRIES is played on an arbitrary graph; see McCurdy [81]. A player picks an appropriately colored cherry from a vertex of minimum degree, which disappears at the same time. Albert, Grossman, McCurdy, Nowakowski, and Wolfe [1] (mistakenly) show that if the graph has a leaf, then the value is an integer. They ask this: Is every RED-BLUE CHERRIES position an integer? Matthew Bardoe and Scott Herman (personal communication) disproved both by finding a 7-cycle with two leaves that has value $\frac{1}{2}$.

A6. (17) Extend the analysis of KOTZIG'S NIM [*WW*, pp. 515–517]. Is the game eventually periodic in terms of the length of the circle for every finite move set? Analyze the misère version of KOTZIG'S NIM.

Let $\Gamma(S; n)$ be the outcome of the game where S lists all the moves and n is the size of the circle. Tan and Ward [118] gave more evidence of the periodicity by showing the following: If $n \in \{1, 3, 5, 7, 15\}$, or if $n \equiv 3 \pmod 5$ and $n \geq 23$, then $\Gamma(1, 4; n) \in \mathcal{P}$; otherwise, $\Gamma(1, 4; n) \in \mathcal{N}$. They also give several conjectures

including these: If n is odd and $n \notin \{9, 11, 17\}$ then $\Gamma(3, 5; n) \in \mathcal{P}$ and is in $\mathcal{N}$ otherwise. Their evidence bolsters the periodicity conjecture but also indicates that there is likely to be a lot of noise when n is small.

A7. (18) Obtain asymptotic estimates for the proportions of $\mathcal{N}$-, $\mathcal{O}$- and $\mathcal{P}$-positions in Epstein's PUT-OR-TAKE-A-SQUARE game [*WW*, pp. 518–520].

A8. GALE'S NIM. This is NIM played with four heaps, but the game ends when three of the heaps have vanished, so that there is a single heap left. Brouwer and Guy have independently given a partial analysis, but the situation where the four heaps have distinct sizes greater than 2 is open. An obvious generalization is to play with h heaps and play finishes when k of them have vanished.

A9. EUCLID'S NIM is played with a pair of positive integers, a move being to diminish the larger by any multiple of the smaller. The winner is the player who reduces a number to zero. Analyses have been given by Cole and Davie [19], Spitznagel [114], Lengyel [75], Collins [20], Fraenkel [34], and Nivasch [91]. Lengyel [75] reports that Schwartz first found that (a, b) is the sequential sum of nim-heaps given by the normal continued fraction of a/b, where the last number in the continued fraction depends on the presence or absence of Fibonacci numbers. Gurvich [49] shows that the nim-value $g^+(a, b)$ for the pair (a, b) in normal play is the same as the misère nim-value $g^-(a, b)$ except for $(a, b) = (kF_i, kF_{i+1})$, where $k > 0$ and F_i is the i-th Fibonacci number. In this case, $g^+(kF_i, kF_{i+1}) = 0$ and $g^-(kF_i, kF_{i+1}) = 1$ if i is even and the values are reversed if i is odd. Aaron Siegel notes that this can be restated as this: EUCLID is tame, and the fickle positions have the form (kF_i, kF_{i+1}). The latter have genus 0^1 when i is even and 1^0 when i is odd.

Collins and Lengyel [21] play EUCLID'S NIM with three integers and solve some special cases.

In PARTIZAN EUCLID, from the position (p, q), where $p = kq + r$, $0 \le r < q$, Left moves to (q, r) and Right to $(q, (k + 1)q - p)$. The outcome classes are investigated in [82]. Determining the values seems hard. The atomic weights may be easier — see Figure 2, where the *mean atomic weights* for $0 < q < p \le 100$ are given, where the coordinates (x, y) are $x = q/p$ and y is the mean atomic weight of (p, q).

A10. (20) DUDENEY [*WW*, pp. 521–523] is NIM, but with an upper bound Y on the number of beans that may be taken, and with the restriction that a player may not repeat his opponent's last move. If Y is even, the analysis is easy. Some advance in the analysis, when Y is odd, has been made by Marc Wallace, Alex Fink and Kevin Saff.
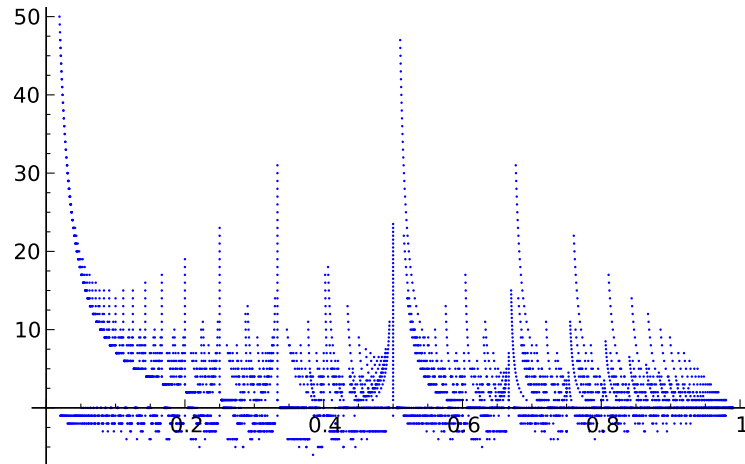
Figure 2. PARTIZAN EUCLID: mean atomic weights for $0 < q < p < 101$.

We can, for example, extend the table of strings of pearls given in [*WW*, p. 523], with the following values of Y which have the pure periods shown, where $D = Y + 2$ and $E = Y + 1$. The first entry corrects an error of $128r + 31$ in *WW*.

$256r + 31$	DEE	$512r + 153$	DEE	$1024r + 415$	DEE
$512r + 97$	DDEDDDE	$512r + 159$	DEE	$512r + 425$	DE
$1024r + 103$	DE	$512r + 225$	DDE	$512r + 487$	DEE
$128r + 119$	DEE	$512r + 255$	E	$1024r + 521$	DDDE
$1024r + 127$	DEEE	$512r + 257$	DDDDE	$1024r + 607$	DDE
$512r + 151$	DDDEE	$512r + 297$	DDEDEDE	$1024r + 735$	DEEE

It seems likely that the string for $Y = 2^{2k+1} + 2^{2k} - 1$ has the simple period E for all values of k. But there is some evidence to suggest that an analysis will never be complete. Indeed, consider the following table showing the fractions, among 2^k cases, that remain undetermined:

$k =$	3	5	6	7	8	9	10	11	12	13	14	15	16	17
	$\frac{1}{2}$	$\frac{5}{16}$	$\frac{9}{32}$	$\frac{11}{64}$	$\frac{21}{128}$	$\frac{33}{256}$	$\frac{60}{512}$	$\frac{97}{1024}$	$\frac{177}{2048}$	$\frac{304}{4096}$	$\frac{556}{8192}$	$\frac{974}{16384}$	$\frac{1576}{32768}$	$\frac{2763}{65536}$

Moreover, the periods of the pearl-strings appear to become arbitrarily long.

A11. (21) SCHUHSTRINGS is the same as DUDENEY, except that a deduction of zero is also allowed, but cannot be immediately repeated [*WW*, pp. 523–524]. In *WW*, it was stated that it was not known whether there is any SCHUHSTRING game in which three or more strings terminate simultaneously. Kevin Saff has found three such strings (when the maximum deduction is $Y = 3430$, the three

strings are multiples of 2793, 3059, 3381 terminate simultaneously) and he conjectures that there can be arbitrarily many such simultaneous terminations.

A12. (22) Analyze **DUDE**, i.e., unbounded DUDENEY, or NIM in which you are not allowed to repeat your opponent's last move.

Let $[h_1, h_2, \ldots, h_k; m]$, $h_i \le h_{i+1}$, be the game with heaps of size h_1 through h_k, where m is the move just made and $m = 0$ denotes a starting position. Then, the $\mathcal{P}$-positions are [22]

$$[(2s+1)2^{2j}; (2s+1)2^{2j}] \quad \text{for } k = 1,$$
$$[(2s+1)2^{2j}, (2s+1)2^{2j}; 1] \quad \text{for } k = 2,$$

and for $k \ge 3$ the heap sizes are arbitrary, the only condition being that the previous move was 1. The nim-values do not seem to show an easily described pattern.

A13. Games with pass moves. NIM WITH PASS. David Gale suggested an analysis of NIM played with the option of a single pass by either of the players, which may be made at any time up to the penultimate move. It may not be made at the end of the game. Once a player has passed, the game is as in ordinary NIM. The game ends when all heaps have vanished. Morrison, Friedman and Landsberg [87] have looked at this game with their renormalization techniques and suggest that obtaining a full solution may be intractable. Despite that warning, Low and Chan [79] made some progress. They show that:

(1) for two heaps, $[2m+1, 2m+2]$ are the only $\mathcal{P}$-positions;

(2) for three heaps, if one is of size 1, then $[1, m, m]$, $m \ne 2$, are the only $\mathcal{P}$-positions.

They have several other results; for example: Suppose $p + 2q + 3r \ge 10$; then the position with p heaps of size 1, q heaps of size 2, and r heaps of size 3 is a $\mathcal{P}$-position if and only if p is odd and q, r are even, or p is even and q, r are odd.

Chan et al. [18] have extended the analysis to 4 heaps.

CHOCOLATE BAR with a pass [86]. CHOCOLATE BAR is a generalization of CHOMP; see D1. Fix a natural number k and a nonnegative integer s. For nonnegative integers y and z such that $y \le \lfloor (z+s)/c \rfloor$, the chocolate bar will consist of $z+1$ columns where the zero column is the bitter square and the height of the i-th column is $t(i) = \min(y, \lfloor (i+s)/k \rfloor + 1)$. A move, as in CHOMP, is to take a square and remove it and every square in the upper-right quadrant based on this square. Let $CB(s, k, y, z, p)$ be the CHOCOLATE BAR game where there is a single pass move which can be used by either player but not at a terminal position. Here, $p = 1$ if the pass move is available and $p = 0$ otherwise. For $s < k$, they solve the game for s odd and ask for a solution when s is even.

In octal games, with a single pass, Horrocks and Nowakowski [57] ask for the regularities that can occur in the $\mathcal{G}$-sequences. They exhibit sequences which are periodic, arithmetic-periodic and split periodic, arithmetic-periodic.

Now what happens if more than one pass is allowed?

Given a game, the *Oslo* (one-sided-loopy) version gives Left the option to pass in any position except in the terminal positions. Angela Siegel [109] shows that the lattice of Oslo games born by day n is a distributive lattice.

The Oslo version of some games becomes almost trivial. OSLO WYTHOFF has the values $W(x, y) = 0$ if $x = y = 0$; $W(x, y) = $ **upon**$*$ if $x = y > 0$ or one of x and $y = 0$; and $W(x, y) = 2.$**upon** otherwise. (See E5 and E7.)

A14. Games with a Muller twist. In such games, each player specifies a condition on the set of options available to their opponent on their next move.

In ODD-OR-EVEN NIM, for example, each player specifies the parity of the opponent's next move. This game was analyzed by Smith and Stănică [111], who propose several other such games which are still open (see also Gavel and Strimling [45]).

The game of BLOCKING NIM proceeds in exactly the same way as ordinary NIM with N heaps, except that before a given player takes his turn, his opponent is allowed to announce a *block*, $(a_1, \ldots a_N)$; i.e., for each pile of counters, he has the option of specifying a positive number of counters which may not be removed from that pile. Flammenkamp, Holshouser and Reiter [56; 55] give the $\mathcal{P}$-positions for three-heap BLOCKING NIM with an incomplete block containing only one number, and ask for an analysis of this game with a block on just two heaps, or on all three. There are corresponding questions for games with more than three heaps. Larsson [69] shows a game with two piles of counters where at most $k - 1$ moves, for some fixed k, may be blocked off at each stage. Then the $\mathcal{P}$-positions are of the form $\{x, y\}$, where either $|y - x| < k$ and $y - x \equiv k - 1 \pmod 2$ or $x + y < k$.

Let S be a set of positive integers. The *complementary subtraction game* $\hat{S}$ is played on a heap where the last act of a move is to say whether the next subtraction is to be a number from S or from its complement. Horrocks and Trenton [58] introduced this variant. They analyzed heaps up to size 8000, the case where S is the Fibonacci numbers, without finding periodicity although there appears to be regularity. They also ask about the case $S = \{x \mid x \equiv a$ or $b \pmod c\}$. They report that the nim-values appear chaotic. The set of $(a_n, a_n + n)$ where $a_1 = 1$ and $a_n = \text{mex}\{a_i : i < n\}$ form the $\mathcal{P}$-positions of Wythoff's game. We ask what happens in this game if $S = \{a_n : n + 1, 2, 3, \ldots\}$.

A15. (13) Misère quaternary and octal games. Misère analysis has been revolutionized by Thane Plambeck and Aaron Siegel [97] with their concept

of the *misère quotient* of a game, though the number of unsolved problems continues to increase. See Section E13 for more theoretical questions and an explanation of some of the concepts mentioned here.

Plambeck and Siegel ask these specific questions:

(1) The misère quotient of ·**07** (DAWSON'S KAYLES) has order 638 at heap size 33. Is it infinite at heap size 34? Even if the misère quotient is infinite at heap size 34, then, by Rédei's theorem [46, p. 142; 99], it must be isomorphic to a finitely presented commutative monoid. Call this monoid D_{34}. Exhibit a monoid presentation of D_{34}, and having done that, exhibit D_{35}, D_{36}, etc., and explain what is going on in general. Given a set of games $\mathcal{A}$, describe an algorithm to determine whether the misère quotient of $\mathcal{A}$ is infinite. Much harder: if the quotient is infinite, give an algorithm to compute a presentation for it.

(2) Give complete misère analyses for any of the (normal-play periodic) octal games that show "algebraic-periodicity" in misère play. Some examples are ·**54**, ·**261**, ·**355**, ·**357**, ·**516**, and ·**724**. Give a precise definition of algebraic periodicity and describe an algorithm for detecting and generalizing it. This is a huge question: if such an algorithm exists, it would likely instantly give solutions to at least a half-dozen unsolved 2- and 3-digit octals.

Plambeck offers prizes of US$500.00 for complete analysis of DAWSON'S CHESS, ·**137** (alias DAWSON'S KAYLES, ·**07**); US $200.00 for the "wild quaternary game" (·**3102**); and US $25.00 each for ·**3122**, ·**3123**, and ·**3312**.

The website miseregames.org contains thousands of misère quotients for octal games.

Siegel notes that Dawson first proposed his problem in 1935, making it perhaps the oldest open problem in combinatorial game theory. (Michael Albert offers the alternative "Is chess a first player win?".) It may be of historical interest to note that Dawson showed the problem to Richard K. Guy around 1947. Fortunately, he forgot that Dawson proposed it as a losing game, was able to analyze the normal play version, rediscover the Sprague–Grundy theory, and get Conway interested in games.

A16. Now called E12.

A17. MEM and MNEM. MEM is played with heaps of tokens. Remove any number of tokens from any one heap. The number of tokens removed must be at least as large as the number that were removed on the previous move from that heap. Equivalently, a "heap" is a pair of integers (n, k), and a move is to any pair $(n - i, i)$, where $k \leq i \leq n$.

MNEM is exactly the same as MEM, with these additional options: Either player may add tokens to a heap instead of removing them. If adding tokens, the

number added must be strictly less than the number of tokens added or removed on the previous move. If removing tokens, the number must be at least as large as the number added or removed on the previous move. So a move from (n, k) is to $(n - i, i)$ for $k \leq i \leq n$, as in MEM, or to $(n + i, i)$ for $1 \leq i < k$.

Conway and (Aaron) Siegel have investigated this game. They conjecture this: *Every position in* MNEM *has finite nim value.* They verified this experimentally up to $n = 10000$. They also conjecture the following: *For both* MEM *and* MNEM, *if* $k^2 \geq n$, *then* $(n, k) = \lfloor n/k \rfloor$. Siegel reports that they have no idea how to prove either, so these may be very difficult problems. Rubinstein-Salzedo reports that, for MEM, he has a proof.

A18. FIBONACCI NIM. We are given $\alpha > 0$, and a heap of n counters. The first move is to take any number except n. On the $(m + 1)$-st move, $m > 0$, the next player can take up to αq_m, where q_m is the number taken on the m-th move. Whinihan [128] attributes this game to R. E. Gaskell, and he solves the case $\alpha = 12$ by showing that the $\mathcal{P}$-positions are Fibonacci numbers. If $\alpha = 1$ then the $\mathcal{P}$-positions are powers of 2. Schwenk [105] solves the general case using enumeration schemes. Let P_α be the set of $\mathcal{P}$-positions for a given α. Extending Schwenk's result, Rubinstein-Salzedo and Sarkar claim that as α increases, then there are intervals on which the $\mathcal{P}_\alpha$ is constant. For example, for $1 \leq \alpha < 2$, $\mathcal{P}_\alpha$ are the powers of 2 together with 0, and for $2 \leq \alpha < 2.5$, $\mathcal{P}_\alpha$ is the set of Fibonacci numbers.

Larsson and Rubinstein-Salzedo [74] define the multiheap game, $\alpha = 2$, in which the first move can remove all of a heap and the restriction on subsequent moves remains regardless of which heap the next player chooses. For two heaps, they characterize the $\mathcal{P}$-positions, again in terms of Fibonacci numbers. The cases for different α and for more heaps are open.

See Levine [76] for the game where there is an upper and/or lower bound on the number that can be taken, where the bounds depend upon the size of the heap.

A19. Self-referential subtraction games. There several games under this heading, but the best-known game under this heading is EUCLID with its variants; see A9. A recent partizan variant is SUBVERSION [31]; see E16. (SUBVERSION is played with a pair of nonnegative integers, (a, b). If $a = 0$ or $b = 0$ then the game is over. Otherwise, if $a \geq b$ then Left can move to $(a, 0)$, if $a < b$ then Left can move to $(a, b - a)$; if $a \leq b$ then Right can move to $(0, b)$, and if $a > b$ then Right can move to $(a, b - a)$.)

Two games, SUSEN and ALBUS, were developed by Larsson and Nowakowski. In SUSEN (SUbtraction Set ENcoded), the nonzero heap sizes form the subtraction set. For example, $(2, 3)$ is a $\mathcal{P}$-position — the moves are to (2) or (3), both followed by $()$; and to $(2, 1)$ followed by $(1, 1)$. The move to $(2, 2)$ is not available.

If there is only one heap, the next player wins. For two heaps, removing a heap is a losing move. If the players never remove a heap, the length of the game (a, b) until they reach (c, c) for some c, is fixed. Call this $L(a, b)$. Then (a, b) is a $\mathcal{P}$-position if and only if $L(a, b)$ is even. Alex Fink gives a proof that (a, b, c) is in $\mathcal{P}$ if and only if (a, b), (a, c), and (b, c) are all in $\mathcal{N}$. Extend the analysis to more heaps.

In ALBUS (ALl BUt set Sizes), the heap sizes are the illegal moves. For one heap, reduce the heap to 1 counter and win. The case of two heaps is still unsolved.

B. Pushing and placing pieces

B1. (5) GO is of particular interest, partly because of the loopiness induced by the "ko" rule, and many problems involve general theory; see E4 and E5.

Elwyn writes:

> I attach one region that has been studied intermittently over the past several years. The region occurs in the southeast corner of the board (Figure 3). At move 85, Black takes the ko at L6. What then is the temperature at N4? This position is copied from the game Jiang and Rui played at MSRI in July 2000. In 2001, Bill Spight and I worked out a purported solution by hand, assuming either Black komaster or White komaster.

Elwyn also writes:

> Nakamura [GONC3] has shown how capturing races in GO can be analyzed by treating liberties as combinatorial games. Like atomic weights, when the values are integers, each player's best move reduces
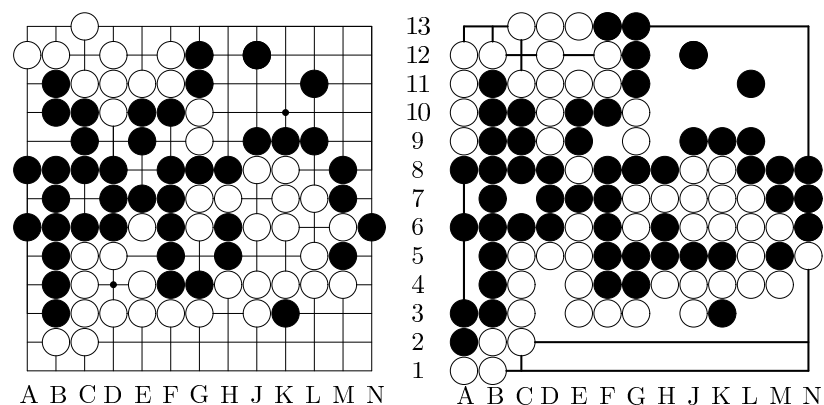

Figure 3. A position from Jiang vs. Rui at MSRI in July 2000.

his opponent's resources by one. The similarities between atomic weights and Nakamura's liberties are striking.

Theoretical problem: Either find a common formulation which includes much or all of atomic weight theory and Nakamura's theory of liberties, *or* find some significant differences.

Important practical applied problem: Extend Nakamura's theory to include other complications which often arise in GO, such as simple kos, either internal and/or external.

B2. WOODPUSH (see [LIP, pp. 214, 275]) is a game that involves ko but is simpler than GO. WOODPUSH is played on a finite strip of squares. Each square is empty or occupied by a black or white piece. A piece of the current player's color retreats — Left retreats to the left and Right to the right — to the next empty square, or off the board if there is no empty square; except, if there is a contiguous string containing an opponent's piece, then it can move in the opposite direction, *pushing* the string ahead of it. Pieces can be moved off the end of the strip. Immediate repetition of a global board positions is not allowed. A "ko-threat" must be played first. For example:

$$
\begin{array}{ccccccccc}
 & \text{Left} & & \text{Right} & & \text{Left} & & \text{Right} & \\
LRR\square & \to & \square LRR & \to & LR\square R & \to & \text{ko-threat} & \to & R\square\square R
\end{array}
$$

Note that Right's first move to $LRR\square$ is illegal because it repeats the immediately prior board position and Left's second move to $\square LRR$ is also illegal so he must play a ko-threat. Also note that in $\square LRR\square$, Right never has to play a ko-threat since he can always push with either of his two pieces — with Left moving first,

$$
\begin{array}{ccccc}
 & \text{Left} & & \text{Right} & \\
\square LRR\square & \to & \square\square LRR & \to & \square LR\square R \\
 & \to & \text{ko-threat} & \to & \text{Right answers ko-threat} \\
 & \to & \square\square LRR & \to & \square LRR\square
\end{array}
$$

Berlekamp, Plambeck, Ottaway, Aaron Siegel and Spight (work in progress) use top-down thermography to analyze the three-piece positions. What about more pieces? Cazenave and Nowakowski [GONC4, pp. 47–56] show that the position $L\square L\square R\square R$ is ± 4 but that $L\square L\square\square\square R\square R$ and $L\square L\square\square\square\square\square R\square R$ are draws by superko (repetition of the board position after more than 2 moves).

B3. (40) CHESS. Noam Elkies [MGONC, pp. 61–78] has examined DAWSON'S CHESS, but played under usual Chess rules, so that capture is not obligatory.

He would still welcome progress with his conjecture that the value $*k$ occurs for all k in (ordinary chess) pawn endings on sufficiently large chessboards.

Thea van Roode has suggested IMPARTIAL CHESS, in which the players may make moves of either color. Checks need not be responded to and kings may be captured. The winner could be the first to promote a pawn.

INFINITE CHESS is played with a regular set of chess pieces but on an infinite board. Evans and Hamkins [28] observe that for each finite number n, there is a position in INFINITE CHESS having value n. They show that positions with values ω, ω^2, $\omega^2 \times k$, and ω^3 occur, and recently, that ω^4 occurs. They ask about 3-dimensional CHESS.

B4. (30) CHESS: king and rook. Low and Stamp [80] have given a strategy in which White wins the king and rook vs. king problem within an 11×9 region. Kanungo and Low [64] show that with the initial position WK$(1, 1)$, WR(x, y), and BK(a, b), where $1 < x < a$, White has a winning strategy on the $(a + b + 3) \times (a + b + 5)$ board.

With the same pieces but on a quarter-infinite board, spanning the first quadrant of the xy-plane, ENTREPRENEURIAL CHESS or ECHESS, introduced by Berlekamp in 2005; see [7], gives Black the additional option of "cashing out", which removes the board and converts the position into the integer $x + y$, where (x, y) are the coordinates of his king's position when he decides to cash out. Berlekamp and Low conjecture that the boiling point of ECHESS is $5\frac{5}{8}$.

B5. NONATTACKING QUEENS. Noon and Van Brummelen [92] alternately place queens on an $n \times n$ chessboard so that no queen attacks another. The winner is the last queen placer. They give nim-values for boards of sizes $1 \le n \le 10$ as 1121312310 and ask for the values of larger boards.

B6. (55) AMAZONS. Müller [88] has shown that the 5×5 game is a first player win and asks about the 6×6 game.

B7. PHUTBALL, more properly, Conway's PHILOSOPHER'S FOOTBALL, is usually played on a GO board with positions (i, j), $-9 \le i, j \le 9$ and the ball starting at $(0, 0)$. For the rules, see [*WW*, pp. 752–755]. The game is loopy (see Section E5 below), and Nowakowski, Ottaway, and Siegel (see [108]) discovered positions that contained tame cycles, i.e., cycles with only two strings, one each of Left and Right moves. Aaron Siegel asks if there are positions in such combinatorial games which are stoppers but contain a *wild cycle*, i.e., one which contains more than one alternation between Left and Right moves. Sarkar (in this volume) has shown that draws can occur on the 19×19 board but it is not clear if they occur without collusion of the players. Demaine, Demaine, and Eppstein [MGONC, pp. 351–360] show that it is NP-complete to decide if a player can win on the next move. Loosen (see DIRECTIONAL PHUTBALL below) raises the question of whether there is any $\mathcal{P}$-position.

PHLAG PHUTBALL is a variant played on an $n \times n$ board with the initial position of the ball at $(0, 0)$ except that now only the ball may occupy the positions $(2i, 2j)$ with both coordinates even. This eliminates "tackling", and is an extension of one-dimensional ODDISH PHUTBALL, analyzed in Grossman and Nowakowski [MGONC, pp. 361–367]. The $(3, 2n + 1)$ board (i.e., (i, j), $i = 0, 1, 2$ and $-n \leq j \leq n$) is already interesting and requires a different strategy from that appropriate to ODDISH PHUTBALL.

Loosen [78] introduces DIRECTIONAL PHUTBALL, a nonloopy version, which is also played on a grid. The ball starts in the bottom-left corner, where Left's goal-line is the right edge and Right's is the top edge. Players can only jump toward their opponent's goal-line, and win by jumping over that goal-line. Men can only be placed ahead of the ball (i.e., in the positive quadrant with origin on the ball). She notes that there is no $\mathcal{P}$-position in this game and that making an off-parity move ("poultry") can be good. She asks for a complete analysis of the $2 \times n$ board. She shows that the atomic weights of the $m \times n$ board for $2 \leq m, n \leq 4$ is 0 and is 0, 1, 1, 2 for $2 \times n$ for $n = 5, 6, 7, 8$, respectively.

B8. HEX. [*LIP*, pp. 264–265]. Nash's strategy-stealing argument shows that HEX is a first-player win but few quantitative results are known. Arneson et al. [131] report that 8×8 is solved, as are most openings on the 9×9.

Garikai Campbell [15] asks:

(1) For each n, what is the shortest path on an $n \times n$ board with which the first player can guarantee a win?

(2) What is the least number of moves that guarantees the first player a win? Campbell showed that this is at least n on an $n \times n$ board. Peng et al. [96] show that it is 7 on the 5×5 board.

B9. (54) FOX AND GEESE. Berlekamp and Siegel [108, Chapter 2] and [*WW* pp. 669–710], "analyzed the game fairly completely, relying in part on results obtained using CGSuite". In [*WW*, p. 710], the following open problems are given:

(1) Define a position's *span* as the maximum occupied row-rank minus its minimum occupied row-rank. Then quantify and prove an assertion such as the following: If the backfield is sufficiently large, and the span is sufficiently large, and if the separation is sufficiently small, and if the Fox is neither already trapped in a daggered position along the side of the board, nor immediately about to be so trapped, then the Fox can escape and the value is *off*.

(2) Show that any formation of three Geese near the center of a very tall board has a "critical rank" with the following property: If the northern Goose is far above, and the Fox is far below, then the value of the position is either positive,

HOT, or *off*, according as to whether the northern Goose is closer, equidistant, or further from the critical rank than the Fox, respectively.

(3) Welton asks what happens if the Fox is empowered to retreat like a bishop, going back several squares at a time in a straight line? More generally, suppose his straight-line retreating moves are confined to some specific set of sizes. Does {1, 3}, which maintains parity, give him more or less advantage than {1, 2}?

(4) What happens if the number of Geese and board widths are changed?

   In Aaron Siegel's thesis there are several other questions:

(5) In the critical position — with Geese at (we use the algebraic CHESS notation of a, b, c, d, ... for the files and 1, 2, 3, ..., n for the ranks) (b, n), (d, n), (e, $n-1$), and (h, $n-1$), and the Fox at (c, $n-1$), which has value $1+2^{-(n-8)}$ on an $n \times 8$ board with $n \geq 8$ in the usual game — is the value $-2n+11$ for all $n \geq 6$ when played with "Ceylonese rules"? (Fox allowed two moves at each turn.)

(6) On an $n \times 4$ board with $n \geq 5$ and Geese at (b, n) and (c, $n-1$), do all Fox positions have value *over*? With the Geese on (b, n) and (d, n) are the only other values 0 at (c, $n-1$) and {*over* | 0} at (b, $n-2$) and (d, $n-2$)?

(7) On an $n \times 6$ board with $n \geq 8$ and Geese at (b, n), (d, n), and (e, $n-1$) do the positions (a, $n-2k+1$), (c, $n-2k+1$), and (e, $n-2k+1$) all have value 0, and those at (b, $n-2k$), (d, $n-2k$), and (f, $n-2k$) all have value Star? And if the Geese are at (b, n), (d, n), and (f, n), are the zeroes and Stars interchanged?

B10. HARE AND HOUNDS. Aaron Siegel asks if the sequences of positions of increasing board length shown in Figure 4, left, are increasingly hot, and in Figure 4, right, if they have arbitrarily large negative atomic weight. He also conjectures that the starting position on a $6n + 5 \times 3$ board, for $n > 0$, has value

$$-(n-1) + \left\{ b, c \mid 0 \| 0 \| 0 \Big\| 0 \dots \Big\| 0 \right\},$$

where there are $2n + 4$ zeroes and slashes and $b = \left\{ 0, a \| 0, \{0 \mid \textit{off}\} \right\}$,

$$c = \left\{ 0 \, \| \downarrow_{[2]} * \mid 0 \| 0 \right\} \quad \text{and} \quad a = \{0, \downarrow_{[2]} * \mid 0, \downarrow_{[2]} *\}.$$

B11. (4) DOMINEERING. [*WW*, pp. 119–122, 138–142; *LIP* pp. 1–7, 260]. Extend the analysis.

   (Left and Right take turns to place dominoes on a checker-board. Left orients her dominoes North–South and Right orients his East–West. Each domino exactly covers two squares of the board and no two dominoes overlap. A player unable to play loses.)
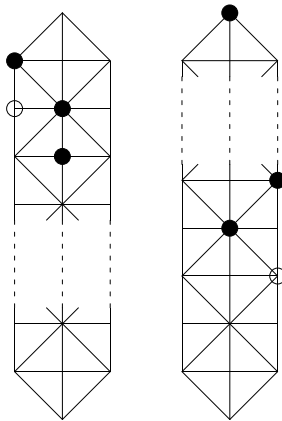
Figure 4. Sequences of hare and hounds positions.

See Berlekamp [5] and the second edition of [*WW*, pp. 138–142], where some new values are given. For example, David Wolfe and Dan Calistrate have found the values (to within "-ish", i.e., infinitesimally shifted) of 4×8, 5×6 and 6×6 boards. The value for a 5×7 board is

$$\left\{ \tfrac{3}{2} \left| \left\{ \tfrac{5}{4} \middle| -\tfrac{1}{2} \right\}, \left\{ \tfrac{3}{2} \middle| -\tfrac{1}{2}, \left\{ \tfrac{3}{2} \middle| -1 \right\} \middle\| -1 \middle| -3 \right\} \middle\| -1, \left\{ \tfrac{3}{2} \middle| -\tfrac{1}{2} \middle\| -1 \right\} \middle| -3 \right\}$$

Lachmann, Moore, and Rapaport [MGONC, pp. 307–315] discover who wins on rectangular, toroidal, and cylindrical boards of widths 2, 3, 5, and 7, but do not find their values. Bullock [11, p. 84] showed that 19×4, 21×4, 14×6, and 10×8 are wins for Left and that 10×10 is a first-player win. Uiterwijk [124] using a new DOMINEERING solver, shows that Left going first wins on 8×15, 11×9, 11×11, 12×8, 12×15, 14×8, and 17×6 boards and loses on 8×12, 9×11, and 11×10 boards. He also shows the following: Left wins on the board 6×15, Right wins on the boards 11×14 and 11×18 [120]; Left wins on the boards 6×19, 10×15, 14×15, and 18×15 [121]; and, in [122] proves that:

(1) all $2k \times n$ boards for $n = 3, 5, 7, 9, 11$ and $2k \times 13$ with $k \geq 3$ are Left wins, and

(2) all $m \times 2k$ boards for $m = 5, 9, 13$, and all $m \times 2k$ boards for $m = 3, 7$ with $k \geq 2$, and all $11 \times 2k$ boards with $k \geq 6$ are Right wins.

He also shows [123] how to recursively construct large positions with given values. Drummond-Cole [25] gives outcome classes for many sets of positions. For example, the boards $6 \times n$ is a Left win for n odd, $n \geq 41$, and $n \neq 43, 45, 51, 59$.
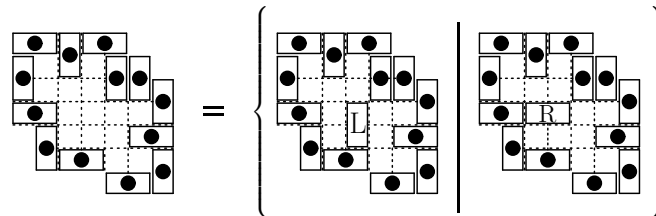
Figure 5. A DOMINEERING position of value $\pm 2*$.

Berlekamp notes that the value of a $2 \times n$ board, for n even, is only known to within "ish", and that there are problems on $3 \times n$ and $4 \times n$ boards that are still open.

Berlekamp asks, as a hard problem, to characterize all hot DOMINEERING positions to within "ish". As a possibly easier problem he asks for a DOMINEER-ING position with a new temperature, i.e., one not occurring in [GONC, Table 1, p. 477]. In an on-going investigation, Gabriel Drummond-Cole found values with temperatures between 1.5 and 2. Figure 5 shows a position of value $\pm 2*$ and temperature 2 [24].

Shankar and Sridharan [106] have found many DOMINEERING positions with temperatures other than those shown in [GONC, Table 1, p. 477]. Blanco and Fraenkel [8] have obtained partial results for the game of TROMINEERING, played with trominoes in place of (or, alternatively, in addition to) dominoes.

B12. NOGO can be found under the name "Anti-Atari Go" at the Sensei Library (see senseis.xmp.net/?AntiAtariGo) and was invented independently by Neil McKay. On a GO board (or on any graph) pieces are placed as in GO; the only restriction is that every connected group of each player must be adjacent to at least one empty intersection. Burke and Hearn [12] have shown that NOGO is NP-hard on a graph. McKay, Nowakowski and (Angela) Siegel found positions of value 1, $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{8}$ on a GO board. Using CGSuite, one can find the nimbers $*$, $*2$, $*4$, and $*8$ on a one-dimensional board. The position $\bullet \cdot \circ \cdot \bullet \cdot \circ \ldots$, where $\bullet$ is a black piece, $\circ$ is a white piece and "$\cdot$" is an empty space, is equivalent to the octal game $.6$ (see Section A2), which is not known to be periodic. We ask for the nim-dimension of both the one- and two-dimensional board. Is there a limit to the denominator of the fractions found on the two-dimensional board?

C. Playing with pencil and paper

C1. (51) DOTS-AND-BOXES. Sierra [110] reports that 3×3 is a first-player win [*WW*], 4×4 (Wilson) and 4×5 (Barker and Korf) are draws, and that William Fraser reports that 5×5 is a first-player win. See Berlekamp's book [6] for more

Figure 6. Starting position for "Faroese" 1×6 Dots-and-Boxes and Dots-and-Triangles

problems about this popular children's (and adult's) game and see also [*WW*, pp. 541–584; *LIP*, pp. 21–28, 260].

Several other opening positions have been contemplated. "Open" is the standard position where there are no edges drawn, "Swedish" has all boundary lines drawn, "Faroese" has three, and "Icelandic" has 2 (see Figure 6). Jobson et al. [62] show that:

(1) Faroese $1 \times n$, $n \neq 2$, DOTS-AND-TRIANGLES is a first-player win, and

(2) the first player can at least tie on both Open and Faroese DOTS-AND-BOXES $1 \times n$ boards.

For the Faroese, they also conjecture that the first player wins by 1 for all $n \geq 9$ and odd. Sierra has also checked this (using David Wilson's analyzer) for Faroese $1 \times n$, $n \leq 19$ and odd (which he calls "staples", S_n), and adds to the conjecture that a winning move for the first player is a vertical move that splits the staple into two staples $S_2 + S_{n-2}$.

Sierra also conjectures for Icelandic DOTS-AND-BOXES with n odd that the first player loses by 1. Again, this is true for $n \leq 19$.

For misère play, Collette et al. [GONC4] show that the open $1 \times n$ position is a first-player win. They also conjecture that *the outcome of misère $1 \times n$ Swedish* DOTS-AND-BOXES *is eventually periodic, starting at $n = 22$ and with a period of* 10.

Elwyn Berlekamp asks for a complete theory of the impartial Icelandic $1 \times n$ game. Here, the player who completes the last box loses since they cannot complete their move by drawing another edge.

C2. (25) SPROUTS. Extend the analysis of this Conway–Paterson game in either the normal or misère form [*WW*, pp. 564–568].

(A move joins two spots, or a spot to itself by a curve which doesn't meet any other spot or previously drawn curve. When a curve is drawn, a new spot must be placed on it. The valence of any spot must not exceed three.)

C3. (26) SYLVER COINAGE. [*WW*, pp. 575–597]. Extend the analysis.

(Players alternately name distinct positive integers, but may not name a number which is a sum, with repetitions allowed, of previously named integers. Whoever names 1 loses.) Sicherman [107] contains the most recent information.

C4. (28) Extend Úlehla's [126] or Berlekamp's [*WW*, pp. 570–572] analysis of VON NEUMANN'S GAME from directed forests to directed acyclic graphs.

(VON NEUMANN'S GAME, or HACKENDOT, is played on one or more rooted trees. The roots induce a direction, towards the root, on each edge. A move is to delete a node, together with all nodes on the path to the root, and all edges incident with those nodes. Any remaining subtrees are rooted by the nodes that were adjacent to deleted nodes.)

C5. (43) INVERTING HACKENBUSH. Thea van Roode has written a thesis [100] investigating both this and REVERSING HACKENBUSH, but there is plenty of room for further analysis of both games.

In INVERTING HACKENBUSH, when a player deletes an edge from a component, the remainder of the component is replanted with the new root being the pruning point of the deleted edge. In REVERSING HACKENBUSH, the colors of the edges are all changed after each deletion. Both games are hot, in contrast to BLUE-RED HACKENBUSH [*WW*, pp. 1–7; *LIP*, pp. 82, 88, 111–112, 212, 266] which is cold, and GREEN HACKENBUSH [*WW*, pp. 189–196], which is tepid.

C6. (42) BEANSTALK and **BEANS-DON'T-TALK** are games invented respectively by John Isbell and John Conway. See Guy in [GONC]. BEANSTALK is played between Jack and the Giant. The Giant chooses a positive integer, n_0. Then Jack and the Giant play alternately $n_1, n_2, n_3, \ldots$, according to the rule

$$n_{i+1} = \begin{cases} \frac{1}{2} n_i & \text{if } n_i \text{ is even,} \\ 3n_i \pm 1 & \text{if } n_i \text{ is odd;} \end{cases}$$

that is, if n_i is even, there's only one option, while if n_i is odd there are just two. The winner is the person moving to 1.

We still don't know if there are any $\mathcal{O}$-*positions* (positions of infinite remoteness).

C7. (63) THE ERDŐS–SZEKERES SEQUENCE GAME [26] (and see Schensted [103]) was introduced by Harary et al. [52]. From a deck of cards labeled from 1 through n, Alexander and Bridget alternately choose a card and append it to a sequence of cards. The game ends when there is an ascending subsequence of a cards or a descending subsequence of d cards.

The game appears to have a strong bias towards the first player. Albert et al. [GONC3] show that for $d = 2$ and $a \leq n$ the outcome is $\mathcal{N}$ or $\mathcal{P}$ accordingly as n is odd or even, and is $\mathcal{O}$ (drawn) if $n < a$. They conjecture that for $a \geq d \geq 3$ and all sufficiently large n, it is $\mathcal{N}$ with both normal and misère play, and also with normal play when played with the rationals in place of the first n integers.

They also suggest investigating the form of the game in which players take turns naming pairs (i, π_i) subject to the constraint that the chosen values form part of the graph of some permutation of $\{1, 2, \ldots, n\}$.

C8. THINNING THICKETS. This game, played on trees consisting of a stalk with at most one leaf at each vertex, is introduced in [60]. They show that the game has infinite nim-dimension and infinite boiling-point. The trees are very thin (a stalk with maybe a leaf at each node) and either all the arcs are green or just blue and red. They ask to extend the analysis to general trees.

D. Disturbing and destroying

D1. (27) Extend the analysis of CHOMP [*WW*, pp. 598–599; *LIP*, pp. 19, 46, 216].

David Gale asks for the solution of the infinite 3D version where the board is the set of all triples (x, y, z) of nonnegative integers, that is, the lattice points in the positive octant of $\mathbb{R}^3$. The problem is to decide whether it is a win for the first or second player.

CHOMP (Gale [43]) is equivalent to DIVISORS (Schuh [104]). CHOMP is easily solved for $2 \times n$ arrays, Sun [115], and indeed a recent result by Byrnes [13] shows that any poset game eventually displays periodic behavior if it has two rows, and a fixed finite number of other elements. See also the Fraenkel poset games mentioned near the end of A2.

Thus, most of the work in recent years has been on three-rowed CHOMP. The situation becomes quite complicated when a third row is added; see Zeilberger [129] and Brouwer et al. [10]. A novel approach (renormalization) is taken by Friedman and Landsberg [GONC3]. They demonstrate that three-rowed CHOMP exhibits certain scaling and self-similarity patterns similar to chaotic systems. Is there a deterministic proof that there is a unique winning move from a $3 \times n$ rectangle? The renormalization approach is based on nonlinear dynamics techniques from physics; its results are highly suggestive but as of yet not fully mathematically rigorous.

TRANSFINITE CHOMP has been investigated by Huddleston and Shurman [MGONC]. An open question is to calculate the nim-value of the position $\omega \times 4$ — they conjecture this to be $\omega \cdot 2$, but it could be as low as 46, or even uncomputable! Perhaps the most fascinating open question in TRANSFINITE CHOMP is their *stratification conjecture*, which states that if the number of elements taken in a move is $< \omega^i$, then the change in the nim-value is also $< \omega^i$.

Andries Brouwer has a CHOMP site: www.win.tue.nl/~aeb/games/chomp.html.

Nakamura and Miyadera [90] introduce a new board. Instead of the columns nonincreasing in size going out from the poisoned square, they are nondecreasing.

They also add a strip going off to the left. Dozier and Perry [23] introduce more poisoned squares and generalize to ideals and Grobner bases.

D2. (33) SUBSET TAKE-AWAY. Given a finite set, players alternately choose proper subsets subject to the rule that once a subset has been chosen no proper subset can be removed. Last player to move wins.

Many people play the dual, that is, a nonempty subset must be chosen and no proper superset of this can be chosen. We discuss this version of the game which now can be considered a poset game with the sets ordered by inclusion.

The $(n; k)$ SUBSET TAKE-AWAY game is played using all subsets of sizes 1 through k of a n-element set. In the $(n; n)$ game, one has the whole set (i.e., the set of size n) as an option, so a strategy-stealing argument shows this must be a first-player win.

Gale and Neyman [44], in their original paper on the game, conjectured that the winning move in the $(n; n)$ game is to remove just the whole set. This is equivalent to the statement that the $(n; n-1)$ game is a second-player win, which has been verified only for $n \leq 5$. A stronger conjecture states that $(n; k)$ is a second-player win if and only if $k + 1$ divides n. This was proved in the original paper only for $k = 1$ or 2.

Andries Brouwer and J. Daniel Christensen [9] show that these are false. The first player loses in $(7; 7)$ if they take the top set. Instead, they win by taking a set of cardinality 4. The second player wins $(7; 3)$, but 4 does not divide 7.

See also Fraenkel and Scheinerman [41].

D3. (39) Sowing or MANCALA games. There appears to have been no advance on the papers mentioned in MGONC, although we feel that this should be a fruitful field of investigation at several different levels.

D4. Annihilation games. *k*-**ANNIHILATION**. Initially place tokens on some of the vertices of a finite digraph. Denote by $\rho_{\text{out}}(u)$ the outvalence of a vertex u. A move consists of removing a token from some vertex u, and "complementing" $t := \min(k, \rho_{\text{out}}(u))$ (immediate) followers of u, say $v_1, \ldots, v_t$: if there is a token on v_j, remove it; if there is no token there, put one on it. The player making the last move wins. If there is no last move, the outcome is a draw. For $k = 1$, there is an $O(n^6)$ algorithm for deciding whether any given position is in $\mathcal{P}$, $\mathcal{N}$, or $\mathcal{O}$; and for computing an optimal next move in the last 2 cases (Fraenkel and Yesha [42]). Fraenkel asks: Is there a polynomial algorithm for $k > 1$? For an application of k-annihilation games to lexicodes, see Fraenkel and Rahat [40].

D5. TOPPLING DOMINOES [*LIP*, pp. 110–112, 274] is played with a row of vertical dominoes, each of which is either blue or red. A player topples one of his/her dominoes to the left or to the right.

we have stacks	0	1&7	2&5	3	4	6
which can be relabeled	0	1&3	1&2	1	2	3

Table 1. Strings of "pearls" in DUDENEY for values of Y of various forms.

See Fink et al. [GONC4, pp. 65–76] for the proofs that every number occurs exactly once and is a palindrome; there are exactly n positions with value $*n$. Several conjectures are listed but the most intriguing seems to be this: *If G is a palindrome then G's value appears uniquely.*

There are several variants of TOPPLING DOMINOES. If all the dominoes must be toppled in the same direction then this is a HACKENBUSH string. TIMBER [95] is an impartial version of TOPPLING DOMINOES played on a directed graph. The dominoes are on the edges. A player chooses a domino which is toppled in the direction of the edge. The dominoes on incident edges are then toppled (regardless of the underlying edge) and the process is iterated. If only outcomes are required then only trees are interesting. In [95], an algorithm is given to determine the outcome class of a tree, which surprisingly requires nim-values. Values, however, appear to be difficult to determine. The normal and misère versions on a path are related to Dyck paths and Catalan numbers; see Section E15 for more details. The partizan version, imaginatively called PARTIZAN TIMBER, has Left and Right dominoes placed on the edges of a directed graph. Any TOPPLING DOMINOES position can be transformed into a PARTIZAN TIMBER position by subdividing each edge and directing the edge toward the new vertex. A similar algorithm to that of TIMBER can be used to determine the outcome class of a tree. The uniqueness of numbers doesn't hold even on a path. In the other direction, are there any values that do not occur in the game?

D6. TOWER OF HANOI. Chappelon et al. [17] consider the TOWER OF HANOI as a two-player game. The first player can win when there are 3 pegs and it is a draw with 4 or more pegs — parity of moves! They introduce a scoring version (see E14) where there are weights on the edges. Almost all are first-player wins. Are there any variations with second-player wins?

HANOI STICK-UP is played with the disks of the TOWERS OF HANOI puzzle, starting with each disk in a separate stack. A move is to place one stack on top of another such that the size of the bottom of the first stack is less than the size of the top of the second; the two stacks then fuse (&) into one. The only relevant information about a stack are its top and bottom sizes, and it's often possible to collapse the labeling of positions: so, for instance, starting with 8 disks and fusing 1&7 and 2&5, in which the legal moves are still the same. John Conway, Alex Fink, and others have found that the $\mathcal{P}$-positions of height ≤ 3 in

normal HANOI STICKUP are exactly those which, after collapsing, are of the form $0^a\,01^b\,1^c\,12^d\,2^e$ with $\min(a+b+c, c+d+e, a+e)$ even, except that if $a+e \leq a+b+c$ and $a+e \leq c+d+e$ then both a and e must be even (02 can't be involved in a legal move so can be dropped).

They also found the normal and misère outcomes of all positions with up to six stacks, but there is more to be discovered.

D7. (56) Are there any draws in BEGGAR-MY-NEIGHBOR? Marc Paulhus showed that there are no cycles when using a half-deck of two suits, but the problem for the whole deck (one of Conway's "anti-Hilbert" problems) is still open. Lakshtanov and Aleksenko [68] showed that the expected number of moves is finite in the modified game where each pile is shuffled when being returned to the deck and that the player to start a new pile is chosen by a Bernoulli trial with $p \in (0, 1)$.

E. Theory of games

E1. (49) Fraenkel updates Berlekamp's questions on computational complexity as follows:

Demaine, Demaine, and Eppstein [MGONC] proved that deciding whether a player can win in a *single* move in PHUTBALL [*WW*, pp. 752–755; *LIP*, p. 212] is NP-complete. Grossman and Nowakowski [MGONC] gave constructive partial strategies for one-dimensional PHUTBALL. Thus, these papers do not show that PHUTBALL has the required properties.

Perhaps NIMANIA (Fraenkel and Nešetřil [39]) and MULTIVISION (Fraenkel [33]) satisfy the requirements. NIMANIA begins with a single positive integer, but after a while there is a multiset of positive integers on the table. At move k, a copy of an existing integer m is selected, and 1 is subtracted from it. If $m = 1$, the copy is deleted. Otherwise, k copies of $m - 1$ are adjoined to the copy $m - 1$. The player first unable to move loses and the opponent wins. It was proved that:

(i) the game terminates, and

(ii) Player 1 can win.

In Fraenkel, Loebl and Nešetřil [38], it was shown that the max number of moves in NIMANIA is an Ackermann function, and the min number satisfies $2^{2^{n-2}} \leq \mathrm{Min}(n) \leq 2^{2^{n-1}}$.

The game is thus intractable simply because of the length of its play. This is a *provable* intractability, much stronger than NP-hardness, which is normally only a *conditional* intractability. One of the requirements for the tractability of a game is that a winner can consummate a win in at most $O(c^n)$ moves, where

$c > 1$ is a constant, and n a sufficiently succinct encoding of the input (this much is needed for nim on 2 equal heaps of size n).

To consummate a win in NIMANIA, player 1 can play randomly most of the time, but near the end of play, a winning strategy is needed, given explicitly. Whether or not this is an "intricate" solution depends on the beholder. But it seems that it's of even greater interest to construct a game with a very *simple* strategy which still has high complexity!

Also every play of MULTIVISION terminates, the winner can be determined in linear time, and the winning moves can be computed linearly. But the length of play can be arbitrarily long. So let's ask the following: Is there a game which has

1. simple, playable rules,

2. a simple explicit strategy,

3. length of play at most exponential, and

4. is NP-hard or harder.

Tung [119] proved the following:

Theorem. *Given a polynomial $P(x, y) \in \mathbb{Z}[x, y]$, the problem of deciding whether for all x there exists $y[P(x, y) = 0]$ holds over $\mathbb{Z}_{\geq 0}$, is co-NP-complete.*

Define the following game of length 2: Player 1 picks $x \in \mathbb{Z}_{\geq 0}$, Player 2 picks $y \in \mathbb{Z}_{\geq 0}$. Player 1 wins if $P(x, y) \neq 0$, otherwise Player 2 wins. For winning, Player 2 has only to compute y such that $P(x, y) = 0$, given x, and there are many algorithms for doing so.

Also Jones and Fraenkel [63] produced games, with small length of play, which satisfy these conditions.

So we are led to the following reformulation of Berlekamp's question: Is there a game which has

1. simple, playable rules,

2. a finite set of options at every move,

3. a simple explicit strategy,

4. length of play at most exponential, and

5. is NP-hard or harder.

E2. Complexity closure. Aviezri Fraenkel asks: Are there partizan games G_1, G_2, G_3 such that:

(i) $G_1, G_2, G_3, G_1 + G_2, G_2 + G_3$, and all their options have polynomial-time strategies, and

(ii) $G_1 + G_3$ is NP-hard?

E3. Sums of switch games. David Wolfe considers a sum of games G, each of the form $a\|b|c$ or $a|b\|c$, where a, b, and c are integers specified in unary. Is there a polynomial time algorithm to determine who wins in G, or is the problem NP-hard?

E4. (52) How does one play *sums* of games with varied overheating operators? Sentestrat and top-down thermography [*LIP*, p. 214]:

David Wolfe would like to see a formal proof that sentestrat works, an algorithm for top-down thermography, and conditions for which top-down thermography is computationally efficient.

Aaron Siegel asks the following generalized thermography questions:

 (1) Show that the Left scaffold of a dogmatic (neutral ko-threat environment [*LIP*, p. 215]) thermograph is decreasing as function of t. (Note, this is *not* true for komaster thermographs.) (Dogmatic thermography was invented by Berlekamp and Spight. See [113] for a good introduction.)

 (2) Develop the machinery for computing dogmatic thermographs of double kos (multiple alternating 2-cycles joined at a single node).

In the same vein as (2):

 (3) Develop a temperature theory that applies to all loopy games.

Siegel thinks that (3) is among the most important open problems in combinatorial game theory. The temperature theory of GO appears radically different from the classical combinatorial theory of loopy games (where infinite plays are draws). It would be a huge step forward if these could be reconciled into a "grand unified temperature theory". Problem (2) seems to be the obvious next step toward (3).

Conway asks for a natural set of conditions under which the mapping $G \mapsto \int^* G$ is the *unique* homomorphism that annihilates all infinitesimals.

E5. Loopy games [*WW*, pp. 334–377; *LIP*, pp. 213–214] are partizan games that do not satisfy the ending condition. A *stopper* is a game that, when played on its own, has no ultimately alternating, Left and Right, infinite sequence of legal moves. Aaron Siegel reminds us of [*WW*, 2nd ed., p. 369], where the authors tried hard to prove that every loopy game had stoppers, until Clive Bach found the CAROUSEL counterexample. Is there an alternative notion of simplest form that works for *all* finite loopy games, and, in particular, for the CAROUSEL? The simplest form theorem for stoppers is in [*WW*, p. 351].

Siegel conjectures that, if Q is an arbitrary cycle of Left and Right moves that contains at least two moves for each player, and is not strictly alternating, then there is a stopper consisting of a single cycle that matches Q, together
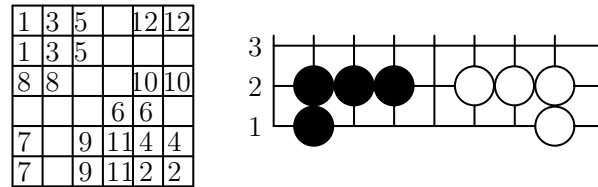
Figure 7. A DOMINEERING position and a chilled GO position of value $*2$.

with various exits to enders, i.e., games which end in a finite, though possibly unbounded, number of moves. (Note that games normally have Left and Right playing alternately, but if the game is a sum, then play in one component can have arbitrary sequences of Left and Right moves, not just alternating ones.)

A long cycle is *tame* if it alternates just once between Left and Right, otherwise it is *wild*. Aaron Siegel writes:

> I can produce wild cycles "in the laboratory," by specifying their game graphs explicitly. So the question is to detect one "in nature", i.e., in an actual game with (reasonably) playable rules such as PHUTBALL (Problem B7).

Siegel also asks under what conditions does a given infinitesimal have a well-defined atomic weight, and asks to specify an algorithm to calculate the atomic weight of an infinitesimal stopper g. The algorithm should succeed whenever the atomic weight is well-defined, i.e., whenever g can be sandwiched between loopfree all-smalls of equal atomic weight.

E6. (45) Elwyn Berlekamp asks for the *habitat* of $*2$, where $*2 = \{0, * \mid 0, *\}$.

In DOMINEERING, Uiterwijk and Barton [125] improve on Drummond-Cole's original $*2$ in that the position is reachable from a standard board (no initial holes) and also exhibits a connected $*3$ position. See, for example, Figure 7, which also shows a GO position, found by Nakamura and Berlekamp [89], whose chilled value is $*2$. The Black and White groups are both connected to life via unshown connections emanating upwards from the second row. Either player can move to $*$ by placing a stone at E2, or to 0 by going to E1. Given a game, let n be the smallest nonnegative integer n such that if a position of the game has value $*k$ then $k < 2^n$.

E7. Partial ordering of games. David Wolfe lets $g(n)$ be the number of games born by day n, notes that an upper bound is given by $g(n+1) \le g(n) + 2^{g(n)} + 2$, and a lower bound for each $\alpha < 0$ is given by $g(n+1) \ge 2^{g(n)^\alpha}$, for n sufficiently large, and asks us to tighten these bounds.

He also asks what group is generated by the all-small games (or — much harder — of all games) born by day 3. Describe the partial order of games born by day 3, identifying all the largest "hypercubes" (Boolean sublattices) and how they are interconnected. These questions have been answered for day 2 [GONC3, pp. 125–130].

Berlekamp suggests other possible definitions for games born by day n, $\mathcal{G}_n$, depending on how one defines $\mathcal{G}_0$. Our definition is 0-based, as $\mathcal{G}_0 = \{0\}$. Other natural definitions are integer-based (where $\mathcal{G}_0$ are integers) or number-based. These two alternatives do not form a lattice, for if G_1 and G_2 are born by day k, then the games

$$H_n := \big\{ G_1, G_2 \parallel G_1, \{ G_2 \mid -n \} \big\}$$

form a decreasing sequence of games born by day $k+2$ exceeding any game $G \geq G_1, G_2$, and the day $k+2$ join of G_1 and G_2 cannot exist. What is the structure of the partial order given by one of these alternative definitions of birthday?

The set of all short games does not form a lattice, but Calistrate, Paulhus, and Wolfe [MGONC] have shown that the games born by day n form a distributive lattice $\mathcal{L}_n$ under the usual partial order. They ask for a description of the exact structure of $\mathcal{L}_3$. Siegel describes $\mathcal{L}_4$ as "truly gigantic and exceedingly difficult to penetrate" but suggests that it may be possible to find its dimension and the maximum *longitude*, $\mathrm{long}_4(G)$, of a game in $\mathcal{L}_4$, which he defines as

$$\mathrm{long}_n(G) = \mathrm{rank}_n(G \vee G^\bullet) - \mathrm{rank}_n(G),$$

where $\mathrm{rank}_n(G)$ is the rank of G in $\mathcal{L}_n$ and $G^\bullet$ is the *companion* of G,

$$G^\bullet = \begin{cases} * & \text{if} \quad G = 0, \\ \{0, (G^L)^\bullet \mid (G^R)^\bullet\} & \text{if} \quad G > 0, \\ \{(G^L)^\bullet \mid 0, (G^R)^\bullet\} & \text{if} \quad G < 0, \\ \{(G^L)^\bullet \mid (G^R)^\bullet\} & \text{if} \quad G \parallel 0. \end{cases}$$

Albert and Nowakowski [3] show that starting with any set of games, instead of just $0 = \{\cdot \mid \cdot\}$, then the games born on the next day form a complete but not necessarily distributive lattice. They ask which sets of games will give a distributive lattice? Is there a set which will give a nondistributive but modular lattice? Carvalho, dos Santos, Dias, Coelho, Neto, Nowakowski, and Vinagre [16] answered this by showing that for any lattice L, there is a set of games which generates L on the next day.

The set of all-small games does not form a lattice, but Siegel forms a lattice $\mathcal{L}_n^0$ by adjoining the least and greatest elements $\triangle$ and $\triangledown$ and asks: Do the elements

of $\mathcal{L}_n^0$ have an intrinsic "handedness" that distinguishes, say, $(n-1) \cdot \uparrow$ from $(n-1) \cdot \uparrow + *$?

A game is *option-closed* [94] (now called hereditarily transitive) if, recursively, each G^{LL} is also $\mathcal{G}^L$ and the same for Right. For example, HACKENBUSH strings are option-closed. McKay, Nowakowski, and (Angela) Siegel [83] show that the option-closed games born on day n form a planar, nondistributive lattice, but the question of *how many?* remains unanswered. Are there other natural families of games that form planar lattices?

E8. Aaron Siegel asks, given a group or monoid, $\mathcal{K}$, of games, to specify a technique for calculating the simplest game in each $\mathcal{K}$-equivalence class.

He notes that some restriction on $\mathcal{K}$ might be needed; for example, $\mathcal{K}$ might be the monoid of games absorbed by a given idempotent.

E9. Siegel also would like to investigate how search methods might be integrated with a canonical-form engine.

E10. (9) Develop a *misère theory* for unions of partizan games [*WW*, p. 312].

E11. Four-outcome games. Guy [50] has given a brute force analysis of a parity subtraction game (start with a subtraction set and an odd number of tokens, the winner is the player who removes an even number of tokens). The Sprague–Grundy theory does not apply because the game wasn't impartial, nor the Conway theory, because it was not last-player-winning. Is there a class of games in which there are four outcomes, $\mathcal{N}$ext, $\mathcal{P}$revious, $\mathcal{L}$eft and $\mathcal{R}$ight, and for which a general theory can be given? The discussion in [73, §2] hints that there will not be an extension of normal- or misère-theory that will solve this type of game.

E12. Impartial misère analysis. See A16. From the works of Plambeck and Siegel: Let $\mathcal{A}$ be some set of games (the *universe*) played under misère rules. Typically, $\mathcal{A}$ is the set of positions that arise in a particular game, such as DAWSON'S CHESS. Games $H, K \in \mathcal{A}$ are said to be equivalent, denoted by $H \equiv K \pmod{\mathcal{A}}$, if $H + X$ and $K + X$ have the same outcome for all games $X \in \mathcal{A}$. The relation $\equiv$ is an equivalence relation, and a set of representatives, one from each equivalence class, forms the *misère quotient*, $\mathcal{Q} = \mathcal{A}/\equiv$. A *quotient map* $\Phi : \mathcal{A} \to \mathcal{Q}$ is defined, for $G \in \mathcal{A}$, by $\Phi : \mathcal{G} = [G]_\equiv$.

(1) A quotient map $\Phi : \mathcal{A} \to \mathcal{Q}$ is said to be *faithful* if, whenever $\Phi(G) = \Phi(H)$, then G and H have the same normal-play Grundy value. Is every quotient map faithful?

(2) Let $(\mathcal{Q}, \mathcal{P})$ be a quotient and $\mathcal{S}$ a maximal subgroup of $\mathcal{Q}$. Must $\mathcal{S} \cap \mathcal{P}$ be nonempty? (Note: it's easy to get a "yes" answer in the special case when $\mathcal{S}$ is the kernel.)

(3) Extend the classification of impartial misère quotients. We have preliminary results on the number of quotients of order $n \leq 18$ but believe that this can be pushed far higher.

(4) Exhibit an impartial misère quotient with a period-5 element. Same question for period 8, etc. We've detected quotients with elements of periods 1, 2, 3, 4, 6, and infinity, and we conjecture that there is no restriction on the periods of quotient elements.

(5) In the flavor of both (3) and (4): What is the smallest quotient containing a period 4 (or 3 or 6) element?

E13. Inverses in partizan misère games. Siegel [Misère canonical forms of partizan games, GONC4] and Allen [GONC4] extend the misère monoid concepts to partizan games. In normal play, games in which both players have a move or neither does are called *all-small* because their values are infinitesimal. This is not true in misère play and the term *dicotic* has been coined to refer to these games in all play conventions. (See Section E14 as well.)

Recall that $-G$ is obtained from the game G by reversing the roles of Left and Right — "turning the board around". In normal play, $G - G = 0$ which is decidedly not true in misère play. To avoid confusion, when dealing with misère play $-G$ is frequently written as $\bar{G}$ and is called the *conjugate* of G.

(1) Allen asks, in the dicotic universe $\mathcal{D}$, when is it true that $G + \bar{G} \equiv 0 \pmod{\mathcal{D}}$? This was shown to hold for all games in [71].

(2) Rebecca Milley asks, in a universe $\mathcal{U}$, closed under sums, conjugation, and subpositions, if $G + H \equiv 0 \pmod{\mathcal{U}}$, is it necessarily true that $H = \bar{G}$? Jason Brown had asked the question about arbitrary universes. Milley [84] has the counterexample: on a finite strip of squares, Left places a 1×1 piece and Right places a 1×2 (domino). In the universe of sums of strips a strip of length 1 plus a strip of length 2 is equivalent to 0. This leads to a different question (see [Milley and Renault, in this volume]): in what universes U do we have no nonconjugate inverses?

A universe $\mathcal{U}$ is *reverse-friendly* if for every $G = \{A|\{\cdot|B\}\}$, there is $H = \{A'|\{\cdot|B\}\}$, where H is invertible modulo $\mathcal{U}$ and $G \geq \{\cdot|B\} \geq \{\cdot|C\}$. Adapting the proofs from [73], it appears that nonconjugate inverses cannot occur in any parental, dense, and reverse-friendly universes.

E14. Scoring games. Instead of "the last play determines the winner", another natural way is to have "whichever player has the higher score wins". Both DOTS-AND-BOXES and GO, for example, are scoring games.

Milnor looked at dicotic scoring games with nonnegative incentives. Milnor [85] defines a "positional game" to be what we would call a dicot (Section E13) scoring game, but only looks at positional games with nonnegative incentives. Hanner [51] looked at the same class of games as Milnor, and invented thermography. Ettinger [27] looked at dicotic scoring games (possibly with negative incentives), which he calls "positional games" following Milnor. Johnson [61] looked at the dicot scoring games that are "well-tempered" in some sense (a subset of Ettinger's games). Stewart [in this volume] looks at general scoring games. Since the games are no longer dicotic, a choice has to be made about how to handle the situation where the current player can't move but their opponent can.

Larsson, Nowakowski, and Santos [72; 71] have developed the theory of guaranteed scoring games. At the end of the game, the score is indicated by $\varnothing^a$ — the empty set to indicate there is no move and a is the score. A Left atomic game has the form $\langle \varnothing^\ell \mid A, B, \ldots \rangle$ — Left has no move and $A, B, \ldots$ are the Right options. In a Right atomic game, it is Right who has no move. A scoring game G is guaranteed if for every Left atomic game $\langle \varnothing^\ell \mid A, B, \ldots \rangle$ in G if $\varnothing^r$ occurs in one of the Right options $A, B, \ldots$ then $\ell \leq r$ and the reverse needs to be true for a Right atomic game. Normal play games can be embedded in scoring games by replacing every occurrence of $\varnothing$ by $\varnothing^0$. This mapping is an order-embedding into the set of guaranteed scoring games. They ask: Is the set of guaranteed scoring games the largest subset of scoring games (closed under disjunctive sum and conjugation) in which normal play games can be order-embedded? This subset of scoring games cannot contain games of the form $\langle \varnothing^\ell \mid \varnothing^r \rangle$ where $\ell > r$.

They ask for a solution of BRUSHES; see ["Scoring games: the state of play", in this volume] for the rules.

E15. Find a formula for the number of $\mathcal{P}$-positions for NIM played with $2n$ tokens. There are none with an odd number of tokens. The On-line Encyclopedia of Integer Sequences (OEIS) has the value for small values of n. Khovanova and Xiong [66] have formulas for when:

(i) the number of heaps is restricted and

(ii) the number in a heap is restricted.

In general, given a game, enumerate the number of $\mathcal{P}$-positions of a given size. We only know of TIMBER (Section D5) where the number of $\mathcal{P}$-positions, both normal and misère, are related to Dyck paths with certain properties (also to Catalan numbers and Fine numbers) and Hetyei [53; 54] which relates the number of $\mathcal{P}$-positions to the Bernoulli numbers of the second kind.

E16. Miller and Guo (see [48] for an overview) consider games played on lattice points in a polyhedra (often a Cartesian grid in 2 or more dimensions). There are finitely many terminal points, and finitely many vectors which indicate the moves. They originally conjectured that all lattice games have strategies that can be computed efficiently. Fink [30] disproved this. In *square-free lattice games*, the vectors only have zeroes and ones as entries. These generalize octal, hexadecimal and many other heap games. Miller and Guo conjecture [48] that there are efficient algorithms for determining the winning moves from any $\mathcal{N}$-position.

E17. A game has *nim-dimension n* if it contains a position $*2^{n-1}$ but not $*2^n$ (Santos [102]). A game has infinite nim-dimension if all the nimbers can be constructed. It has null, or $\varnothing$, nim-dimension if $*$ cannot be constructed. This was motivated by E6. (See also THINNING THICKETS [60].)

One can ask for the nim-dimension for each game one encounters. TOPPLING DOMINOES (Section C5), THINNING THICKETS [60], and GENERALIZED KONANE [Carvalho and Santos, in this volume] all have infinite nim-dimension, COL has nim-dimension 1 [*WW*], it is conjectured that DOMINEERING (Sections B11, E6) has nim-dimension $= 1$; CLOBBER has nim-dimension $= 1$; and AMAZONS (Section B6) has nim-dimension 2 (only $*4$ is known).

E18. Atomic weight calculus and nim-dimension. SUBVERSION (see A19) is a dicotic game in which both players have exactly one move and, in all positions, at least one of the players can end the game. The nim-dimension is 0 since the only nimbers that can occur are 0 and $*$. Fisher et al. [31] show that the exceptional case in the atomic weight calculations involve only a finite number of cases. They ask if this is true for any game with bounded nim-dimension. Of course, if true then the number of cases will probably increase with the nim-dimension.

E19. Universal partizan games. NIM is *universal* for impartial games in that for every impartial value there is (at least) one NIM of that value. Carvalho and Santos show that GENERALIZED KONANE [in this volume] is universal for the short Conway's group. Are there any other universal games? (Other than sliding a coin down a game-tree.) TOPPLING DOMINOES (D5) is a candidate but dealing with reversible options has proven difficult.

A related question: are there any general techniques that, given a game, show that certain values do not occur?

E20. Temperature. Nowakowski and Santos define the *boiling point* of a rule set R as $BP(R) = \limsup_G \text{temp}(G)$. (See B11 and E17.)

As with nim-dimension, one can ask for the boiling point for each game one encounters. TOPPLING DOMINOES (D5), THINNING THICKETS [60], and

GENERALIZED KONANE [Carvalho and Santos, in this volume] all have infinite boiling point. It is conjectured that the boiling point of DOMINEERING (B11, E6) is 2. See Berlekamp [in this volume].

Find feasible methods that give good upper bounds on the temperatures. Nowakowski and Santos show that if the length of the confusion intervals are bounded by k then the boiling point is no more than $3k/2$.

E21. Joins of games. When playing game A, a position may be a valid position for another game B.

This has been utilized in *conjoined games*; see Huggan and Nowakowski [in this volume]. Start playing A and when a terminal position is reached, start playing B. The two games must be compatible in some not-well-defined sense. This is a possible mechanism to generate a starting position for games without a prescribed start.

Heinrich (with Duchene, Larsson, and Parreau) makes it an option for a player to switch games (from A to B) at any point. Tennenhouse makes it an option to switch from normal to misère (or the reverse depending on the rules of the original game). Conway [*WW*, p. 534] introduced WHIM, which is NIM with the extra option, to be only be exercised once, of deciding if the outcome convention is normal or misère.

Farr and Ho [29] explore another possible join.

E22. Multiplayer games. (Propp [98] and [67] are good places to start.) If a player cannot win but can make either of the other two the winner, how should play proceed?

One way is to invoke a preference. Nowakowski, Santos, and Silva consider the podium rule (each player wants to be highest on the podium). They consider THREE-PLAYER NIM and determine the equivalence classes of the associated monoid. Koki Suetsugu gives each player a preference ordering. An intriguing aspect is that a player doesn't necessarily want themselves to win.

Using the podium rule, misère THREE-PLAYER NIM with one pile might be easier to analyze. Both Kelly [65] and Zhao and Liu [130] consider this game. Liu and Duan [77] finally solve the one heap and also the case when the number of players is at least as large as the number of heaps. The other case is still open. They ask for an analysis under other preferences.

We can consider cooperative two- or more player games such that all players want that one of the players make the last move. For two-player games this is a "puzzle" rather than a game. However, disjunctive sums still occur and the understanding about the recursive construction, comparison procedures, and canonical forms is an open field.

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