A nontrivial surjective map onto the short Conway group

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This paper explores the general question "Is there a natural habitat for the short Conway group?" by looking for a ruleset with a legal position for each short game value. Surprisingly, a ruleset with this property exists in combinatorial game theory literature and it is implemented in Siegel's CGSuite software. A proof that KONANE is an affirmative answer to the question is presented, making it the first known universal ruleset.

1. Introduction

The main result of this paper states that all the short combinatorial games are game values of particular positions of KONANE (Theorem 14 in Section 4). To prove this result constructively, two instrumental lemmas giving the needed building "pieces" are used (Lemmas 12 and 13 in Section 3). Some fundamental results of combinatorial game theory, like reduction concepts and the largest game value of the day n, are also used. Readers can find these results in Section 3. The next paragraphs of this introduction and Section 2 have some basics: the rules of KONANE, the state of the art and motivation for this research, and the definitions of *habitat* and *universality of a ruleset* (Definitions 3 and 7). Readers who know the rules of KONANE and are fluent in combinatorial game theory may wish to proceed to Lemma 12, Lemma 13 and Theorem 14 (Sections 3 and 4).

One of the principal goals of combinatorial game theory (CGT) is the study of combinatorial rulesets with the following properties ([1; 2; 5; 15] are fundamental references, [7] is a complete survey):

- There are two players who take turns moving alternately.

- No chance devices such as dice, spinners, or card deals are involved, and each player is aware of all the details at all times.

- Even ignoring the alternating condition, the play must end in a finite number of moves, and the winner is often determined on the basis of who made the last move. Under normal play, the last player wins, while in misère play, the last player loses.

We distinguish between multiple meanings of the word *game* by using the words *ruleset* and *game*. The word *ruleset* has a concrete meaning related to some particular set of rules (KONANE, AMAZONS, NIM are examples of rulesets). The word *game*, by contrast, has the abstract mathematical meaning defined by Conway [2; 5]. When we speak of the *game value* of a game, we are emphasizing that it is being considered in this latter sense, as an algebraic object which can be compared for equality with, or added to, other games.

The *options* of a game are all those positions which can be reached in one move. In CGT, games can be expressed recursively as $G = \{\mathcal{G}^L \mid \mathcal{G}^R\}$ where $\mathcal{G}^L$ are the Left options and $\mathcal{G}^R$ are the Right options of G. The *followers* of G are all the games that can be reached by all the possible sequences of moves from G.

Conway made a recursive construction based in a transfinite sequence of *days* [5]. His inductive definition constructs the proper class of combinatorial games (we will only consider the normal play). The games with finite sets of options $\mathcal{G}^L$ and $\mathcal{G}^R$ are called *short games* and are born before the day ω. The games born on day ω or after are called *long games*.

Founding fathers of CGT observed that independent components naturally arise in several rulesets. The analysis of these decompositions led to the definition of a *disjunctive sum*.

Definition 1 (disjunctive sum). Let G and H be games. Then,

$$G + H = \{\mathcal{G}^L + H, G + \mathcal{H}^L \mid \mathcal{G}^R + H, G + \mathcal{H}^R\}$$

(note that if G is a game and $\mathcal{S}$ a set of games, $A + \mathcal{S} = \{G + H : H \in \mathcal{S}\}$).

The proper class of combinatorial games (short and long) with the disjunctive sum is an abelian group. In fact, with a suited order relation (motivated by the game practice), it is an abelian group with a partial order.

Definition 2 (short Conway group). The *short Conway group* is the subgroup of the proper class of combinatorial games containing the short Normal-play games.

A classical example of a combinatorial ruleset is NIM, first studied by C. Bouton [3]. NIM is played with piles of stones. On his turn, each player can remove any number of stones from any pile. The winner is the player who takes the last stone. NIM is an example of an impartial ruleset: Left options and Right options are the same for the game and all its followers. The values involved in NIM are

called nimbers (stars):

$$*k = \{0, *, \ldots, *(k-1) \mid 0, *, \ldots, *(k-1)\}.$$

It is a surprising fact that all impartial rulesets take only nimbers as values (Sprague–Grundy theorem; see [8; 16]).

2. NIM **dimension and the concept of habitat**

The Sprague–Grundy theorem states that for every impartial G there is a nonnegative integer n such that $G = *n$. It is also well known that in partizan rulesets we still can construct nimbers (see [1; 2; 5; 15]). This motivates some very natural questions.

Definition 3. Let $\mathscr{S}$ be a set of combinatorial game values. A ruleset A is a *habitat* of $\mathscr{S}$ if for every $G \in \mathscr{S}$ there is a position of A with game value equal to G.

Berlekamp asked the question "What is the habitat of $*2$?" [9]. It was possible to generalize the question: "For a given ruleset, what is the largest n such that $*2^n$ is the game value of a legal position?" [10; 11; 12; 13]. This led to the definition of nim dimension and to the proposal of some processes to analyze the problem.

Definition 4. A combinatorial ruleset has *nim dimension n* if it contains a position with game value $*2^n$ but not $*2^{n+1}$. A ruleset has *infinite* nim dimension if all the nimbers can be constructed. It has *null*, or $\varnothing$, nim dimension if $*$ cannot be constructed.

Observation 5. A combinatorial ruleset A has nim dimension n if it is a habitat of $\{0, *, *2, *4, \ldots, *2^n\}$ and it is not habitat of $\{0, *, *2, *4, \ldots, *2^n, *2^{n+1}\}$.

The rulesets KONANE and AMAZONS are "case studies" related to *nim dimension*. KONANE is a classical Hawaiian ruleset, considered very interesting by CGT researchers [4; 6]. In the starting position, the checkered board is filled in such a way that no two stones of the same color occupy adjacent squares. In the opening, two adjacent pieces of the board are removed. After this, a player moves by taking one of his stones and jumping orthogonally over an opposing stone into an empty square. The jumped stone is removed. A player can make multiple jumps on his turn but cannot change direction mid-turn. Multiple jumps are not mandatory. The winner is the player who makes the last move. KONANE is implemented in Siegel's CGSuite, a fundamental computational tool for CGT research [14]. CGSuite, as well as the main result of this paper, regards the generalized version of KONANE.

Convention. In the *generalized* version of KONANE two stones of the same color can occupy adjacent cells.
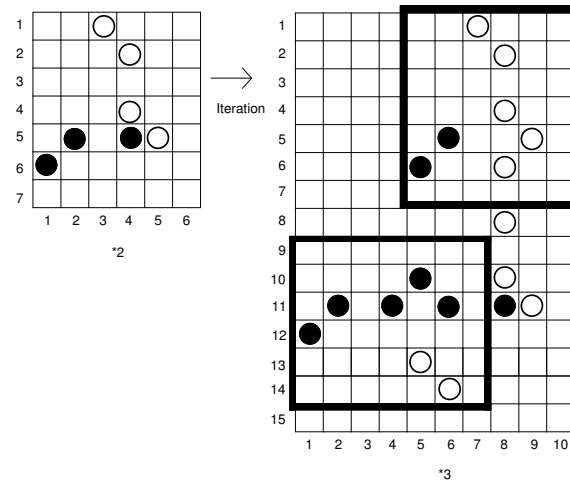
Figure 1. Nimbers in KONANE.

AMAZONS is a combinatorial ruleset invented in 1988 by Walter Zamkauskas. It is played on a checkered board and each move consists of two parts: moving one of one's own amazons one or more empty cells in a straight line (orthogonally or diagonally), exactly as a queen moves in CHESS; it may not cross or enter a cell occupied by an amazon of either color or a stone. After moving, the amazon shoots a stone from its landing cell to another cell, using another queen-like move. This stone may travel in any orthogonal or diagonal direction and, like an amazon, cannot cross or enter a cell where another stone has landed or an amazon of either color stands. The winner is the player who makes the last move. As usual in CGT, Left plays with the black stones and Right with the white ones in both rulesets.

In [12], the *fractal process* was proposed in order to prove that the nim dimension of KONANE is infinite. The basic idea was to construct the $*n$ using the previous $*(n-1)$. Figure 1 shows an iteration to obtain a $*3$ from a previous $*2$.

In [13], the *algebraic process* was proposed. The idea was the implementation of an algebraic table where the entries were the "needed stuff" to construct a $*n$. The amazons moves "created" the entries of the table splitting the initial position into two disjoint components. With this process, it was possible to construct the first known $*4$ in AMAZONS (Figure 2).

Conjecture 6. *The nim dimension of* AMAZONS *is* 2.

Finally, in [10], the *embedding process* was proposed. Considering an initial ruleset A, the idea was to find a construction process in some other ruleset B and embed it in A. Obviously, B should be somehow better understood than A. It was shown that the nim dimension of TRAFFIC LIGHTS is infinite (embedding in
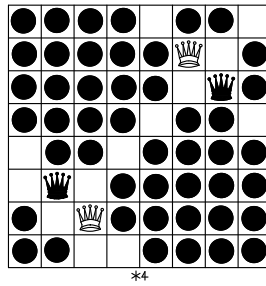
Figure 2. $*4$ in AMAZONS.

it constructions made in REGIO).

It is possible to generalize Berlekamp's original question even further.

Definition 7. A ruleset A is *universal* if it is a habitat of the short Conway group.

Natural is the new question "Is there a natural habitat for the short Conway group?". A classical universal ruleset was not known. In the following sections this state changes. With a constructive idea very similar to the *fractal process*, a universal ruleset is revealed.

3. Preliminary results

The main purpose of this text is to answer the general question "Is there a habitat for the short Conway group?". In Section 4, we prove that all the short combinatorial games are game values of KONANE's positions, and so, this well-known ruleset is a habitat of the short Conway group. In this section, some needed previous lemmas are proved.

KONANE is a very rich combinatorial ruleset with several interesting values; see Figure 3, left and middle. Figure 3, right, shows an example with value 1 in the generalized version.

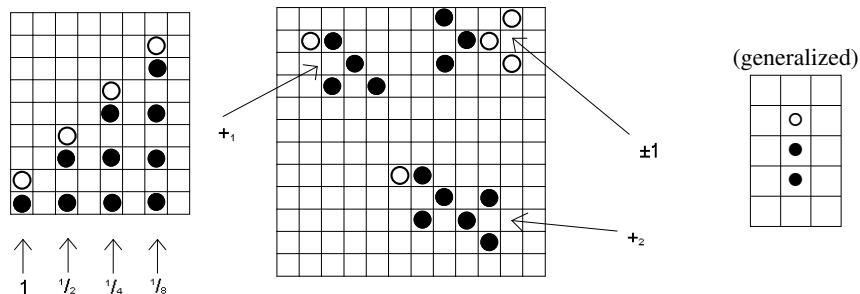We now recall some important results of combinatorial game theory.

Figure 3. Left and middle: several game values in KONANE. Right: the value 1 in generalized KONANE.

Definition 8. For a game G, suppose $G_1^L, G_2^L \in \mathcal{G}^L$ with $G_2^L \geq G_1^L$. Then the Left option G_1^L is said to be *dominated* by G_2^L (or that G_2^L *dominates* G_1^L).

Definition 9. For a game G, suppose there is $G^L \in \mathcal{G}^L$ and $G^{LR} \in (G^L)^{\mathcal{R}}$ with $G^{LR} \leq G$. Then the Left option G^L is *reversible*, and sometimes, to be specific, G^L is said to be *reversible through* its Right option G^{LR}. In addition, G^{LR} is called a *reversing* option for G^L and the set of Right options of G^{LR}, $(G^{LR})^{\mathcal{L}}$, is a *replacement set* for G^L (eventually empty).

Theorem 10 (reductions of combinatorial games; see [2, pp. 60–63]). (a) *Let G be a game and suppose $G_1^L, G_2^L \in \mathcal{G}^L$ such that G_1^L is dominated by G_2^L. Then $G = \{\mathcal{G}^L \setminus \{G_1^L\} \mid \mathcal{G}^R\}$ in the sense of the equality of games.*

(b) *Let G be a game and suppose that G^L is a Left option of G reversible through G^{LR}. Then $G = \{\mathcal{G}^L \setminus \{G^L\}, (G^{LR})^{\mathcal{L}} \mid \mathcal{G}^L\}$ in the sense of the equality of games.*

Analogous versions of (a) *and* (b) *hold for Right.*

A reversible move for Left is one which Right can promise to respond to in such a way that prospects are at least as good as they were before. In any context, Right promises "if you ever choose option A_1 of G then I will immediately move to A_1^R". So, Left just chooses the option A_1 if he intends to follow up Right's move to A_1^R with an immediate response to one of A_1^R's Left options. If he plans some other move elsewhere, he might just as well start with that. We say that A_1 *reverses through* A_1^R to the Left options of A_1^R. If the Left options of A_1^R are empty then we say that A_1 *reverses out*. G is in *canonical form* if G and all of G's followers have no dominated or reversible options.

The next one is also very well known.

Theorem 11 [1, p. 119]. *The largest game born by day n is n.*

Because the proof for the main result is constructive, we will need some patterns to build the construction. The next two lemmas are useful KONANE patterns.

Lemma 12 (rubber bands). *Consider the KONANE pattern shown in Figure 4.*

The position P_n is a $(2n+3) \times 5$-rectangle. The black pieces occupy the cells $(1, 2), (2, 2), (1, 4), (2, 4), \ldots, (1, 2n), (2, 2n)$.

The white pieces occupy the cells $(1, 0), (1, 2n+2)$, the cells $(1, 1), (1, 3), \ldots, (1, 2n + 1)$, and the cells $(3, 2), (4, 2), (3, 4), (4, 4), \ldots, (3, 2n), (4, 2n)$.

Let $P_n \setminus (1, 2k + 1)$ be the exposed positions without *the white piece in the position $(1, 2k + 1)$. Call $a(k)$ the number of black stones above the line $2k + 1$ and $b(k)$ the number of black stones bellow the line $2k + 1$. The game values of P_n and $P_n \setminus (1, 2k + 1)$ are the following*:

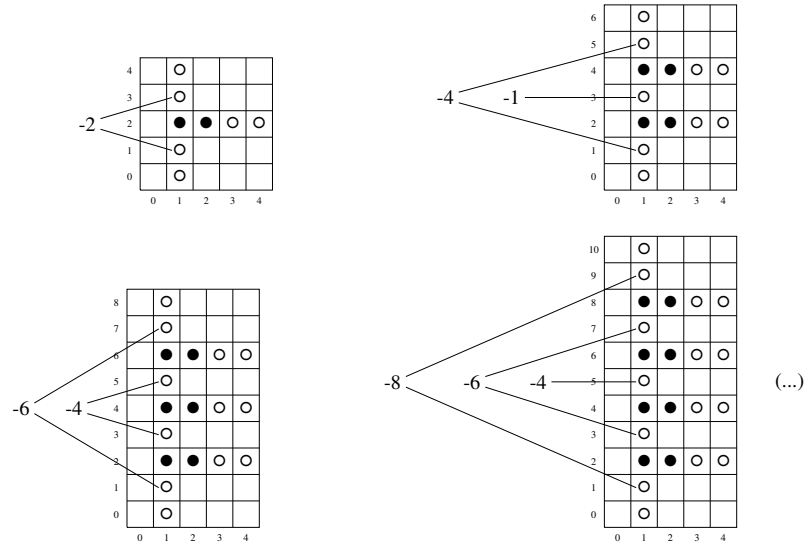(1) *P_n has value 0 (there are no moves).*
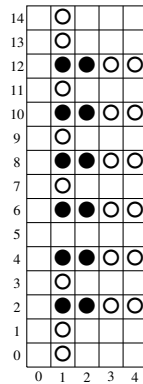
Figure 4. KONANE pattern for rubber bands.

Figure 5. Illustrative case.

(2) $P_2 \backslash (1, 3)$ *has value* -1.

(3) $P_n \backslash (1, 2k + 1)$ *different from* $P_2 \backslash (1, 3)$ *and has value* $-\max(a(k), b(k))$.

Previous observation: For ease, let us consider a very particular case to understand the idea behind the general case; see Figure 5.

If Right plays $(1, 7) \mapsto (1, 5)$ the value becomes $\{-8 \mid -6\} = -7$ (simplicity rule). If Right plays $(1, 3) \mapsto (1, 5)$ the value becomes $\{-4 \mid -2\} = -3$ (simplicity rule).

So, the original position is $\{\mid -7, -3\}$ which, by domination, is equal to $\{\mid -7\} = -8$. Right's move must be made in the opposite direction of the largest number of black stones.
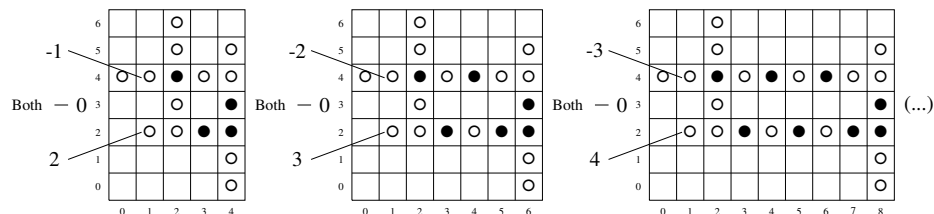
Figure 6. KONANE pattern for taps.

Proof. The cases P_1 and P_2 can be analyzed with pure calculations. Consider P_n with $n > 2$.

The idea for the general case P_n with $n > 2$ follows as in the previous observation: Right's move must be made in the opposite direction of the largest number of black stones. After removing the piece in the cell $(1, 2k+1)$, the game value is $\{\, | -\max(a(k), b(k)) + 1\}$ which is equal to $-\max(a(k), b(k))$.   □

Lemma 13 (taps). *Consider the* KONANE *pattern shown in Figure 6.*

The position P_n is a $7 \times (2n + 3)$-rectangle.

The black pieces occupy the cells $(2n + 2, 2)$, $(2n + 2, 3)$ *and the cells* $(2, 4)$, $(3, 2)$, $(4, 4)$, $(5, 2)$, $\ldots$, $(2n, 4)$, $(2n + 1, 2)$.

The white pieces occupy the cells $(0, 4)$, $(1, 2)$, $(1, 4)$, $(2, 2)$, $(2, 3)$, $(2, 5)$, $(2, 6)$, *the cells* $(2n + 2, 0)$, $(2n + 2, 1)$, $(2n + 2, 4)$, $(2n + 2, 5)$, *and the cells* $(3, 4)$, $(4, 2)$, $(5, 4)$, $\ldots$, $(2n, 2)$, $(2n + 1, 4)$.

Let $P_n \backslash (1, 2)$ and $P_n \backslash (1, 4)$ be the exposed positions without *the white pieces in the positions* $(1, 2)$ *and* $(1, 4)$, *respectively. Let $P_n \backslash \{(1, 2), (1, 4)\}$ be the exposed positions* without both *white pieces in the positions* $(1, 2)$ *and* $(1, 4)$. *The game values of P_n, $P_n \backslash (1, 2)$, $P_n \backslash (1, 4)$ and $P_n \backslash \{(1, 2), (1, 4)\}$ are the following*:

(1) *P_n has value 0 (there are no moves).*

(2) *$P_n \backslash (1, 2)$ has value $n + 1$.*

(3) *$P_n \backslash (1, 4)$ has value $-n$.*

(4) *$P_n \backslash \{(1, 2), (1, 4)\}$ has value 0.*

Proof. In $P_n \backslash (1, 4)$, Right can take, one by one, all the n black pieces of the line 4. So, the game value of $P_n \backslash (1, 4)$ is $-n$.

In $P_n \backslash (1, 2)$, Left can take, one by one, all the n white pieces of the line 2 plus the white piece of the cell $(2, 3)$, leaving the value 0 in the line 4. The game value of $P_n \backslash (1, 2)$ is $n + 1$.

In $P_n \backslash \{(1, 2), (1, 4)\}$, there are exactly n moves for both players (the capture of the white stones of the line 2 and the capture of the black stones of the line 4). So, the first player loses and the game value of $P_n \backslash \{(1, 2), (1, 4)\}$ is 0.   □
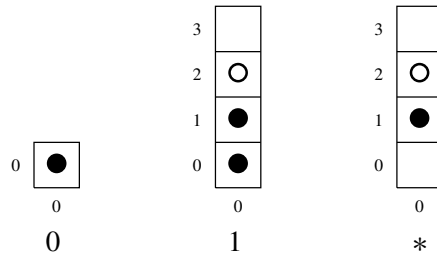
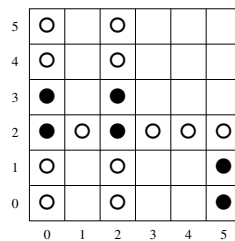**Figure 7.** Initial games 0, 1, and $*$ (day 1).

Figure 8. Joining of rubber bands.

4. KONANE **is a universal ruleset**

The proof for the main result is constructive. The games of day n are constructed with games of the previous days, generating a recursive process Γ such that

$$G = \Gamma(\mathcal{G}^L, \mathcal{G}^R).$$

The "design type" of $\Gamma(\mathcal{G}^L, \mathcal{G}^R)$ must be the same as the "design types" of the games in $\mathcal{G}^L$ and $\mathcal{G}^R$. In our proof, the "design type" is basically the shape. All the games are constructed in rectangular areas. The rectangular areas of the set of the options generate a new game in a larger rectangular area which will be useful to build new rectangles.

Theorem 14. *All the short combinatorial games are game values of particular positions of* KONANE.

Proof. First, we present the initial games 0 (day 0) and 1 and $*$ (day 1) in Figure 7. It is possible to join rubber bands without changing the values. For example, consider the position for the value 1 shown in Figure 8.

Second, we present the *connecting scheme* in Figure 9. If, at some point, a black stone moves to the cell $(5, 1)$ removing the stones in $(0, 1)$, $(2, 1)$ and $(4, 1)$, the game value becomes 1. Moreover, *and this is one of the principal ideas of the proof*, it is possible to choose rubber band sizes such that the incomplete captures reverse out (Theorem 11 and Lemma 12).

Third, we present the general idea of the recursion in Figure 10. In the initial
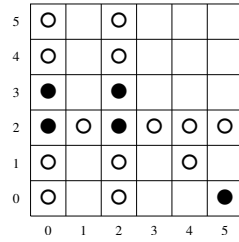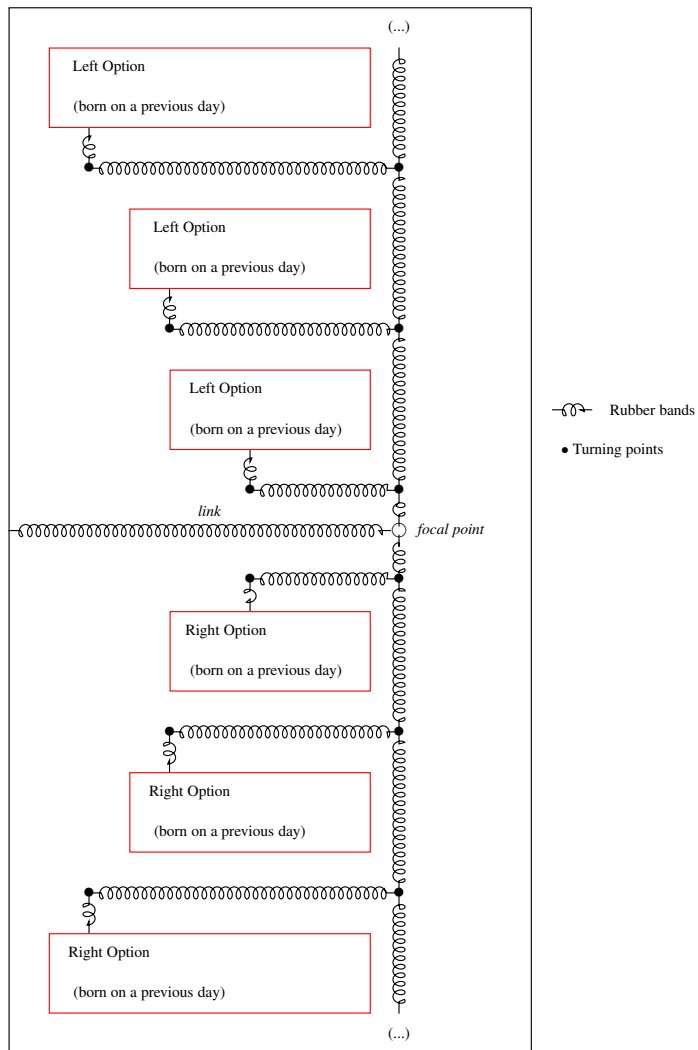
Figure 9. Connecting scheme example.
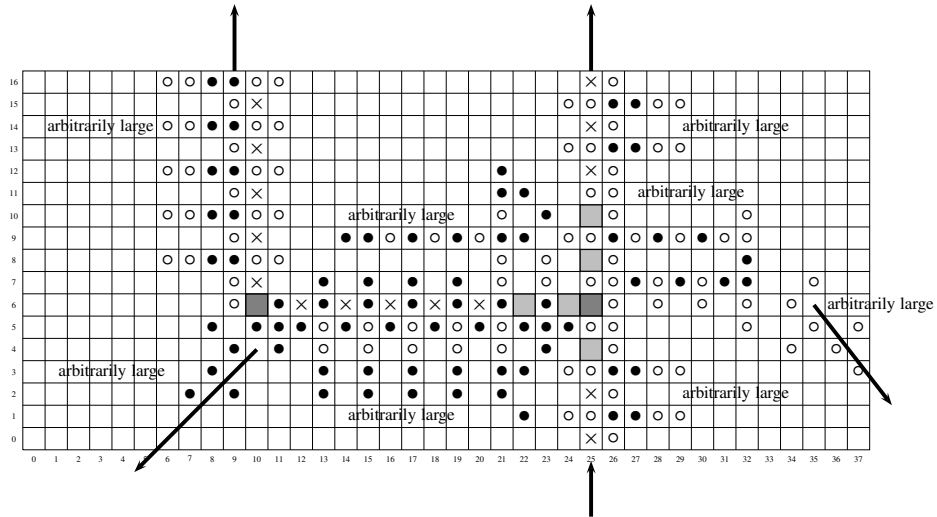

Figure 10. Recursion example.

Figure 11. Turning point example.

position, the *focal point* is the pair of cells occupied by the only pair of stones that can move.

The construction of each rectangular position is made with *rubber bands*, *turning points* based in *taps*, *one link* (itself a rubber band) and *rectangular positions* of the previous days. It is important to observe that there are no moves in these rectangular positions. The focal point is occupied by opposing stones only after a complete capture through the link; after that, a game value of the previous days is obtained.

The rubber bands are arbitrarily large. So, when we construct a game of the day n with options of the previous days, we choose the rubber band sizes in such a way that for every incomplete Left capture (incomplete Right capture) exists an integer $G^{LR} \leq -n$ ($G^{RL} \geq n$). Therefore, these options reverse out. This is possible due to Lemma 12.

The turning points are constructed with the following idea: when Left (Right) captures to a turning point he makes a threat larger than or equal to n (smaller than or equal to $-n$). To defend the threat, Right (Left) has only one good option, creating again a threat smaller than or equal to $-n$ (larger than or equal to n). After these two forced moves, the only way for Left (Right) to defend the second threat is to choose an option of the previous days. The idea is based again in reversibility: instead of constructing directly an option G^L (G^R), we construct an option $\{T \geq n, \ldots \mid \{G^L, \ldots \mid T' \leq -n, \ldots\}\}$ ($\{\{T \geq n, \ldots \mid G^R, \ldots\} \mid \ldots, T' \leq -n\}$).

Fourth, we present the details of the turning points. Consider the picture showing a turning point in Figure 11. The crosses represent options adjacent to

rubber bands that reverse out. So, when we have a sequence of captures made by a dark piece in the column 25, we only need to analyze the moves to the gray and dark gray cells. The turning point is the cell $(25, 6)$ (dark gray).

(1) If Left captures to the cell $(25, 4)$, Right replies by capturing to the cell $(22, 4)$ obtaining more than or equal to $-n$ points (Lemma 12). So, the Left move to $(25, 4)$ reverses out.

(2) If Left captures to the cell $(25, 8)$, Right replies by capturing to the cell $(25, 7)$ obtaining more than or equal to $-n$ points. This happens due to Lemma 13 (the tap is arbitrarily large) So, the Left move to $(25, 8)$ reverses out.

(3) If Left captures to the cell $(25, 10)$, she opens completely the tap, turning it equal to zero (Lemma 13). Right replies capturing to the cell $(22, 10)$ and, afterwards, Right's stone in $(22, 10)$ will capture to $(22, 8)$, obtaining at least $-n$ points.

After the Left move to $(25, 6)$, a threat in $(35, 6)$ is created (at least n points).

(1) If Right captures to the cell $(24, 6)$, Left replies by capturing to the cell $(24, 7)$. Afterwards, the move from $(24, 7)$ to $(20, 7)$ will get at least n points. So, the Right move to $(24, 6)$ reverses out.

(2) If Right captures to the cell $(22, 6)$, Left replies by capturing to the cell $(22, 7)$, obtaining at least n points. So, the Right move to $(22, 6)$ reverses out.

The good Right move is to $(10, 4)$, creating a threat in $(10, 4)$ (at least $-n$ points). Left has to go up to an option that is a game of the previous days.   □

The proof that KONANE is an universal ruleset was constructive. Using the process illustrated in the proof, as an example, we finish the paper presenting the impressive construction of the game $\{0, \uparrow *, \pm 1 \,||\, -1, \{* \,|\, -1\}\}$ (game of the day 3). Figure 12 illustrates a terminal position useful for future constructions (it is a rectangle). If we remove the white stone of the cell $(37, 29)$ while putting a black stone in the cell $(38, 29)$, a situation that occurs after a capture of the white pieces of the row 29, we get the game $\{0, \uparrow *, \pm 1 \,||\, -1, \{* \,|\, -1\}\}$. Observe that the colored rectangles are the options of the previous days.

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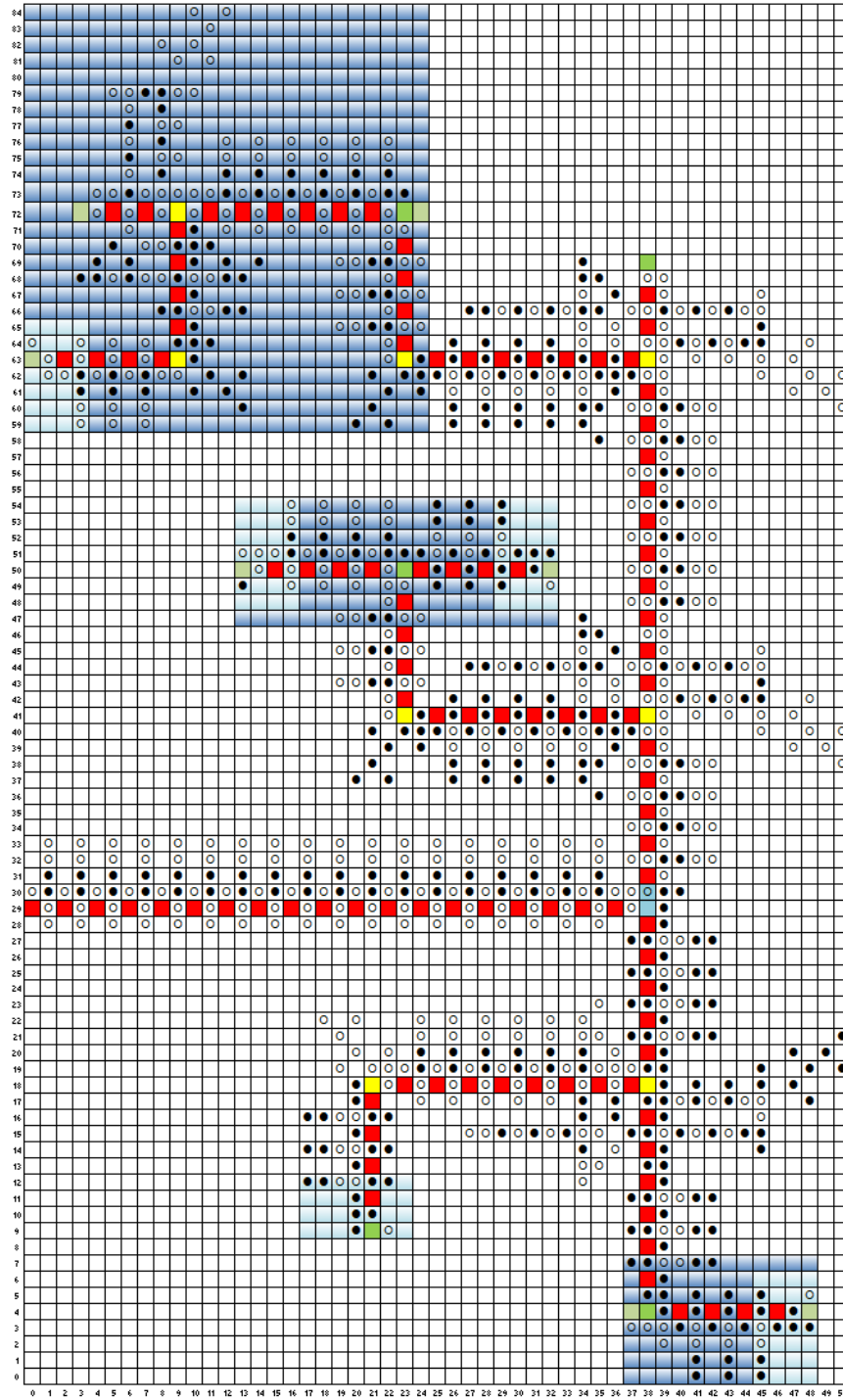[3]   C. L. Bouton. "Nim, a game with a complete mathematical theory", *Ann. Math.* 3 (2), 35–39, 1902.

Figure 12. Terminal position for future constructions.

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