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Take-away games on Beatty's theorem and the notion of k-invariance

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We formulate three reasonably short game rules for three two-pile take-away games, which share one and the same set of P-positions. This set is comprised of a pair of complementary homogeneous Beatty sequences together with $(0, 0)$. We relate the succinctness of the game rules with the complexity of the P-positions by means of a notion dubbed k-invariance, and introduce the *game-invariance number* for a set of P-positions.

1. Introduction

The first author posed an intriguing problem at the GONC-workshop: "Describe nice/short rulesets for games with so-called complementary Beatty sequences as sets of P-positions." The problem is the opposite of the main field of research in this area, which is to, given a game, search for its set of P-positions. Here we describe three such rule sets, which resolves the question for any pair of complementary homogenous Beatty sequences.

Let us recall the rules of d-Wythoff [10], d a fixed positive integer. The available positions are (x, y), x and y nonnegative integers. The legal moves are

(I) Nim-type: $(x, y) \to (x - t, y)$, if $x - t \geq 0$ and $(x, y) \to (x, y - t)$, if $y - t \geq 0$; $t > 0$.

(II) Extended diagonal type: $(x, y) \to (x - s, y - t)$ if $|t - s| < d$ and $x - s \geq 0$, $y - t \geq 0$; $s > 0$, $t > 0$.

This game is a so-called impartial take-away game [2], vol. 1. We restrict attention to *normal* play; that is, the player first unable to move loses. For our games it means that the player called upon to move from $(0, 0)$ loses.

Rules (I) and (II) imply that d-Wythoff is a so-called *invariant* [5; 16] (take-away) game; that is, each available move is legal from any position as long as the resulting position has nonnegative coordinates. Every move in any invariant

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game is an *invariant move*. In this note we study another type of take-away game, where certain positions have some local restrictions on the set of otherwise invariant moves. Such games are called *variant* [5; 16]. We define these notions in Section 4.

Central to our investigation is Beatty's theorem [1] (predated by Lord Rayleigh [19]): Let $\beta > 2$ be an irrational number and define its *complement*, $\hat{\beta}$, by the equality $\hat{\beta}^{-1}+\beta^{-1}=1$, so that $\hat{\beta}=\beta/(\beta-1)$. This clearly implies $1<\hat{\beta}<2<\beta$. Let $A_n=\lfloor n\hat{\beta}\rfloor$, $B_n=\lfloor n\beta\rfloor$, $A=\cup_{n\geq 1}\{A_n\}$, $B=\cup_{n\geq 1}\{B_n\}$. Beatty's theorem then asserts that A and B are *complementary* sets, that is, $A\cup B=\mathbb{Z}_{\geq 1}$, $A\cap B=\varnothing$. Since $\beta>\hat{\beta}>1$, the (homogeneous) *Beatty sequences* (A_n) and (B_n) are strictly increasing.

1.1. ***Three games.*** We formulate three game rules. Let $\beta>2$ be a fixed irrational and let $d=\lfloor\beta\rfloor$. Fix a pair of nonnegative integers (x, y). Recall that $B_n=\lfloor n\beta\rfloor$ for all n:

(G1) The moves are as in nim on two piles (I), except that, if $B\cap\{x, y\}=\varnothing$, then in addition to the nim-type move a player may also take away $s\in\{0, \dots, d\}$ from the other pile in the same move. This game is denoted by *β-nim*.

(G2) The moves are as in d-Wythoff, subject to (I) and (II), except that if $B\cap\{x, y\}\neq\varnothing$, then only nim-type moves (I) are permitted. This game is denoted by *β-Wynim*.

(G3) The moves are as in d-Wythoff, subject to (I) and (II), except that if $B\cap\{x, y\}\neq\varnothing$, then the pair (s, t), with s and t as in (II), cannot belong to the pair of *β-triangles* defined by

$$\big\{(x, y), (y, x) \mid (x, y)\in\{(1, \lfloor\beta\rfloor), (2, \lfloor\beta\rfloor), (2, \lfloor\beta\rfloor+1)\}\big\}.$$

This game is denoted by *β^{T}-Wynim*.

The name Wynim derives from Wythoff-nim; in (G3) the T in β^{T} stands for triangles. The main result of this note is as follows.

Theorem 1. *The set of P-positions of β-nim, β-Wynim and β^{T}-Wynim is the same. It is*

$$\mathcal{P}:=\bigcup_{n\geq 0}\{(A_n, B_n)\}\cup\bigcup_{n\geq 0}\{(B_n, A_n)\},$$

where $A_n=\lfloor n\hat{\beta}\rfloor$, $B_n=\lfloor n\beta\rfloor$.

We prove this result in Section 3. In Section 4 we develop the distinction between invariant and variant games and relate our findings to certain complexity issues. In the section to come we give some examples.

2. Examples and tables of P-positions

In the proof of the main result and in the examples of sets of P-positions to come, we use the following illustrative notation.

Notation 2. For every $n \geq 0$:

(1) $\Delta A_n := A_{n+1} - A_n$, $\Delta B_n := B_{n+1} - B_n$ are the *gaps*.

(2) $\Delta_n := B_n - A_n$.

(3) $\Delta_n^2 := \Delta_{n+1} - \Delta_n$.

For some (invariant) take-away games on two heaps where short formulas for both the rules and the P-positions are known, such as [10; 12; 14], the coordinates of the P-positions are defined via certain algebraic numbers together with the floor function. Our first example rather uses a well-known transcendental number.

Example 3. In the game of π-Wynim, a player may move as in nim on two piles (I), or, if the position does not contain a coordinate of the form $\lfloor \pi n \rfloor$, deviate at most $\lfloor \pi \rfloor - 1 = 2$ positions from the "main diagonal" as given by the game d-Wythoff; that is use (II) with $d = 3$. The result of this note implies that the P-positions of this game are the set

$$\cup_{n \geq 0} \{ (\lfloor \hat{\pi} n \rfloor, \lfloor \pi n \rfloor), (\lfloor \pi n \rfloor, \lfloor \hat{\pi} n \rfloor) \},$$

the first few of which are displayed in Table 1.

Example 4. Example 3 illustrates Theorem 1 for a member of our second game family, β-Wynim. A further example: Let $d = 2$ in the formula $\beta =$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
A_n	0	1	2	4	5	7	8	10	11	13	14	16	17	19	20
B_n	0	3	6	9	12	15	18	21	25	28	31	34	37	40	43
Δ_n	0	2	4	5	7	8	10	11	14	15	17	18	20	21	23
Δ_n^2	2	2	1	2	1	2	1	3	1	2	1	2	1	2	2

n	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
A_n	22	23	24	26	27	29	30	32	33	35	36	38	39	41	42
B_n	47	50	53	56	59	62	65	69	72	75	78	81	84	87	91
Δ_n	25	27	29	30	32	33	35	37	39	40	42	43	45	46	49
Δ_n^2	2	2	1	2	1	2	2	2	1	2	1	2	1	3	1

Table 1. The first few P-positions (A_n, B_n) for β-nim, β-Wynim and β^{T}-Wynim, $\beta = \pi = 3.14159\ldots$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
A_n	0	1	2	4	5	7	8	9	11	12	14	15	16	18
B_n	0	3	6	10	13	17	20	23	27	30	34	37	40	44
n	14	15	16	17	18	19	20	21	22	23	24	25	26	27
A_n	19	21	22	24	25	26	28	29	31	32	33	35	36	38
B_n	47	51	54	58	61	64	68	71	75	78	81	85	88	92

Table 2. The first few P-positions (A_n, B_n) for 2-Wythoff, β-nim β-Wynim and β^{T}-Wynim; β as in Example 4.

$\frac{1}{2}(2+d+\sqrt{d^2+4})$ (see d-Wythoff and paper [10]) and with $\hat{\beta} = \beta - d$. Then $\beta = \sqrt{2}+2$, $\hat{\beta} = \beta - 2$; note that $\lfloor\beta\rfloor = 3$ as in Example 3. The first few P-positions are shown in Table 2. Since $\beta - \hat{\beta} = d = 2$, we have $\Delta_n = dn = 2n$, so $\Delta_n^2 = d = 2$ for all $n \geq 0$, and the β-triangles, as in (G3), for both these games, will be $\{(1, 3), (2, 3), (2, 4)\} \cup \{(3, 1), (3, 2), (4, 2)\}$.

Remark 5. It is remarkable that, for $\beta = \frac{1}{2}(2+d+\sqrt{d^2+4})$, d-Wythoff has the same set of P-positions as our three games. In particular, for $d = 1$, 1-Wythoff is the classical Wythoff game [2]. For $d = 2$, the first few P-positions of the games are displayed in Table 2. In [4] it was shown that from the classical Wythoff game no move can be deleted while preserving the set of P-positions of the classical Wythoff game. In the present note, Wythoff moves were deleted, and the P-positions are still preserved. The difference is that in [4] only *invariant* moves were permitted. See Section 4 for more on the latter topic.

3. Proof of the main result

We preface the proof of Theorem 1 by collecting some facts on the sets $\{A_n\}$ and $\{B_n\}$.

Proposition 6. *For every* $n \geq 0$:

(i) *The only possible gap pairs are*

$$(\Delta A_n, \Delta B_n) \in \big\{(1, \lfloor\beta\rfloor),\ (1, \lfloor\beta\rfloor+1),\ (2, \lfloor\beta\rfloor),\ (2, \lfloor\beta\rfloor+1)\big\}.$$

(ii) $\Delta_n^2 = \Delta B_n - \Delta A_n$.

(iii) $\Delta_n^2 \in \{\lfloor\beta\rfloor - 2, \lfloor\beta\rfloor - 1, \lfloor\beta\rfloor\}$.

Proof. (i) This is a well-known result.

(ii) $\Delta_n^2 = (B_{n+1} - A_{n+1}) - (B_n - A_n) = (B_{n+1} - B_n) - (A_{n+1} - A_n) = \Delta B_n - \Delta A_n$.

(iii) Follows directly from (i) and (ii). □

Example 7. Notice that in Example 3, Table 1, Δ_n^2 assumes all three possible values $\{1, 2, 3\} = \{\lfloor\beta\rfloor - 2, \lfloor\beta\rfloor - 1, \lfloor\beta\rfloor\}$. In Example 4, Δ_n^2 assumes only the value $2 = \lfloor\beta\rfloor - 1$.

Proof of Theorem 1. Since our games are acyclic, it suffices to demonstrate the following two properties for each game:

P → N: Every move from any position of the form

$$(A_n, B_n) \quad \text{or} \quad (B_n, A_n) \tag{1}$$

results in a position outside (1).

N → P: Given any position outside (1), there exists a move into (1).

For the direction P → N we use the same argument for the games (G1) β-nim and (G2) β-Wynim, namely: Suppose that we play from a position of the form (1). The game rules imply that only nim-type moves (I) are permitted so that by complementarity, there is no move to a position of the same form.

For the game (G3) β^{T}-Wynim, we have to show that both

(i) $(A_n, B_n) \to (A_m, B_m)$ and

(ii) *cross moves* $(A_n, B_n) \to (B_m, A_m)$ are blocked for every $0 \le m < n$.

(i) By Proposition 6, $(B_n - B_m) - (A_n - A_m) = \Delta_n - \Delta_m \ge \Delta_n - \Delta_{n-1} = \Delta_{n-1}^2 \in \{\lfloor\beta\rfloor - 2, \lfloor\beta\rfloor - 1, \lfloor\beta\rfloor\}$, where the $\ge$ follows from the fact that $\beta > \hat{\beta}$, which implies that Δ_i is a nondecreasing function of i. Therefore the move $(A_n, B_n) \to (A_m, B_m)$ is either blocked by the triangle move restriction of β^{T}-Wynim (if $\Delta_{n-1}^2 \le \lfloor\beta\rfloor$), or by the $\lfloor\beta\rfloor$-Wythoff constraint (if $\Delta_{n-1}^2 \ge \lfloor\beta\rfloor$).

(ii) Notice that this move is possible only if $A_n > B_m$. Now $(B_n - A_m) - (A_n - B_m) = \Delta_n + \Delta_m$. Similarly to (i), if $\Delta_n + \Delta_m \in \{\lfloor\beta\rfloor - 2, \lfloor\beta\rfloor - 1, \lfloor\beta\rfloor\}$, this forces $m = 0$ and $n = 1$ so that the move is blocked by the β-triangle move restriction; otherwise by the $\lfloor\beta\rfloor$-Wythoff constraint.

For the direction N → P, let (x, y), $0 \le x \le y$ be a position not of the form (1). We assume first that this position has a coordinate of the form B_n, so for each game it suffices to show that a nim-type (I) move suffices for moving into (1). If $x = B_n$ then move $y \to A_n$. If $y = B_n$ and $x > A_n$ then move $x \to A_n$. If $y = B_n$ and $x < A_n$, complementarity implies that there exists $m < n$ such that either $x = A_m$ so the move $y \to B_m$ restores (1); or else $x = B_m$, so the move $y \to A_m$ does it.

Hence we may assume that both x and y are in A, say $x = A_m \le A_n = y$. If $y > B_m$, then the nim-type (I) move $y \to B_m$ suffices for each game. We may therefore assume that

$$x = A_m \le A_n = y < B_m. \tag{2}$$

Since each of (A_i) and (B_i) is strictly increasing, a nim-type move to a position (1) does not exist, so we have to find a (II) extended diagonal type move for the games β-Wynim and β^{T}-Wynim. Observe that for both these games, this type of move is now unrestricted with $k = \lfloor\beta\rfloor$.

Let $d := y - x$. Then $d = A_n - A_m < B_m - A_m = \Delta_m$. By Proposition 6, Δ_i grows from 0 to Δ_m as i grows from 0 to m, in steps $\Delta_i^2 = \Delta B_i - \Delta A_i \in \{\lfloor\beta\rfloor - 2, \lfloor\beta\rfloor - 1, \lfloor\beta\rfloor\}$, bounded above by $\lfloor\beta\rfloor$. Hence there exists j such that $0 \le d - \Delta_j < \lfloor\beta\rfloor$. Then move $(x, y) \to (A_j, B_j)$. We need to show three things:

(i) $j < m$,

(ii) $y > B_j$,

(iii) $|(y - B_j) - (A_m - A_j)| < \lfloor\beta\rfloor$.

(i) $\Delta_j \le d = y - A_m < B_m - A_m = \Delta_m$. Since Δ_i is an increasing function of i, we have $j < m$.

(ii) $y = A_m + d > A_j + d \ge A_j + \Delta_j = B_j$.

(iii) $|(y - B_j) - (A_m - A_j)| = |(y - A_m) - (B_j - A_j)| = |d - \Delta_j| < \lfloor\beta\rfloor$.

On the other hand, for the game β-nim and a position of the form in (2), by Proposition 6(i) a nearest lower P-position is attainable by an extended "horizontal" nim-type move. Precisely, since $\Delta B_n \in \{\lfloor\beta\rfloor, \lfloor\beta\rfloor + 1\}$ we can lower $y = A_n$ to B_i, where $i \ge 0$ is such that $B_i < A_n < B_{i+1}$, and $x = A_m$ to A_i, that is move $(A_m, A_n) \to (A_i, B_i)$. By the (I) nim-type move we have to show that $A_i < A_m$. But the definition of i together with (2) give $B_i < A_n < B_m$ which, by $i < m$, implies $A_i < A_m$.

Thus the set $\mathcal{P}$ is indeed the set of P-positions for our three games. □

4. The notion of *k*-invariance and game complexity

Let us continue our discussion of variant versus invariant games from the introduction and Remark 5, and relate it to the complexity of numbers and games. We think of an integer as the simplest number, followed by the rationals, algebraic numbers and transcendental numbers, the most complex numbers.

Our three games are, in fact, "minimally variant" in the sense that all their positions can be partitioned into precisely two sets, namely,

$$\{(A_i, A_j) \mid i, j \in \mathbb{Z}_{>0}\} \quad \text{and} \quad \{(B_i, A_j), (A_i, B_j), (B_i, B_j) \mid i, j \in \mathbb{Z}_{\ge 0}\},$$

such that, for each game, for each set, the possible moves are invariant. This observation motivates a weakening of the notion of invariance to *k-invariance*, $k \in \mathbb{Z}_{>0}$.

Definition 8. Let X be a subset of the set of positions (j-tuples of nonnegative integers) of a game G on j heaps. Then m (also a j-tuple of nonnegative integers, but not all 0) is an *invariant move* in X, if for all $x \in X$, $x - m$ is an option, provided $x - m$ is a position in G.

Definition 9. Let G be a game and X a subset of all positions in G. Then m is a *variant move* in X if there exist $x, y \in X$ such that both $x - m$ and $y - m$ are positions in G, and $x - m$ is an option but $y - m$ is not.

Definition 10. Let $k \in \mathbb{Z}_{>0}$. A game G is *k-invariant* if

- its set of positions can be partitioned into k subsets, such that, within each subset X, each allowed move is invariant in X;
- for any partition $\sqcup X_i$ of G's positions into $< k$ subsets, there is an i and an m such that m is a variant move in X_i.

If a game G is not k-invariant for any $k \geq 1$, then it is *∞-invariant*. If $k = 1$, then G is *invariant* (the second item does not apply). If $k \neq 1$, then G is *variant*.

The games in this paper are all 2-invariant. The "mouse game" in [6] is 2-invariant. On the other hand, the $\star$-operator for invariant subtraction games [16; 15] produces an invariant game, the "mouse trap" [13], with the same sets of P-positions as the mouse game. The game Mark [7; 8] is ∞-invariant (of course, any ∞-invariant game is variant). It is played on the nonnegative integers: from position n, either subtract one, or move to $\left\lfloor \frac{1}{2}n \right\rfloor$.

Let S be a set of d-tuples of nonnegative integers. Suppose that $P(G) = S$ where G is a k-invariant game, and for all $l < k$ there is no l-invariant game H such that $P(H) = S$, then we say that the set of P-positions S is *k-game-invariant*. Note that, for any set S there is a *trivial $|S|$-invariant game* (perhaps $|S| = \infty$): no move is possible from a position in S and each position not in S has a move to 0. Hence, we may ask, for any given set S of P-positions (at least there is the trivial game), what its game-invariance number is (a positive integer or infinity).

For example, we wonder if the game-invariance number for the P-positions of Mark is finite.

Let us return to the setting in this paper, and let $\gamma = \beta - \hat{\beta}$. It appears that the complexity of γ, the simplicity of the game rules and the game-invariance number are related. If γ is an integer, the game-invariance number is one, and moreover, by (I) and (II) the game rules are "short"; see also Example 4. For our three games, γ is not necessarily an integer, the game rules are longer and the 1-invariance is replaced by 2-invariance. To shed more light on these suggested relationships, it might be well to investigate whether γ rational, algebraic [5; 17], or transcendental has any effect on the length of the game rules and the game-invariance number.

We close this section with a problem which requires a little background. The succinct input size of a given ordered pair of integers (x, y) is $\log(xy)$. The time complexity of deciding whether a given ordered pair (x, y) is of the form (A_n, B_n) is polynomial in $\log(xy)$; see [10, §3]. In [16, Main Theorem] it is demonstrated that, given the set $\mathcal{P}$ in Theorem 1, there is an *invariant* game for which the time complexity of determining whether a given ordered pair (s, t) is a legal move, but it is exponential in $\log(st)$. The game invariance number is one, but in general short game rules are not known. In [5; 17] polynomial time and invariant game rules are determined for the set $\mathcal{P}$ when γ is restricted to some specific algebraic numbers of degree 2.

We state a problem related to the games in this note, concerning the relation between game-invariance number and short game rules.

Problem. Let $\beta > 2$ be irrational, and let

$$\mathcal{Q} = \bigcup_{n\geq 0}\{(A_n, B_n)\} \cup \bigcup_{n\geq 0}\{(B_n, A_n)\},$$

where $(A_n)_{n\geq 1}$ and $(B_n)_{n\geq 1}$ are complementary Beatty sequences defined by β, and $A_0 = B_0 = 0$. Suppose that $\gamma = \beta - \hat{\beta}$ is transcendental, and consider a game G with $\mathcal{Q}$ as its set of P-positions. Can G have game-invariance number 1, if there is a polynomial time algorithm, in $\log(x, y)$, for finding a winning move from any given N-position (x, y)?

By the results of this paper, we know that game invariance number 2 is attainable. Perhaps this problem exists even if γ is algebraic, or even if γ is a noninteger rational number. Perhaps it holds even if the Beatty sequences A and B are not complementary [9].

The notion of k-invariance is also interesting in a somewhat different context. In [18] certain k-invariant 2-heap subtraction games with a finite number of (variant) moves are studied and it was shown that they embrace computational universality.

Many heap games in the literature have move-size dynamic rules (e.g., Fibonacci nim), blocking maneuvers (e.g., blocking Wythoff nim), depend on positions moved to, rather than position moved from, and so on, and new variations yet to come. The notion of k-invariance in this paper is only intended as a small guide for a larger classification in the future.

5. Discussion

We have formulated three reasonably short game rules for three 2-invariant games, which have identical sets of P-positions. Observe however that the rules contain a partial information about the set of P-positions (but neither $\hat{\beta}$ nor the density 1 is disclosed). Is it possible to find short 2-invariant game rules, without disclosing any part of the P-positions?

Suppose that we fix irrational $\beta > 2$ and then increase the density of the pairs of sequences from 1 to say an arbitrary number $\zeta > 1$ (or decreases to a density < 1) where α is defined via $1/\alpha + 1/\beta = \zeta$, and for all n, $A_n = \lfloor \alpha n \rfloor$. (That is, for all β, $\alpha \neq \hat{\beta}$.) Given the new sets of P-positions, is there still a short/succinct 2-invariant description for game rules without disclosing irrationals or/and the joint density of the sequences? It is unknown to us whether there exist invariant rules for such games (see also [16], [9] for similar problems). What are the game-invariance numbers for such sets of P-positions?

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