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Scoring play combinatorial games

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In this paper we will discuss scoring play games. We will give the basic definitions for scoring play games, and show that they form a well-defined set, with clear and distinct outcome classes under these definitions. We will also show that under the disjunctive sum these games form a monoid that is closed and partially ordered. We also show that they form equivalence classes with a canonical form, and even though it is not unique, it is as good as a unique canonical form.

Finally we will define impartial scoring play games. We will then examine the game of nim and all octal games, and define a function that can help us analyse these games. We will finish by looking at the properties this function has and give many conjectures about the behaviour this function exhibits.

1. Introduction

Combinatorial games where the winner is determined by a “score”, rather than who moves last, have been largely ignored by combinatorial game theorists. As far as this author is aware, there have been four previous studies of scoring play combinatorial games, all of which focused on the universe of “well-tempered” scoring games.

There are the works of Milnor [9], Hanner [6], Ettinger [3; 4] and most recently, Johnson [7]. The definition of a scoring game that all of them used is the following.

Definition 1. A scoring game is defined as

$$G = \begin{cases} \text{a real number} & \text{if } G^L = G^R = \varnothing, \\ \langle G^L | G^R \rangle, & \text{if } G^L \text{and } G^R \neq \varnothing. \end{cases}$$

The authors would say that a game G, where $G^L = G^R = \varnothing$, is *atomic* and all other games are not atomic. Using this terminology, they were able to define concepts such as the disjunctive sum, and the “left outcome” and “right outcome”, which are the score at the end of a game under optimal play, when Left and Right move first respectively. Their mathematical definitions are given here.

Definition 2. The disjunctive sum is defined as follows:

$$G+H = \begin{cases} G+H, & \text{if } G \text{ and } H \text{ are atomic,} \\ \langle G^L+H, G+H^L | G^R+H, G+H^R \rangle, & \text{otherwise.} \end{cases}$$

Definition 3. The left outcome $L(G)$ and right outcome $R(G)$ are defined as follows:

$$L(G) = \begin{cases} G, & \text{if } G \text{ is atomic,} \\ \max_{G^L} R(G^L), & \text{otherwise;} \end{cases}$$
$$R(G) = \begin{cases} G, & \text{if } G \text{ is atomic,} \\ \min_{G^R} L(G^R), & \text{otherwise.} \end{cases}$$

Effectively they showed that under the disjunctive sum, this class of games forms a nontrivial monoid, and that with certain restrictions, it is equivalent to the set of all small normal play combinatorial games.

However, these games all share one thing in common, they are all *dicot* scoring games, or a subset of dicot scoring games. Meaning that if one player has an option, so does the other, and if one player has no options, then neither does his opponent.

For this paper, we will be considering the most general class of scoring games that it is possible to define. The definitions given in this paper, are effectively, equivalent to the definitions given by Milnor, Hanner, Ettinger and Johnson. That is to say, the class of games studied by these four authors is a proper subset of the class of games we will be defining and analysing in this paper.

2. Scoring play games

In this paper, we will be looking at the structure of scoring play games under the disjunctive sum, since it is by far the most commonly used operator in combinatorial game theory. Intuitively, we want all scoring play games to have the following four properties:

(1) The rules of the game clearly define what points are, and how players either gain or lose them.

(2) When the game ends, the player with the most points wins.

(3) For any two games G and H, a points in G are equal to a points in H, where $a \in \mathbb{R}$.

(4) At any stage in a game G, if Left has L points and Right has R points then the score of G is $L-R$, where $L, R \in \mathbb{R}$.

For example, in the game Go you get one point for each of your opponents stones that you capture, and for each piece of area you successfully take. In Mancala you get one point for each bean you place in your Kala. So when comparing these games, we would like one point in Mancala to be worth one point in Go.

Mathematically, scoring games are defined in the following way.

Definition 4. A scoring play game $G = \{G^L \mid G^S \mid G^R\}$, where G^L and G^R are sets of games and $G^S \in \mathbb{R}$, the base case for the recursion is any game G where $G^L = G^R = \emptyset$. $G^L = \{\text{all games that Left can move to from } G\}$, $G^R = \{\text{all games that Right can move to from } G\}$, and for all G there is an $S = (P, Q)$ where P and Q are the number of points that Left and Right have on G respectively. Then $G^S = P - Q$, and for all $g^L \in G^L$, $g^R \in G^R$, there is a $p^L, p^R \in \mathbb{R}$ such that $g^{LS} = G^S + p^L$ and $g^{RS} = G^S + p^R$.

A quick note about the notation. One thing the reader will notice, especially after we introduce the disjunctive sum, is that if both G^L and G^R are nonempty, then the value of G^S does not appear to be relevant.

However, it is useful for several reasons. The first is that it tells us how many points a player gains or loses on their turn, i.e., Left gains $G^{LS} - G^S$ points, and Right gains $G^S - G^{RS}$ points. The second is that if we are playing games under the short rule (i.e., the game ends when a player cannot move on any one component), then the value of G^S can change everything. It is also worth keeping it so that we can use "standard" notation for scoring play games.

The reader may also feel that it is perhaps better to write $\{. \mid G^S \mid G^R\}$ as $\{G^{SL} \mid G^R\}$, and likewise if $G^R = \emptyset$. However, we feel that this is simply a matter of personal choice, and from a mathematical perspective, not really relevant.

A concept we will be using throughout this paper is the game tree of a game. While it may be intuitively obvious to the reader, nonetheless, we feel it is important to define it mathematically.

Definition 5. The game tree of a scoring play game $G = \{G^L \mid G^S \mid G^R\}$ is a tree with a root node, and every node has children either on the Left or the Right, which are the Left and Right options of G respectively. All nodes are numbered, and are the scores of the game G and all of its options.

We also need to define a concept that we call the "final score". This is something which hopefully the reader finds relatively intuitive. When the game ends, which it will after a finite amount of time, the score is going to determine whether a player won, lost or tied.

From a combinatorial game theory perspective we want to know "what is the best that a player can do?". Left is trying to maximise the value of the score, while Right is trying to minimise it. Since this is going to be the backbone of our theory it is important to get it right, and so we use the following definition.

Definition 6. We define the following:

- G_F^{SL} is called the Left final score, and is the maximum score — when Left moves first on G — at a terminal position on the game tree of G, if both Left and Right play perfectly.
- G_F^{SR} is called the Right final score, and is the minimum score — when Right moves first on G — at a terminal position on the game tree of G, if both Left and Right play perfectly.

The reason we define it this way is because the terminal position can vary dramatically, depending on the rules of the game and the operator being used. For instance under the long rule the game ends when a player cannot move on all components, but under the short rule the game ends when a player cannot move on any one component. These two rules will clearly give different results when computing the final score of a game.

Since we want our definition to be as general as possible, i.e., cover every possibility, it makes sense to define the final score in this way. For the purposes of this paper we will be using standard combinatorial game theory convention. That is, a game ends when it is a player's turn and he has no options.

It is also important to note that we will only be considering finite games, i.e., for any game G the game tree of G has finite depth and finite width. This means that G_F^{SL} and G_F^{SR} are always computable, and cannot be infinite or unbounded.

There is also the case where a game may have a form of aggregate scoring. For example players may play two games in sequence, and the winner would be the player who gets the most points over both games. This gives scoring play games an additional dynamic, where in the event of a tie after two games, the winner may be determined by the player who managed to accumulate more points in one of the games.

However, as far as this paper is concerned, we will not be considering games of this type. We will only look at games where the winner is determined after one game ends. Games with aggregate scoring would be an interesting area to look at for further research.

There are two conventions that we will be using throughout this paper. The first is that in all examples given we will take the initial score of the game to be 0, unless stated otherwise. The second is that if for a game G, $G^L = G^R = \emptyset$, we will simply write G as G^S, rather than $\{.\mid G^S \mid .\}$. For example the game $G = \{\{.\mid 0 \mid .\} \mid 1 \mid \{.\mid 2 \mid .\}\}$, will be written as $\{0 \mid 1 \mid 2\}$. The game $\{.\mid n \mid .\}$, will be written as n and so on. This is simply for convenience and ease of reading.

2.1. ***An example.*** Before we continue we will give an example of a scoring play game to demonstrate how to use the notation. So consider the game Toad and Frogs from Winning Ways [1], under scoring play. The rules are as follows:

(1) The game is played on a horizontal grid.

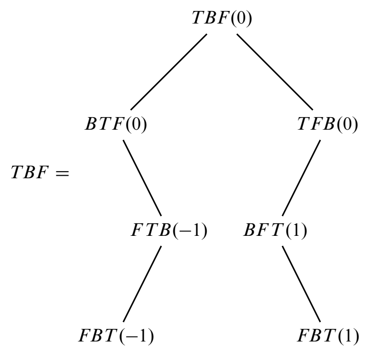


Figure 1. $TBF = \{\{. \mid 0 \mid \{-1 \mid -1 \mid .\}\} \mid 0 \mid \{\{. \mid 1 \mid 1\} \mid 0 \mid .\}\}$.

(2) Left moves Toads and Right moves Frogs.

(3) Toads move from left to right and Frogs move from right to left.

(4) Toads can only jump Frogs and Frogs can only jump Toads.

(5) The player who jumps the most pieces wins.

So consider the game *TBF* as shown in Figure 1, where *B* represents a blank space, *T* represents toads and *F* represents frogs. The numbers in brackets are the current score.

The game in Figure 1 has value $\{\{. \mid 0 \mid \{-1 \mid -1 \mid .\}\} \mid 0 \mid \{\{. \mid 1 \mid 1\} \mid 0 \mid .\}\}$. This game is in "canonical form", that is it neither has a dominated or reversible option. For more details see Section 3.

2.2. *Outcome classes.* In combinatorial game theory we would like to know who wins under optimal play, e.g., if $G \in \mathcal{L}$, then that means Left has a winning strategy moving first or second, if he plays his optimal strategy for both normal and misère play. Under scoring play the outcome classes are a little different, since in scoring play we allow ties, i.e., games where neither player wins.

Before we can define what the outcome classes precisely, we first need a new definition. The definition we are about to give is very important for scoring play combinatorial game theory. It, together with the definition of the final score, forms the core of our theory.

Definition 7.

$$L_> = \{G \mid G_F^{SL} > 0\}, \quad L_< = \{G \mid G_F^{SL} < 0\}, \quad L_= = \{G \mid G_F^{SL} = 0\};$$
$$R_> = \{G \mid G_F^{SR} > 0\}, \quad R_< = \{G \mid G_F^{SR} < 0\}, \quad R_= = \{G \mid G_F^{SR} = 0\};$$
$$L_\geq = L_> \cup L_=, \quad L_\leq = L_< \cup L_=, \quad R_\geq = R_> \cup R_=, \quad R_\leq = R_< \cup R_=.$$

Since we would like to classify every game by an outcome class it is also important that every game belongs to exactly one outcome class. So we define the five outcome classes as follows.

Definition 8. The outcome classes of scoring games are defined as:

- $\mathcal{L} = (L_> \cap R_>) \cup (L_> \cap R_=) \cup (L_= \cap R_>)$.
- $\mathcal{R} = (L_< \cap R_<) \cup (L_< \cap R_=) \cup (L_= \cap R_<)$.
- $\mathcal{N} = L_> \cap R_<$.
- $\mathcal{P} = L_< \cap R_>$.
- $\mathcal{T} = L_= \cap R_=$.

The reason that we chose the outcome classes in this way is because if you have a game $G = \{1 \mid 0 \mid 0\}$, then it is more natural to say that it belongs to the outcome $\mathcal{L}$, since Right cannot win, but Left can if he moves first. In this way we also keep the usual convention of calling a game $G \in \mathcal{N}$ a "next player win" and a game $H \in \mathcal{P}$ a "previous player win".

An interesting distinction is that while $\mathcal{L}$ means the set of games where Left can win moving first or second in both normal and misère play, in scoring play, it means that if Left wins moving first he does not lose, and may win, moving second, and vice-versa. Another distinction is the addition of the outcome class $\mathcal{T}$, which of course does not exist in either normal or misère play, and means that the game ends in a tied score regardless of who moves first.

Theorem 9. *Every game G belongs to exactly one outcome class.*

Proof. This is clear since every game belongs to exactly one of $L_>$, $L_<$, $L_=$ and exactly one of $R_>$, $R_<$, $R_=$. Therefore, every game belongs to exactly one of the nine possible intersections of $L_>$, $L_<$, $L_=$ and $R_>$, $R_<$, $R_=$. Since each outcome class is simply the union of one or more of these, then each game can only be in exactly one outcome class. □

2.3. ***The disjunctive sum.*** As we mentioned earlier, the disjunctive sum is by far the most commonly used operator in combinatorial game theory. This is because many well-known games, such as Go, naturally break up into the disjunctive sum of two or more components. For scoring play the disjunctive sum needs to be defined a little differently; this is because in scoring play games when we combine them together we have to sum the games and the scores separately.

For this reason we will be using two symbols $+_\ell$ and $+$. The ℓ in the subscript stands for "long rule". This comes from [2], and means that the game ends when a player cannot move on any component on his turn. The "short rule" means that the game ends when a player cannot move on at least one component on his turn.

In this paper we will only be considering the disjunctive sum played with the long rule.

Definition 10. The disjunctive sum is defined as follows:

$$G +_\ell H = \{G^L +_\ell H, G +_\ell H^L \mid G^S + H^S \mid G^R +_\ell H, G +_\ell H^R\},$$

where $G^S + H^S$ is the normal addition of two real numbers.

As with the disjunctive sum of normal and misère play games we abuse notation by making the comma mean set union, and $G^L +_\ell H$ means take the disjunctive sum of all $g^L \in G^L$ with H.

We would also like to know when one game is "better" than another one. That is, given several options to play, which one is the best? In normal play and misère play the definitions of "$\geq$" and "$\leq$" are relatively easy to define, since players either win or lose; however, for scoring play we have to take into account tied scores. So for this reason we will redefine "$\geq$" and "$\leq$".

Definition 11. We define the following:

- $-G = \{-G^R \mid -G^S \mid -G^L\}$.
- For any two games G and H, $G = H$ if $G +_\ell X$ has the same outcome as $H +_\ell X$ for all games X.
- For any two games G and H, $G \geq H$ if $H +_\ell X \in O$ implies $G +_\ell X \in O$, where $O = L_\geq$, $R_\geq$, $L_>$ or $R_>$, for all games X.
- For any two games G and H, $G \leq H$ if $H +_\ell X \in O$ implies $G +_\ell X \in O$, where $O = L_\leq$, $R_\leq$, $L_<$ or $R_<$, for all games X.
- $G \cong H$ means G and H have identical game trees.
- $G \approx H$ means G and H have the same outcome.

Theorem 12. *$G \geq H$ if and only if $H \leq G$.*

Proof. First let $G \geq H$, and let $G +_\ell X \in O$ for some game X, where O is one of $L_\leq$, $R_\leq$, $L_<$ or $R_<$. This means that $H +_\ell X \notin O'$, where O' is one of $L_\geq$, $R_\geq$, $L_>$ or $R_>$, since if it was this would mean that $G +_\ell X \in O'$, since $G \geq H$; therefore $H +_\ell X \in O$, and hence $H \leq G$.

A completely identical argument can be used for $H \leq G$, and hence $G \geq H$ if and only if $H \leq G$ and the theorem is proven. □

Theorem 13. *Scoring play games are partially ordered under the disjunctive sum.*

Proof. To show that we have a partially ordered set we need 3 things:

(1) *Transitivity*: If $G \geq H$ and $H \geq J$ then $G \geq J$.

(2) *Reflexivity*: For all games G, $G \geq G$.

(3) *Antisymmetry*: If $G \geq H$ and $H \geq G$ then $G = H$.

(1) Let $G \geq H$ and $H \geq J$. $G \geq H$ means that if $H +_\ell X \in O$ this implies $G +_\ell X \in O$, where $O = L_\geq$, $R_\geq$, $L_>$ or $R_>$, for all games X. $H \geq J$ means that if $J +_\ell X \in O$ this implies that $H +_\ell X \in O$. Since $G \geq H$, then this implies that $G +_\ell X \in O$, therefore $J +_\ell X \in O$ implies that $G +_\ell X \in O$ for all games X, and $G \geq J$.

(2) Clearly $G \geq G$, since if $G +_\ell X \in O$ then $G +_\ell X \in O$, where $O = L_\geq$, $R_\geq$, $L_>$ or $R_>$, for all games X.

(3) First let $G \geq H$ and $H \geq G$. $G = H$ means that $G +_\ell X \approx H +_\ell X$ for all X. So first let $G +_\ell X \in L_=$, then this implies that $H +_\ell X \in L_\geq$, since $H \geq G$. However, $H +_\ell X \in L_=$, since if $H +_\ell X \in L_>$, then this implies that $G +_\ell X \in L_>$, since $G \geq H$, therefore $G +_\ell X \in L_=$ if and only if $H +_\ell X \in L_=$.

An identical argument can be used for all remaining cases, therefore $G +_\ell X \approx H +_\ell X$ for all games X, i.e., $G = H$. □

Theorem 14. *For any three outcome classes $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$, there is a game $G \in \mathcal{X}$ and $H \in \mathcal{Y}$ such that $G +_\ell H \in \mathcal{Z}$.*

Proof. Consider the games $G = \{\{\{d \mid c \mid e\} \mid b \mid .\} \mid a \mid .\}$ and $H = \{. \mid f \mid \{. \mid g \mid h\}\}$. The final scores of G are $G_F^{SL} = a$ and $G_F^{SR} = b$, and the final scores of H are $H_F^{SL} = f$ and $H_F^{SR} = g$. Now consider the game $G +_\ell H$ shown in Figure 2.

The final scores of $G +_\ell H$ are $(G +_\ell H)_F^{SL} = e+g$ or $d+h$ and $(G +_\ell H)_F^{SR} = e+h$. Since e, d and h can take any value we can select them so that: $e+g$, $d+h$ and $e+h > 0$ and $G +_\ell H \in \mathcal{L}$; $e+g$, $d+h$ and $e+h < 0$ and $G +_\ell H \in \mathcal{R}$; $e+g$, $d+h > 0$ and $e+h < 0$ and $G +_\ell H \in \mathcal{N}$; $e+g$, $d+h < 0$ and $e+h > 0$ and $G +_\ell H \in \mathcal{P}$ or finally $e+g = d+h = e+h = 0$ and $G +_\ell H \in \mathcal{T}$.

Since the outcomes of G and H depend on the values of a, b, f and g, we can select them so that G and H can be in any outcome class, and thus the theorem is proven. □

From the theory of normal play games, we have the following theorem.

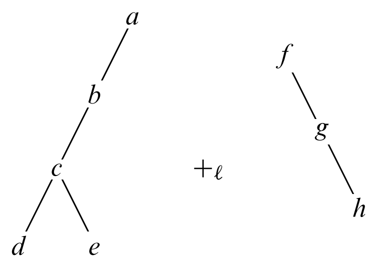


Figure 2. The game $G +_\ell H$, $G = \{\{\{d \mid c \mid e\} \mid b \mid .\} \mid a \mid .\}$ and $H = \{. \mid f \mid \{. \mid g \mid h\}\}$.

Theorem 15 (the greediness principle). *Let $G = \{G^L \mid G^R\}$ and $H = \{H^L \mid H^R\}$ be two combinatorial games. If $H^L \subseteq G^L$ and $G^R \subseteq H^R$, then $G \geq H$.*

A direct consequence of Theorem 14 is that this principle will not hold for scoring play games.

Under normal play combinatorial games form an abelian group under the disjunctive sum. The identity that is used is the set $\mathcal{P}$, that is if $I \in \mathcal{P}$ then $G +_\ell I \approx G$ for all games G. In this case the entire set $\mathcal{P}$ has a single unique representative, the game $\{. \mid .\}$. This of course also means that $G = H$ if and only if $G +_\ell (-H) \in \mathcal{P}$.

Under misère play, the identity set contains only one element, which is the same game $\{. \mid .\}$. That is, if $G \not\cong \{. \mid .\}$, then $G \neq \{. \mid .\}$. This was proven by Paul Ottaway (personal communication, 2007). This of course means that there is no easy or equivalent method for determining if two games are equivalent under misère play.

For scoring play games, we have an equivalent theorem. That is our identity set contains only one element, namely the game $\{. \mid 0 \mid .\}$, which we will call 0. It should be clear that $0 +_\ell G \approx G$ for all games G, and so 0 is the identity.

Theorem 16. *For any game G, if $G \not\cong 0$ then $G \neq 0$.*

Proof. The proof of this is very simple, first let $G^L \neq \varnothing$, since the case $G^R \neq \varnothing$ will follow by symmetry. Next let $P = \{. \mid a \mid b\}$, and note that $P_F^{SL} = a$, since Left has no move on P. So let $a > 0$; if $G = 0$ then this means that $(G +_\ell P)_F^{SL} \approx P$. However, since G is a combinatorial game we know from the definition that G has both finite depth, and finite width. So we can choose $b < 0$ such that $|b|$ is greater than any score on the game tree of G.

Therefore when Left moves first on $G +_\ell P$ he must move to the game $G^L +_\ell P$. Right will respond by moving to $G^L +_\ell b$, since $(G +_\ell P)_F^{SL} < 0$ by choice of b. This implies that $G +_\ell P \not\approx P$, and $G \neq 0$. □

What is interesting is that unlike misère games, some scoring games do have an inverse, namely the set of games $\{. \mid n \mid .\}$, where n is a real number. It should be clear that these are the only games which are invertible under scoring play, and any other nontrivial game cannot be inverted.

Another important consequence of this theorem is that under normal play if we wish to know if $G > H$ for any two games G and H, we simply play $G + (-H)$, where "+" here means the disjunctive sum. However, because no nontrivial scoring games are invertible, we can no longer use this technique to compare them.

3. Canonical forms

Canonical forms are important, because if we can show that these games can be split up into equivalence classes with a unique representative for each class, then it makes these games much easier to analyse and compare. We do not have to consider each game individually, but only the equivalence class to which it belongs.

Theorem 17. *There exist two games G and H such that $G \not\cong H$ and $G = H$.*

Proof. Consider the games G and H, where $a, b, c, d, e, f \in \mathbb{R}$, shown in Figure 3.

This example is a variant of a similar example used to prove the same theorem for misère games in [8].

For any two games G and H, $G = H$ if $G +_\ell X \approx H +_\ell X$ for all games X. The easiest way to prove this is to show that $G \geq H$ and $H \geq G$. Right can do at least as well playing $H +_\ell X$ as he can playing $G +_\ell X$, by simply copying his strategy from $G +_\ell X$ and not playing the left-hand string on H. Right cannot do better on $H +_\ell X$ than he can on $G +_\ell X$, since the string on the left hand side of H can be copied on $G +_\ell X$ by simply not moving to e. So therefore if $H +_\ell X \in O$ then this implies that $G +_\ell X \in O$ where $O = L_\geq$, $R_\geq$, $L_>$ or $R_>$, i.e., $G \geq H$.

Left can also do at least as well playing $H +_\ell X$ as he can playing $G +_\ell X$, since if Right can achieve a lower final score playing the left-hand string on $H +_\ell X$, then he can also do so by choosing not to move to e on $G +_\ell X$. Similarly if Right copies his strategy from $G +_\ell X$ onto $H +_\ell X$ then their final scores will be the same. So if $G +_\ell X \in O$ then this implies that $H +_\ell X \in O$ where $O = L_\geq$, $R_\geq$, $L_>$ or $R_>$, i.e., $H \geq G$. So therefore, $G = H$ and the proof is finished. □

For both normal and misére play games, the standard way to reduce a game to its canonical form is to use two concepts. These are called domination and reversibility, and are defined as follows.

Definition 18. Let $G = \{A, B, C, \ldots \mid G^S \mid D, E, F, \ldots\}$, if $A \geq B$ or $D \leq E$ we say that A dominates B and D dominates E.

Definition 19. Let $G = \{A, B, C, \ldots \mid G^S \mid D, E, F, \ldots\}$, an option A is reversible if $A^R \leq G$. An option D is also reversible if $D^L \geq G$.

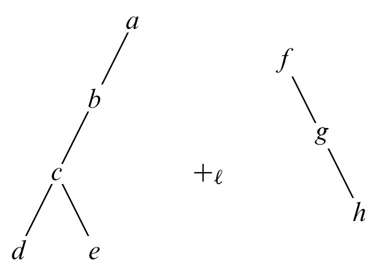


Figure 3. Two games G and H, where $G \not\cong H$, but $G = H$.

Theorem 20. *Let* $G = \{A, B, C, \dots \mid G^S \mid D, E, F, \dots\}$, *and let* $A \geq B$, *then* $G' = \{A, C, \dots \mid G^S \mid D, E, F, \dots\} = G$. *By symmetry if* $D \leq E$ *and* $G'' = \{A, B, C, \dots \mid G^S \mid D, F, \dots\}$ *then* $G'' = G$.

Proof. Let $G = \{A, B, C, \dots \mid G^S \mid D, E, F, \dots\}$ such that $A \geq B$; further let $G' = \{A, C, \dots \mid G^S \mid D, E, F, \dots\}$. First suppose that $G +_\ell X \in O$, where $O = L_{\geq}, R_{\geq}, L_{>}$ or $R_{>}$ if Left moves to $B +_\ell X$. This implies that $G' +_\ell X \in O$, since $A \geq B$. Hence if $G +_\ell X \in O$ this implies that $G' +_\ell X \in O$, and since the Right options of G and G', this implies that $G' \geq G$.

Next suppose that $G' +_\ell X \in O'$ where $O' = L_{\leq}, R_{\leq}, L_{<}$ or $R_{<}$. This implies that $G +_\ell X \in O'$, since the only option in G^L that is not in G'^L is B and $B \leq A$, therefore $G' \leq G$, and $G = G'$. So this means that the option B may be disregarded and the proof is finished. □

Theorem 21. *Let* $G = \{A, B, C, \dots \mid G^S \mid D, E, F, \dots\}$, *and let A be reversible with Left options of* $A^R = \{W, X, Y, \dots\}$. *If* $G' = \{W, X, Y, \dots, B, C, \dots \mid G^S \mid D, E, F, \dots\}$, *then* $G = G'$. *By symmetry if D is reversible with Right options of* $D^L = \{T, S, R, \dots\}$. *If* $G'' = \{A, B, C, \dots \mid G^S \mid T, S, R, \dots, D, E, F, \dots\}$, *then* $G = G''$.

Proof. Let $G = \{A, B, C, \dots \mid G^S \mid D, E, F, \dots\}$, where the Left options of $A^R = \{W, X, Y, \dots\}$ and let $G' = \{W, X, Y, \dots, B, C, \dots \mid G^S \mid D, E, F, \dots\}$, further let $A^R \leq G$. If $G +_\ell X \in O$, where $O = L_{\geq}, R_{\geq}, L_{>}$ or $R_{>}$, when Left does not move to A on G, then clearly $G' +_\ell X$ is also in O, since all other options for Left on G are available for Left on G'.

So consider the case where $G +_\ell X \in O$ if Left moves to $A +_\ell X$, then this implies that $A^R +_\ell X$ must also be in O. This means that $G' +_\ell X \in O$ because $A^{RL} \subset G'^L$, and since all other options on G' are the same as G, then $A^R +_\ell X \in O$ implies that $G' +_\ell X \in O$. Hence if $G +_\ell X \in O$ then this implies that $G' +_\ell X \in O$ for all games X, i.e., $G' \geq G$.

Next assume that $G +_\ell X \in O'$, where $O' = L_{\leq}, R_{\leq}, L_{<}$ or $R_{<}$, for all games X. However, $A^R \leq G$, i.e. $G +_\ell X \in O'$ implies that $A^R +_\ell X \in O'$, and since $A^{RL} \subset G'^L$, and all other options on G' are identical to options on G, this means that $G +_\ell X \in O'$, implies that $G' +_\ell X \in O'$, for all games X, i.e. $G' \leq G$. Therefore $G = G'$ and the theorem is proven. □

Definition 22. A vertex v on the game tree of a game G is called a termination vertex if there is a game X, such that $G +_\ell X$ ends if the players reach vertex v.

The reason why we called it a termination vertex is because it is a place where a game could potentially end. If both Left and Right have an option at a particular vertex, then under the disjunctive sum a game cannot end at that point.

Definition 23. We say that G is equivalent to H, or $G \equiv H$, if the underlying game trees of G and H are identical, and all termination vertices have the same score.

Theorem 24. *If $G \equiv H$, then $G = H$.*

Proof. To prove this, first let $G \equiv H$ and let $G +_\ell X \in O$, where $O = L_>$, $L_\geq$, $R_>$ or $R_\geq$. Since the underlying game trees of G and H are identical, and all termination vertices have the same score, then Left can do at least as well on $H +_\ell X$ simply by copying his strategy from $G +_\ell X$. If he does so, then he will arrive at the same termination vertex on $H +_\ell X$ as he did on $G +_\ell X$, and therefore the games will end with identical scores.

Therefore, if $G +_\ell X \in O$ then this implies that $H +_\ell X \in O$, i.e., $H \geq G$. By a totally symmetrical argument we also have that $G \geq H$. So, $G = H$ and the theorem is proven. □

Equivalence is a little stronger than equality, and a little weaker than saying two games are identical. The reason we need it is because it is possible for two games, say $G = \{1 \mid 1 \mid 1\}$ and $H = \{1 \mid 0 \mid 1\}$, to be equal to each other, not identical and neither has a dominated or reversible option.

However, we still want to use domination and reversibility to achieve a "canonical form", so we will say that nontermination vertices are not important in the sense of determining the winner. So while it is not a "true" canonical form in the sense that it is not necessarily unique, it is still useful for studying games.

Definition 25. A game G is in canonical form if it has no dominated or reversible options.

Theorem 26. *For any two games G and H, if $G = H$, and both G and H are in canonical form, then $G \equiv H$.*

Proof. Let G and H be two games such that $G = H$ and neither G nor H has a dominated or reversible option.

So first let $H +_\ell X \in O$, where $O = L_<$, $R_<$, $L_\leq$ or $R_\leq$. Since $G = H$, this implies that $G +_\ell X \in O$. However, if Left moves to $G^L +_\ell X$ then $G^{LR} +_\ell X$ cannot be in O. If it were, this would mean that $H +_\ell X \in O$ implies $G^{LR} +_\ell X \in O$, i.e., $G^{LR} \leq H$, and G would have a reversible option, which means that $G^L +_\ell X^R \in O$.

This implies that $H^L +_\ell X^R \notin O'$, where $O' = L_>$, $R_>$, $L_\geq$ or $R_\geq$. Since if it were, then H would have a dominated option. Therefore, $G^L +_\ell X^R \in O$ if and only if $H^L +_\ell X^R \in O$, i.e., for all $g^L \in G^L$ there is an $h^L \in H^L$ such that $g^L \leq h^L$, and for all $h^L \in H^L$ there is a $g^{L'} \in G^L$ such that $h^L \leq g^{L'}$.

So that means $g^L \leq h^L \leq g^{L'}$. However, g^L and $g^{L'}$ must be identical, otherwise g^L is a dominated option. So every Left option of G is equivalent to a Left option of H, i.e., $G^L \subseteq H^L$, and by a symmetrical argument $H^L \subseteq G^L$. Therefore, $H^L \equiv G^L$, and similarly $H^R \equiv G^R$.

Since all options of G and H are equivalent, we can conclude that the only differences between the game trees of G and H are on nonterminating vertices. Therefore, $H \equiv G$ and the proof is finished. □

It is also important to note that a game may have more than one canonical form. For example, consider the game $G = \{\{3 \mid 0 \mid 4\}, \{3 \mid 1 \mid 4\} \mid 0 \mid .\}$. This game has two canonical forms, namely $\{\{3 \mid 0 \mid 4\} \mid 0 \mid .\}$ and $\{\{3 \mid 1 \mid 4\} \mid 0 \mid .\}$. However, both of these games are equivalent, so either can be used as the canonical form and it will not affect the analysis of this game.

Theorem 27. *Let $G \to G_1 \to G_2 \to \cdots \to G_n$ represent a series of reductions on a game G to a game G_n, which is in canonical form. Further let $G \to G'_1 \to G'_2 \to \cdots \to G'_m$ represent a different series of reductions on G to a game G'_m which is also in canonical form, then $G_n \equiv G'_m$*

Proof. Since each reduction preserves equality, then $G_n = G'_m$ and they are both in canonical form. By Theorem 26 $G_n \equiv G'_m$, and so the theorem is proven. □

Finally, it is important to note that it is certainly possible to define a unique canonical form, for example we could simply set all nonterminating vertices on a game tree to 0. However, we feel that it is more important to keep the original values as this gives a lot of information about the games.

Consider the games $G = \{3 \mid 0 \mid 4\}$ and $H = \{3 \mid 10 \mid 4\}$. In the game G Left moves and gains 3 points, while Right moves and loses 4, but in the game H Left moves and loses 7 points, while Right moves and gains 6. If we set H^S to zero then this information would be lost. So for this reason, we feel the only ways you should reduce a game is using domination and reversibility.

4. Impartial scoring games

The definition of an impartial scoring play game is less intuitive than for normal and misère play games. The reason for this is because we have to take into account the score; for example, consider the game $G = \{4 \mid 3 \mid 2\}$. On the surface the game does not appear to fall into the category of an impartial game, since Left wins moving first or second, however this game is impartial since both players move and gain a single point, i.e., they both have the same options.

So we will use the following definition for an impartial game.

Definition 28. A scoring game G is impartial if it satisfies the following:

(1) $G^L = \varnothing$ if and only if $G^R = \varnothing$.

(2) If $G^L \neq \varnothing$ then for all $g^L \in G^L$ there is a $g^R \in G^R$ such that $g^L +_\ell -G^S = -(g^R +_\ell -G^S)$.

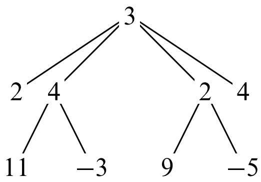
$G =$ 

Figure 4. The impartial game $G = \{2, \{11 \,|\, 4 \,|\, -3\} \,|\, 3 \,|\, , 4, \{9 \,|\, 2 \,|\, -5\}\}$.

An example of an impartial game is shown in Figure 4. This game satisfies the definition since

$$2 +_\ell -3 = -(4 +_\ell -3),$$
$$\{11 \,|\, 4 \,|\, -3\} +_\ell -3 = -(\{9 \,|\, 2 \,|\, -5\} +_\ell -3) = \{8 \,|\, 1 \,|\, -6\} \text{ and}$$
$$11 +_\ell -4 = -(-3 +_\ell -4) = 9 +_\ell -2 = -(-5 +_\ell -2) = 7.$$

The reader may be confused about why we choose the name "impartial". The reason for this is because under normal play a game G is impartial if both Left and Right have the same options *at all stages* in G. The phrase "at all stages" is crucial here. If a scoring play game G is impartial, then the options of G must also be impartial. If the reader checks, he will find that our definition is exactly analogous to the definition used for normal play.

As stated in the introduction, all of the work into scoring play games in the past focused exclusively on dicot games. Since impartial games are merely a subset of dicot games, we can deduce much of the structure of these games from Ettinger's work.

In particular [5, Theorem 20, p. 20; Corollary 1, p. 22; Theorem 14, p. 48] give us the structure of impartial games. That is to say, they form a nontrivial monoid. However, we have the following conjecture.

Conjecture 29. Not all impartial scoring play games have an inverse.

To prove this one needs to show that given an impartial game G, for all impartial games Y there is an impartial game P such that $G +_\ell Y +_\ell P \not\approx P$. This is very difficult to show, however it is extremely likely that this conjecture is true because for normal play games the inverse of any game G is $-G$, and as we will now show there are impartial games H where $-H$ is not the inverse.

So consider the game $G = \{2, \{1 \,|\, 2 \,|\, 3\} \,|\, 0 \,|\, -2, \{-3 \,|\, -2 \,|\, -1\}\}$, in this case $-G = G$. If G is the inverse of itself then $G +_\ell G +_\ell 0 \approx 0$; in other words, $G +_\ell G \in \mathcal{T}$. However, $G +_\ell G \in \mathcal{P}$, this is easy to see since if Left moves first and moves to $2 +_\ell G$, then Right can respond by moving to $2 +_\ell \{-3 \,|\, -2 \,|\, -1\}$ and Left must move to $2 +_\ell -3$ and loses. If Left moves to $\{1 \,|\, 2 \,|\, 3\} +_\ell G$, then Right will move to $\{1 \,|\, 2 \,|\, 3\} +_\ell -2$ and Left must move to $1 +_\ell -2$ and again loses. Obviously the opposite will be true if Right moves first on $G +_\ell -G$. So $G +_\ell -G +_\ell 0 \not\approx 0$ and $G +_\ell -G \notin I$.

So, because $-G$ is not the inverse of G in this case then it is very unlikely that any other impartial game could be G's inverse, and while we do not have a proof of that, this simple example shows that it is probably true.

It is also worth noting that impartial scoring games can belong to any of the five outcomes for scoring games, i.e., $\mathcal{L}$, $\mathcal{R}$, $\mathcal{P}$, $\mathcal{N}$ and $\mathcal{T}$. This is in stark contrast to both normal play and misére play games, where impartial games can only belong to either $\mathcal{P}$ or $\mathcal{N}$.

It is easy to see that this is true by considering an impartial game of the form $\{a \mid G^S \mid b\}$. Clearly when $G^S = 0$ then $b = -a$ and the outcome can only be $\mathcal{N}$, $\mathcal{P}$ or $\mathcal{T}$. However, we can set $G^S \neq 0$ and either large enough that both a and b are greater than zero, or less than zero, depending on if we make G^S a very large positive or negative number. In these cases the outcome will either be $\mathcal{L}$ or $\mathcal{R}$.

5. Nim

Nim is a classic combinatorial game. It has been studied under both normal and misère play extensively, and for that reason we wish to study it, or at least variations of it, under scoring play. We will define scoring play nim by the following rules:

(1) The initial score is 0.
(2) The game is played on heaps of beans, and on a players turn he may remove as many beans as he wishes from any one heap.
(3) A player gets 1 point for each bean he removes.
(4) The player with the most points wins.

It should be clear that the best strategy for this game is simply to remove all the beans from the largest possible heap, and keep doing so until the game ends.

Another thing to note is that, under normal play, for every single impartial game G there is a nim heap of size n such that $G = n$. This not the case with scoring play games, but as we will show in the next section, these games are still relatively easy to solve, regardless of the rules and of the scoring method.

5.1. ***Scoring Sprague-Grundy theory.*** Sprague–Grundy theory is a method that is used to solve any variation of a game of nim. The function for normal play $\mathcal{G}(n)$ is defined in a such a way that if for a given heap n, played under some rules, if $\mathcal{G}(n) = m$ then this means that the original heap n is equivalent to a nim heap of size m.

For scoring play games this function is going to be defined slightly differently. Rather than telling us equivalence classes of different games, it will tell us the final scores of games. While this may not be as powerful as normal play Sprague–Grundy theory, it is still a very useful function and can be used to solve many different variations of scoring play nim.

One of the standard variations that have been used widely in books such as Winning Ways [1] are a group of games called octal games. These games cover a very large portion of nim variations, including all subtraction games. For scoring games we will use the following definition.

Definition 30. A scoring play octal game $O = (n_1 n_2 \dots n_k,\ p_1 p_2 \dots p_k)$, is a set of rules for playing nim where if a player removes i beans from a heap of size n he gets p_i points, $p_i \in \mathbb{R}$, and he must leave $a, b, c \dots$ or j heaps, where $n_i = 2^a + 2^b + 2^c + \cdots + 2^j$.

By convention we will say that a nim heap $n \in O$ means that n is played under the rule set O. We will now define the function that will be the basis of our theory.

Definition 31. Let $n \in O = (t_1 \dots t_f,\ p_1 \dots p_f)$ and $m \in P = (s_1 \dots s_e,\ q_1 \dots q_e)$:

- $\mathcal{G}_s(0) = 0$.
- $\mathcal{G}_s(n) = \max_{k,i}\{p_k - \mathcal{G}_s(n_1 +_\ell n_2 +_\ell \cdots +_\ell n_i)\}$, where $n_1 + n_2 + \cdots + n_i = n - k$, $t_k = 2^a + 2^b + \cdots 2^p$ and $i \in \{a, b, \dots, p\}$.
- $\mathcal{G}_s(n +_\ell m) = \max_{k,i,l,j}\{p_k - \mathcal{G}_s(n_1 +_\ell n_2 +_\ell \cdots +_\ell n_i +_\ell m),\ q_l - \mathcal{G}_s(n +_\ell m_1 +_\ell m_2 +_\ell \cdots +_\ell m_j)\}$, where $n_1 + n_2 + \cdots + n_i = n - k$, $t_k = 2^a + 2^b + \cdots 2^p$ and $i \in \{a, b, \dots, p\}$, $m_1 + m_2 + \cdots + m_j = m - l$, $s_l = 2^c + 2^d + \cdots + 2^q$ and $j \in \{c, d, \dots, q\}$.

The first thing to prove is that this function gives us the information we want, namely the final score of a game. So we have the following theorem.

Theorem 32.

$$\mathcal{G}_s(n) = n_F^{SL} = -n_F^{SR} \quad \textit{and} \quad \mathcal{G}_s(n +_\ell m) = (n +_\ell m)_F^{SL} = -(n +_\ell m)_F^{SR}.$$

Proof. The proof of this will be by induction on all heaps $n_1, n_2, \dots, n_i$, $m_1, \dots, m_j$, such that $n_1 + n_2 \cdots + n_i, m_1 + \cdots + m_j \le K$ for some integer K, the base case is trivial since $\mathcal{G}_s(0 +_\ell 0 +_\ell 0 +_\ell \cdots +_\ell 0) = 0$ regardless of how many zeroes there are.

So assume that the theorem holds for all $n_1, n_2, \dots, n_i, m_1 \dots, m_j$, such that $n_1 + n_2 + \cdots + n_i, m_1 + \cdots + m_j \le K$ for some integer K, and consider $\mathcal{G}_s(n +_\ell m)$, where $n + m = K + 1$.

$\mathcal{G}_s(n +_\ell m) = \max_{k,i,l,j}\{p_k - \mathcal{G}_s(n_1 +_\ell n_2 +_\ell \cdots +_\ell n_i +_\ell m), q_l - \mathcal{G}_s(n +_\ell m_1 +_\ell m_2 +_\ell \cdots +_\ell m_j)\}$, but since $n_1 + n_2 + \cdots + n_i + m$ and $n + m_1 + m_2 + \cdots +$

$m_j \leq K$, then by induction

$$\max_{k,i,l,j}\{p_k - \mathcal{G}_s(n_1 +_\ell n_2 +_\ell \cdots +_\ell n_i +_\ell m), q_l - \mathcal{G}_s(n +_\ell m_1 +_\ell m_2 +_\ell \cdots +_\ell m_j)\}$$
$$= \max_{k,i,l,j}\{p_k - (n_1 +_\ell n_2 +_\ell \cdots +_\ell n_i +_\ell m)_F^{SL}, q_l - (n +_\ell m_1 +_\ell \cdots +_\ell m_j)_F^{SL}\}$$
$$= (n +_\ell m)_F^{SL},$$

and the theorem is proven. □

5.1.1. *Subtraction games.* Subtraction games are a very widely studied subset of octal games. A subtraction game is a game of nim where there is a predefined set of integers and a player may only remove those numbers of beans from a heap. This set is called a subtraction set. From our definition of an octal game this means that each n_i is either 0 or 3. In this section we will also say that if a player removes i beans then he gets i points.

Lemma 33. *Let S be a finite subtraction set, then for all $s \in S$, $\mathcal{G}_s(s + 2ik) = k - \mathcal{G}_s(s + (2i - 1)k)$ for all $i \in \mathbb{N}$, where $k = \max\{S\}$.*

Proof. We will split the proof of this into three parts:

Part 1: For all $i \in \mathbb{Z}^+$, $\mathcal{G}_s(r + 2ik) \leq r$.

The first thing to show is that for each $0 \leq r \leq k$, $\mathcal{G}_s(r) \leq r$ and $\mathcal{G}_s(r+2ik) \leq r$ for all $i \in \mathbb{Z}^+$. First let $r \leq k$, $\mathcal{G}_s(r) = \max_j\{j - \mathcal{G}_s(r - j)\}$ and since each j in the set is less than or equal to r, and each $\mathcal{G}_s(r - j) \geq 0$, this implies that $\mathcal{G}_s(r) \leq r$.

Next let $\mathcal{G}_s(r + 2ik) \leq r$ for smaller i, and consider $\mathcal{G}_s(r + 2ik) = \max_j\{j - \mathcal{G}_s(r + 2ik - j)\}$. If $j \leq r$, then since $\mathcal{G}_s(r + 2ik - j) \geq 0$, we have $j - \mathcal{G}_s(r + 2ik - j) \leq j \leq r$. If $j > r$, then $\mathcal{G}_s(r + 2ik - j) = \mathcal{G}_s(r + k - j + (2i - 1)k) \geq k - (r + k - j) = j - r$, by induction, therefore $j - \mathcal{G}_s(r + 2ik - j) \leq j - (j - r) = r$. So therefore $\mathcal{G}_s(r + 2ik) \leq r$ for all i.

Part 2: For all $i \in \mathbb{Z}^+$, $\mathcal{G}_s(r + (2i + 1)k) \geq k - r$.

We also need to show that for each $0 \leq r \leq k$, $\mathcal{G}_s(r + (2i+1)k) \geq k - r$ for all $i \in \mathbb{N}$. Clearly $\mathcal{G}_s(r + k) \geq k - \mathcal{G}_s(r) \geq k - r$. Again let $\mathcal{G}_s(r + (2i+1)k) \geq k - r$ for smaller i, then $\mathcal{G}_s(r + (2i + 1)k) \geq k - \mathcal{G}_s(r + 2ik)$ and from above we know that $\mathcal{G}_s(r + 2ik) \leq r$ and hence $\mathcal{G}_s(r + (2i + 1)k) \geq k - \mathcal{G}_s(r + 2ik) \geq k - r$ for all i.

Part 3: For all $s \in S$ and $i \in \mathbb{Z}^+$, $\mathcal{G}_s(s + 2ik) \geq s$ and $\mathcal{G}_s(s + (2i + 1)k) \leq k - s$.

Let $s \in S$, then $\mathcal{G}_s(s) \geq s - \mathcal{G}_s(0) = s$, since we know from Part 1 that $\mathcal{G}_s(s) \leq s$; this means that $\mathcal{G}_s(s) = s$. So consider $\mathcal{G}_s(s + k) = \max_j\{j - \mathcal{G}_s(s + k - j)\}$, if $j \leq s$ then $j - \mathcal{G}_s(s + k - j) \leq j - k + \mathcal{G}(s - j) \leq j - k + s - j \leq s - k \leq k - s$. If $j > s$ then $j - \mathcal{G}_s(s + k - j) \leq j - s + \mathcal{G}_s(k - j) \leq j - s + k - j = k - s$. From Part 2 we know that $\mathcal{G}_s(s + k) \geq k - \mathcal{G}_s(s) = k - s$, so $\mathcal{G}_s(s + k) = k - s$.

So assume that the theorem holds up to $i \geq 1$, and consider $\mathcal{G}_s(s + (2i + 1)k) = \max_j\{j - \mathcal{G}_s(s + (2i + 1)k - j)\}$. If $j \leq s$ then $j - \mathcal{G}_s(s + (2i + 1)k - j) \leq$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\mathcal{G}_s(n)$	0	0	0	0	4	5	5	5	5	1	0	0	0	3	4	5

Table 1. A game with subtraction set {4, 5}.

$j-k+\mathcal{G}_s(s+2ik-j)$, and from Part 2 we know that $\mathcal{G}_s(s+2ik-j)\le s-j$; therefore $j-k+\mathcal{G}_s(s+2ik-j)\le j-k+s-j\le s-k\le k-s$.

If $j>s$ then $j-\mathcal{G}_s(s+(2i+1)k-j)=j-\mathcal{G}_s(s+k+2ik-j)\le j-s+\mathcal{G}_s(k-j+2ik)\le j-s+k-j$, by induction, which is equal to $k-s$.

Finally consider $\mathcal{G}_s(s+(2i+2)k)\ge k-\mathcal{G}_s(s+(2i+1)k)$, and from before we know that $\mathcal{G}_s(s+(2i+1)k)\le k-s$; therefore $k-\mathcal{G}_s(s+(2i+1)k)\ge k-(k-s)=s$. So therefore $\mathcal{G}_s(s+(2i+2)k)=s$ and the lemma is proven. □

The obvious question to ask is, does the lemma hold for all n? The answer is no. While it is clear that our function is eventually periodic for subtraction games at least, there are many examples where simply taking the largest number of beans, as in the lemma, is not always the best move. For example consider a game with subtraction set {4, 5}. The table of this game's $G_s(n)$ values are given in Table 1.

In particular consider the value of $\mathcal{G}_s(13)$, this is $\max\{4-\mathcal{G}_s(9), 5-\mathcal{G}_s(8)\}=4-\mathcal{G}_s(9)=3$. Therefore, for this game taking 4 beans and gaining 4 points is preferable to taking 5 beans and gaining 5 points. This is a very simple example to illustrate the point that we cannot say playing greedily would always work. In other words, we need to show that if n is large enough then taking the largest number of beans available *is* the best strategy. So we make the following conjecture.

Conjecture 34. Let S be a finite subtraction set, then there exists an N such that $\mathcal{G}_s(n+2k)=\mathcal{G}_s(n)$ for all $n\ge N$, where $k=\max\{S\}$.

It seems plausible that this conjecture is true, given the lemma, however it is also possible that there is an n such that $\mathcal{G}_s(n+2ik)=J$ and $\mathcal{G}_s(n+(2i+1)k)=k-j$, where $J>j$. What we have seen from the data is that often if $n\notin S$ the values of $\mathcal{G}_s(n+2ik)$ and $\mathcal{G}_s(n+(2i+1)k)$ will alternate as in the lemma, but then you will reach an i where the values change, and this switch might happen several times before it settles down.

A proof of the conjecture or a counterexample would be a very big step forward in understanding how the function operates.

5.2. ***Taking-no-breaking games.*** Taking-no-breaking games are a more general version of subtraction games, and cover a fairly wide range of octal games. The rules of these games are fairly basic, when a player removes a certain number of beans from a heap, he will have one of three options:

(1) Leave a heap of size zero, i.e., remove the entire heap.

(2) Leave a heap of size strictly greater than zero.

(3) Leave a heap of size greater than or equal to zero.

From the definition of an octal game this means that each n_i is either 0, 1, 2 or 3, also an octal game $O = (n_1 n_2 \dots n_k,\ p_1 p_2 \dots p_k)$ is finite if k is finite.

It should be clear that for a fixed $m \in P$ and finite O, where P and O are two taking-no-breaking games, then the function $\mathcal{G}_s(n +_\ell m)$ must always be eventually periodic. The reason is that we always compute each value from a finite number of previous values, and since O is finite this implies that $\mathcal{G}_s(n +_\ell m)$ is bounded, and both of these facts together mean that the function will be eventually periodic.

The real question that one needs to answer however is not "Is it periodic?", but "What is the period?". We believe we can answer that question for a particular class of taking-no-breaking games, and that is the class of games where if you remove i beans you get i points. We make the following conjecture.

Conjecture 35. Let $O = (n_1 n_2 \dots n_t,\ p_1 p_2 \dots p_t)$ and $P = (m_1 m_2 \dots m_l,\ q_1 q_2 \dots q_l)$ be two finite taking-no-breaking octal games such that there is at least one $n_s \neq 0$ or 1, and if n_i and $m_j = 1, 2$ or 3 then $p_i = i$ and $q_j = j$, and $p_i = q_j = 0$, otherwise; then

$$\mathcal{G}_s(n + 2k +_\ell m) = \mathcal{G}_s(n +_\ell m),$$

where O is finite and k is the largest entry in O such that $n_k \neq 0, 1$.

There is very strong evidence that this conjecture will hold. Since m is a constant it changes the value of $\mathcal{G}_s(n +_\ell m)$, but not the period. We have checked the theorem for many examples and not yet found a counterexample, which suggests that it is probably true.

Unfortunately, proving it is surprisingly difficult. The conjecture says that if n is large enough, then your best move is to simply remove the maximum available beans from the heap n, so a proof would need to show that for any given m, there are only finitely many places where moving on m or removing fewer than k beans from n is a better move.

There are several problems with this, the first is that the function $\mathcal{G}_s(n +_\ell m)$ only tells us the maximum possible value from the set of possible values. This makes it very difficult to do a proof that first shows $\mathcal{G}_s(n+2k +_\ell m) \geq \mathcal{G}_s(n +_\ell m)$ and vice-versa. The second is to understand *why* removing a lower number of beans would be better than playing greedily, in some instances.

The last problem is that induction is hard, because what may hold for lower values may not hold at higher values, making a proof by induction difficult.

However, since the function is recursively defined an inductive proof seems to be more natural than a deductive proof.

We believe that a proof of this theorem would also help in finding the period, and proving it for the more general case, where i beans are worth k points, $k \in \mathbb{R}$.

Of course, it is natural to ask what happens in the general case. Unfortunately, in the general case the conjecture does not hold. To see why consider the game $O = (3333, 2222)$. The values of $G_s(n)$ are given in the following table:

n	0	1	2	3	4	5	6	7	8	9	10
$\mathcal{G}_s(n)$	0	2	2	2	2	0	2	2	2	2	0

This game has period 5, which does not correspond to a possible value of k, i.e., 1, 2, 3 or 4. While all taking-no-breaking games are periodic as we can see from the example, it is not clear what the period is, since we can take our p_i's to be any real number. So we make the following conjecture.

Conjecture 36. Let $O = (n_1 n_2 \dots n_t,\ p_1 p_2 \dots p_k)$ and $P = (m_1 m_2 \dots m_l,\ q_1 q_2 \dots q_l)$ be two finite taking-no-breaking octal games; then there exists a t such that

$$\mathcal{G}_s(n + t +_\ell m) = \mathcal{G}_s(n +_\ell m).$$

5.3. ***Taking-and-breaking.*** Another type of nim game we can examine are taking-and-breaking games. That is, games where after the player removes some beans from a heap, he must break the remainder into two or more heaps. This is more general than taking-no-breaking games, since taking-no-breaking games are a subset of taking-and-breaking games.

There are several problems with examining taking-and-breaking scoring games. The first is that we cannot even say that the function $\mathcal{G}_s(n +_\ell m)$ is bounded. The reason is that with each iteration you are increasing the number of heaps, which may increase the value of the function as n increases. So we cannot put a bound on the function as we could with subtraction games and taking-no-breaking games.

Another problem is that if we were to say, examine the game 0.26, which means take one bean and leave one nonempty heap, or take two beans and leave either two nonempty heaps, or one nonempty heap, the number of computations required to find $\mathcal{G}_s(n)$ increases exponentially with n. Since a heap of size $n-2$ may be broken into two smaller heaps n_1 and n_2, we must therefore also compute the value of $\mathcal{G}_s(n_1 +_\ell n_2)$.

However, if $n_1 - 2$ or $n_2 - 2$ may also be broken into two smaller heaps, say n_1', n_1'', n_2' and n_2'', then we must compute the value of $\mathcal{G}_s(n_1' +_\ell n_1'' +_\ell n_2)$ and $\mathcal{G}_s(n_1 +_\ell n_2' +_\ell n_2'')$. This process will continue until we have heaps that are too small to be broken up. So this means that computing $\mathcal{G}_s(n)$ for a taking-and-breaking game is a lot harder than for a taking-no-breaking game, simply due to the number of computations involved.

So we have the following conjecture.

Conjecture 37. Let $O = (n_1n_2 \dots n_k,\ p_1p_2 \dots p_k)$ and $P = (m_1m_2 \dots m_l,\ q_1q_2 \dots q_l)$ be two finite octal games; then there exists a t such that

$$\mathcal{G}_s(n + t +_\ell m) = \mathcal{G}_s(n +_\ell m).$$

While we feel that this conjecture may be true, it is certainly not as strong as Conjecture 35, for the reasons previously given. However, studying these games would certainly be interesting, and anything anyone could find out about them would be useful.

Conclusion

We hope that we have given the readers some interesting new ideas about the types of games that can be studied with scoring play theory, as well as opening up a whole new world of impartial games that can be researched. We have simply introduced the ideas, but there is still much to be learned from these fascinating games.

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