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# Generalized misère play

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We introduce a new framework by which to view impartial games. Instead of thinking of normal play and misère play as differing in their winning condition, we instead view them as differing in which set of positions are "in the field of play". This leads to a generalization producing an infinite array of *game boards*.

Throughout this paper, the natural numbers $= \{0, 1, 2, \ldots\}$ are denoted $\mathbb{N}$, the set of sequences of natural numbers (integers) are denoted $\mathbb{N}^\infty$ ($\mathbb{Z}^\infty$) and the origin is denoted $\boldsymbol{O}$.

For an introduction to combinatorial game theory, including impartial games, see [WW]. For a detailed description of misère quotient monoids, see [PS]. An algorithm for computing quotient monoids is found in [W].

## 1. Introduction

Traditionally, normal play impartial games and misère play impartial games are thought to differ by their respective *winning conditions*, that is, the goal of normal play is to make the final move to the empty game, whereas the goal of misère play is to force the opponent to make the final move. Both versions end when the last bean is removed. Here we present a different perspective, that normal play and misère play differ in the set of positions which are legal, so that both end with the same winning condition: having your opponent unable to move to a legal position.

Also, the traditional description of impartial games relies on the notion of the *sum* of two games and the theory is typically presented in terms of arbitrary sums of games. Here we adopt a notation that includes arbitrary sums of games under a particular ruleset as a set of lattice points $\subseteq \mathbb{N}^\infty$. Hence we need not talk about *sums* of positions; all discussion will be localized to legal moves from a particular lattice point, since the notion of sum is inherent in the lattice point definition of a position.

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Our basic framework is to play on several (but a finite number of) heaps of beans, where the rules allow a heap to be replaced by specified finite multisets of heaps of smaller sizes.

## 2. Heap games

Our definitions are motivated by the game of Nim, in which the set of all legal positions is generated by the set of heaps of various sizes. We will denote the heap of size $i$ as $h_i$ and call the set of all heap sizes the *heap alphabet*, denoted $H$. Thus every position in Nim is the sum of (perhaps repeated) elements of $H$. Therefore, we can represent any position in Nim as a sequence $(x_1, x_2, \ldots)$, where $x_i \in \mathbb{N}$ counts the number of copies of $h_i$ in the sum.

Games not typically played with heaps of beans can still be thought of in this setting (see [GM1, §5] for a general description. Also see the Appendix for a specific example.)

In Nim, any heap can be moved to a heap of a smaller size or removed all together, thus a typical move[1] can be represented as a sequence where all entries are zero except for a single coordinate = $-1$ and perhaps a single coordinate = 1 which is to the left of the $-1$.

Normal play variations of Nim (subtraction games, octal games, etc.) use the same heap alphabet and the same set of legal positions, but have a different set of moves. To ensure our rules obey the termination condition of combinatorial games, we require that a *move* be a sequence with a rightmost, nonzero entry of $-1$ and all entries to the left of the $-1$ be elements of $\mathbb{N}$.

**Example 1.** Let's examine moves in Dawson's Chess, which has octal code 0.137. Imagine a row of bowling pins, and you may remove any bowling pin along with it's neighbors, if any. One pin can be removed only when it is isolated. Two pins can be removed only from the end of a row. Three pins can be removed anywhere, including the possibility of splitting the row into two separate rows. For example, a row of size 5 can be split into two rows of size 1 by removing the middle three pins. Using the notation above, $(2, 0, 0, 0, -1, 0, \ldots)$ is a move.

In normal play, the legal positions are all sequences whose entries are in $\mathbb{N}$. From the position $(1, 1, 0, \ldots)$, the standard way of thinking about possible moves is to list the moves from heaps of size 1 and 2 only, since those are the only heaps present in the position. We wouldn't think that a player could make the move $(0, 0, -1, \ldots)$ since there aren't any heaps of size 3 present in the current position. Such a move would be "off the board". We can get around the difficulty of moves

[1] The notation used here is in the opposite direction from that used in [GM1]. In this paper, when $p$ is a position and $m$ is a move, then the options of $p$ are of the form $p + m$, whereas in [GM1], the options are $p - m$.

to "off the board" positions by declaring that any sequence with a coordinate $< 0$ is a **D**efeated position, (a "D"-position). We typically think of a normal play contest ending when a player moves to the origin, thereby winning. We now think of the move to the origin as a good strategy, since the opponent has no option but to move to a defeated position, thereby losing. A contest ends when a player moves to a defeated position. By making this adjustment, the moves are now *invariant*.

We call the set of positions which are not defeated positions the *game board*, thus for normal play Nim, the game board is $\mathbb{N}^\infty$. If we wish to use the misère play convention, all we need to do is remove the origin from the game board (thereby making the origin a defeated position). The remainder of the language stays the same. For misère play, the game board is $\mathbb{N}^\infty \setminus \boldsymbol{O}$ and as always, a player loses by moving to a defeated position.

**Definition 2.** A *heap game* is a triple $(H, B, \Gamma)$, where $H$ is the *heap alphabet*, $B$ is the *game board*, and $\Gamma \subset \mathbb{Z}^\infty$ is the set of *moves*.

We require the following:

- $H$ is a countable set;
- $B \subseteq \mathbb{N}^\infty$;
- $\forall\, \gamma \in \Gamma,\ \gamma$ has a rightmost nonzero entry which is $-1$ and all other entries of $\gamma \in \mathbb{N}$.

**2.1.** ***Heap games restricted to heaps of a fixed finite size d.*** If we restrict the heap alphabet to heaps of size $\leq d$, we often can employ the quotient monoid approach [PS] to describe the strategy. To each position $p \in B$, the *outcome function*[2] $o : B \to \{P, N\}$ assigns the outcome $o(p) = N$ if there exists an option $p + \gamma$ ($\gamma \in \Gamma$) with outcome $P$ and $o(p) = P$ if each option $p + \gamma$ has outcome $N$ or $D$. We then find the *quotient monoid* $Q$ by starting with the free monoid generated by the heap alphabet $H = \{h_1, \ldots, h_d\}$ and modding out by the equivalence $p \equiv r$ if

$$\forall x \in \mathbb{N}^d,\, o(p + x) = P \iff o(r + x) = P.$$

Finally, we define the *monoid outcome function* $\mathbb{O} : Q \to \{P, N\}$ making the following diagram commute:

$$\begin{tikzcd} B \arrow[r, "o"] \arrow[d, "\equiv"'] & \{P, N\} \\ Q \arrow[ur, "\mathbb{O}"'] & \end{tikzcd}$$

[2]In [PS], the normal play outcome function is denoted $o^+$ and the misère play outcome function is denoted $o^-$. Here, we have no need to distinguish the two; normal play and misère play differ in their game board, not in their winning condition.

It is commonplace for the game board to be generated by the heaps $\{h_1, h_2, h_3, \dots\}$, but the quotient monoid $Q$ to be generated by $\{a, b, c, \dots\}$. As an intermediate step, we may first pass from the game board on heaps of size $\leq n$ to the free monoid on $n$ generators via the map

$$h_1 \to a,\ h_2 \to b,\ h_3 \to c \dots,$$

and then find the quotient produced by the equivalence relation on the generators $\{a, b, c, \dots\}$.

Aaron Siegel's *MisèreSolver* computer program finds misère quotients when $|Q| < \infty$. An algorithm for finding misère quotients is also given in [W].

If you wish to be proficient at beating your opponent, you will need to know which positions are $P$-positions, which in the monoid presentation, requires you to memorize $\mathbb{O}^{-1}(P)$, the preimage of $P$ for the monoid outcome function.[3] As a very simple example, in misère Nim[4], if you are to move from the position $(3, 4)$, first reduce the game $a^3b^4$ by using the relations $a^2 = 1$ and $b^3 = b$ to conclude that $a^3b^4$ is equivalent to $ab^2$. You are happy to see that $ab^2$ is an $N$-position, since it is in $Q$, but not in $\mathbb{O}^{-1}(P)$. You now must find an option equivalent to either $a$ or $b^2$. After a brief search, you realize that $(2, 4) \to a^2b^4$ reduces to $b^2$, thus you remove a heap of size 1.

**Example 3.** Normal play Nim restricted to heaps of size $\leq 2$ has game board $B = \mathbb{N}^2$ and ruleset

$$\Gamma = \{(-1, 0), (0, -1), (1, -1)\}.$$

Table 1 shows the outcomes of positions $(x_1, x_2) \in B$.

The quotient monoid is $\langle a, b \mid a^2 = 1, b^2 = 1\rangle$ with $\mathbb{O}^{-1}(P) = \{1\}$.

[3] An alternate approach for defining the winning strategy via the generating function of $P$-positions can be found in [GM2], where in many cases it is called a "rational strategy".

[4] For the presentation of the quotient monoid for misère Nim, see Example 4 below.

| $x_1 =$ | $<0$ | 0 | 1 | 2 | 3 | 4 | 5 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|
| $x_2 < 0$ | $D$ | $D$ | $D$ | $D$ | $D$ | $D$ | $D$ | $\cdots$ |
| $x_2 = 0$ | $D$ | **P** | $N$ | **P** | $N$ | **P** | $N$ | $\cdots$ |
| $x_2 = 1$ | $D$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $\cdots$ |
| $x_2 = 2$ | $D$ | **P** | $N$ | **P** | $N$ | **P** | $N$ | $\cdots$ |
| $x_2 = 3$ | $D$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $\cdots$ |
| $x_2 = 4$ | $D$ | **P** | $N$ | **P** | $N$ | **P** | $N$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

**Table 1.** Normal play Nim.

| $x_1 =$ | $< 0$ | 0 | 1 | 2 | 3 | 4 | 5 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|
| $x_2 < 0$ | $D$ | $D$ | $D$ | $D$ | $D$ | $D$ | $D$ | $\cdots$ |
| $x_2 = 0$ | $D$ | $D$ | **P** | $N$ | **P** | $N$ | **P** | $\cdots$ |
| $x_2 = 1$ | $D$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $\cdots$ |
| $x_2 = 2$ | $D$ | **P** | $N$ | **P** | $N$ | **P** | $N$ | $\cdots$ |
| $x_2 = 3$ | $D$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $\cdots$ |
| $x_2 = 4$ | $D$ | **P** | $N$ | **P** | $N$ | **P** | $N$ | $\cdots$ |
| $x_2 = 5$ | $D$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

**Table 2.** Misère play Nim.

**Example 4.** Misère play Nim restricted to heaps of size $\leq 2$ has game board $B = \mathbb{N}^2 \setminus \boldsymbol{O}$ and ruleset

$$\Gamma = \{(-1, 0), (0, -1), (1, -1)\}.$$

Table 2 shows the outcomes of positions $(x_1, x_2) \in B$.

The quotient monoid is $\langle a, b \mid a^2 = 1, b^3 = b\rangle$ with $\mathbb{O}^{-1}(P) = \{a, b^2\}$.

## 3. Generalized misère play

There is no reason why $\mathbb{N}^d$ and $\mathbb{N}^d \setminus \boldsymbol{O}$ are the only choices for $B$.[5] We could instead play Nim with the restriction that no move can be made which reduces the number of heaps to one.

**Example 5.** The game board $B = \mathbb{N}^2 \setminus \{(0, 0), (0, 1), (1, 0)\}$ and the ruleset

$$\Gamma = \{(-1, 0), (0, -1), (1, -1)\}.$$

Table 3 shows the outcomes of positions $(x_1, x_2) \in B$.

The quotient monoid is $\langle a, b \mid a^3 = a, ba^2 = b, b^2 = a^2\rangle$ with $\mathbb{O}^{-1}(P) = \{a^2\}$. Note that $|Q| = 5$. Quotients of odd order never occur in either in misère or normal play impartial games (see [PS, Theorem 4.5]).

For another variation, the set of defeated positions in $\mathbb{N}^d$ need not include the origin.

**Example 6.** The gameboard $B = \mathbb{N}^2 \setminus \{(2, 0)\}$ and the ruleset

$$\Gamma = \{(-1, 0), (0, -1), (1, -1)\}.$$

[5] This concept is also used in [GFL] where the authors restrict the number of heaps instead of restricting the size of the heaps

| $x_1 =$ | $<0$ | 0 | 1 | 2 | 3 | 4 | 5 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|
| $x_2 < 0$ | $D$ | $D$ | $D$ | $D$ | $D$ | $D$ | $D$ | $\cdots$ |
| $x_2 = 0$ | $D$ | $D$ | $D$ | **P** | $N$ | **P** | $N$ | $\cdots$ |
| $x_2 = 1$ | $D$ | $D$ | $N$ | $N$ | $N$ | $N$ | $N$ | $\cdots$ |
| $x_2 = 2$ | $D$ | **P** | $N$ | **P** | $N$ | **P** | $N$ | $\cdots$ |
| $x_2 = 3$ | $D$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $\cdots$ |
| $x_2 = 4$ | $D$ | **P** | $N$ | **P** | $N$ | **P** | $N$ | $\cdots$ |
| $x_2 = 5$ | $D$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

**Table 3.** $B = \mathbb{N}^2 \setminus \{(0,0), (0,1), (1,0)\}$.

| $x_1 =$ | $<0$ | 0 | 1 | 2 | 3 | 4 | 5 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|
| $x_2 < 0$ | $D$ | $D$ | $D$ | $D$ | $D$ | $D$ | $D$ | $\cdots$ |
| $x_2 = 0$ | $D$ | **P** | $N$ | $D$ | **P** | $N$ | **P** | $\cdots$ |
| $x_2 = 1$ | $D$ | $N$ | **P** | $N$ | $N$ | $N$ | $N$ | $\cdots$ |
| $x_2 = 2$ | $D$ | $N$ | $N$ | **P** | $N$ | **P** | $N$ | $\cdots$ |
| $x_2 = 3$ | $D$ | **P** | $N$ | $N$ | $N$ | $N$ | $N$ | $\cdots$ |
| $x_2 = 4$ | $D$ | $N$ | **P** | $N$ | **P** | $N$ | **P** | $\cdots$ |
| $x_2 = 5$ | $D$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $\cdots$ |
| $x_2 = 6$ | $D$ | **P** | $N$ | **P** | $N$ | **P** | $N$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

**Table 4.** $B = \mathbb{N}^2 \setminus \{(2,0)\}$.

Table 4 shows the outcomes of positions $(x_1, x_2) \in B$.

The quotient monoid is $\langle a, b \mid a^4 = a^2, b^3a^3 = b^3a, b^4a^2 = b^4, b^7 = b^5\rangle$ with $\mathbb{O}^{-1}(P) = \{1, a^3, ba, b^2a^2, b^3, b^4a, b^6\}$.

## 4. Victorious positions

We achieved misère play by inserting a defeated position inside $\mathbb{N}^d$ at the origin. We achieved other variations by similarly forcing players to avoid certain positions inside $\mathbb{N}^d$. These defeated ($D$) positions are worse than $N$ positions. In an $N$ position, you are losing, but could still win if your opponent makes a mistake. In a $D$ position, you have lost. We can do the same for $P$ positions. In a $P$ position, you are winning, but might wind up losing if you don't follow optimal play. The counterpart to the defeated position is a position in which you are declared the winner. We will call such a position a *victorious* position. A victorious ($V$)

position is a position in $\mathbb{N}^d$ such that when a player moves to that position, the contest ends with the player making the move declared the winner.

As a motivation for victorious positions, we can think of this being somewhat analogous to the premature ending of the battle in chess via checkmate compared to the battle in checkers. When two imperfectly programmed computers that never resign play checkers, they battle until one side captures all of the opponent's pieces, similar to whittling a Nim game down to the empty position. The addition of $D$-positions is akin to "resigning". Once you realize you are going to lose the battle, you may wish to quit the game in a losing manner without fighting the battle to it's conclusion. If the rules of chess treated the king as an ordinary piece whose mate did not end the game, the battle would rage on until one army was vanquished as in checkers. The checkmate provides a position that ends the contest in victory prematurely, despite the winner perhaps having a disadvantage in terms of material and position on the rest of the board.

Formally, we choose a game board $B \subseteq \mathbb{N}^d$, with all positions in $\mathbb{Z}^d \setminus B$ declared to be defeated positions. Then we choose the set of victorious positions $V \subset B$. For all $p \in B \setminus V$, the outcome $\mathbb{O}(p) = N$ if any option of $p$ has outcome $P$ or $V$ and the outcome $\mathbb{O}(p) = P$ if all options of $p$ have outcome $N$ or $D$. Thus $\mathbb{O}$ is a function $\mathbb{O} : \mathbb{Z}^d \to \{P, N, D, V\}$. We define the equivalence relation

$$p \equiv r \text{ if } \mathbb{O}(p+x) \in \{P, V\} \Longleftrightarrow \mathbb{O}(r+x) \in \{P, V\}, \quad \forall x \in \mathbb{N}^d.$$

**Example 7.** Let the game board $B = \mathbb{N}^2$ with victorious position $V = \{(2, 1)\}$ and ruleset

$$\Gamma = \{(-1, 0), (0, -1), (1, -1)\}.$$

Table 5 shows the outcomes of positions $(x_1, x_2) \in B$.

The quotient monoid is $\langle a, b \mid a^5 = a^3, b^2a^4 = b^2a^2, b^4a^3 = b^4a, b^6a^2 = b^6, b^9 = b^7\rangle$ with $\mathbb{O}^{-1}(\{P, V\}) = \{1, a^2, a^4ba^2, b^2, b^2a^3, b^3a, b^4a^2, b^5, b^6a, b^8\}$.

## 5. Open questions

- Example 5 provides a quotient of order 5. Quotients of odd order do not occur in either normal play or misère play. Are there game boards which produce quotients of other odd orders? In particular, is there a game board with $|Q| = 3$? (The only way to produce $|Q| = 1$ is for $B = \varnothing$.) Are there any positive integers $k$ for which it is impossible to find a quotient monoid with $|Q| = k$?
- What happens when the defeated positions in $\mathbb{N}^d$ are not within a neighborhood of the origin? Most misère quotient monoids have the property that once we are far enough away from the origin, the outcomes resemble those in normal play. For example, in misère Nim, once there is a heap of size

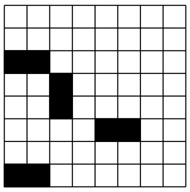
| $x_1 =$ | $< 0$ | 0 | 1 | 2 | 3 | 4 | 5 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|
| $x_2 < 0$ | $D$ | $D$ | $D$ | $D$ | $D$ | $D$ | $D$ | $\cdots$ |
| $x_2 = 0$ | $D$ | **P** | $N$ | **P** | $N$ | **P** | $N$ | $\cdots$ |
| $x_2 = 1$ | $D$ | $N$ | $N$ | V | $N$ | $N$ | $N$ | $\cdots$ |
| $x_2 = 2$ | $D$ | **P** | $N$ | $N$ | **P** | $N$ | **P** | $\cdots$ |
| $x_2 = 3$ | $D$ | $N$ | **P** | $N$ | $N$ | $N$ | $N$ | $\cdots$ |
| $x_2 = 4$ | $D$ | $N$ | $N$ | **P** | $N$ | **P** | $N$ | $\cdots$ |
| $x_2 = 5$ | $D$ | **P** | $N$ | $N$ | $N$ | $N$ | $N$ | $\cdots$ |
| $x_2 = 6$ | $D$ | $N$ | **P** | $N$ | **P** | $N$ | **P** | $\cdots$ |
| $x_2 = 7$ | $D$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $\cdots$ |
| $x_2 = 8$ | $D$ | **P** | $N$ | **P** | $N$ | **P** | $N$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

**Table 5.** $V = \{(2, 1)\}$.

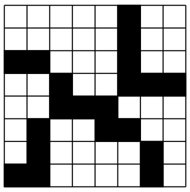
at least two, the outcomes agree with the outcomes in normal play Nim. Will certain patterns of defeated positions extending arbitrarily far from the origin prevent us from being able to find any equivalence relations?

## Appendix

We will use the game *Cram* [WW] to illustrate how an arbitrary impartial game can be expressed as a heap game. *Cram* is the impartial version of *Domineering*, which is played by placing $2 \times 1$ dominoes on a board (typically starting with an $8 \times 8$ board). In *Cram*, either player may place their domino either vertically or horizontally. After two moves for each player, the board may look like this:

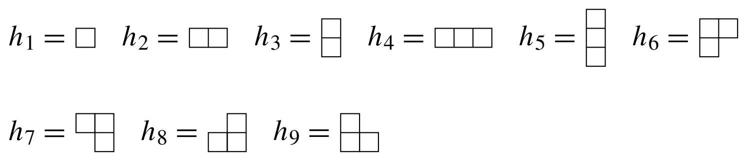


Eventually, the board will be partitioned into connected components, such as this:



Any *Cram* position can be decomposed into to the sum of its connected components, so the various connected components can serve as a generating set for *Cram* positions. We will have the heap alphabet once we have ordered the connected components.

For example, we could begin by ordering connected components of *Cram* based on the number of cells they contain, and then arbitrarily extending that partial order to a total order. One example being that the first several heaps are

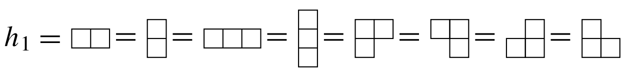


Of course, we could interchange $h_2$ and $h_3$. All that is required is that when we play in one connected component, the resulting position is composed of connected components that occur earlier in the heap alphabet, which they will, since the resulting connected components will have fewer cells.

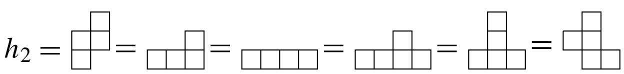
Depending on the setting, we may wish to further simplify the heap alphabet. For example, in normal play, the rotational symmetry allows us to think of $h_2$ and $h_3$ as identical, so we only need to include one of them in the heap alphabet. Also, since there is no move from $h_1$, we don't need to include it in the heap alphabet at all, $h_1 = 0$. Additionally, under normal play, any move from $h_4, \ldots, h_9$ is to $h_1$, so we don't need to include those in the heap alphabet either, since they act in the same manner as $h_2$ in that they have a single option to the endgame.

There is, however, no harm in leaving these heaps in from the perspective of the quotient monoid; $h_2, \ldots, h_9$ will be equated in the quotient $Q$ and $h_1$ and 0 will likewise be equated in $Q$ for normal play.

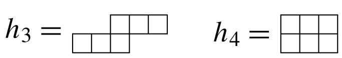
In misère play, we can declare that arbitrary collections consisting solely of disconnected single cells are also defeated positions and then make the same simplifications so that the heap alphabet begins



Next come the positions containing four and five cells. Some of them, such as  have moves only to $h_1$, and as such, are reversible to 0. Others, such as  can be moved only to 0, and are equivalent to $h_1$. Others have moves to both $h_1$ and 0 and will be used as the heap $h_2$:



Possible choices for the next two heaps are



The two positions have the following moves: $h_3$ can be moved to $h_2$, $2h_1$, or $h_1 + 0$. $h_4$ can be moved to $h_2$ or $2h_1$. (Note that $2h_1$ is reversible to 0.)

Using the map from $B$ to $Q$,

$$h_1 \to a,\ h_2 \to b,\ h_3 \to c,\ h_4 \to d.$$

the quotient monoid is $\langle a, b, c, d \mid a^2 = 1, b^3 = b, c = a, d = ab\rangle$ with $\mathbb{O}^{-1}(P) = \{a, b^2\}$.

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