Critical Issues in Mathematics Education
Volume 10 • Workshop 11 • March 2014

The Role of the Mathematics Department in the Mathematical Preparation of Teachers

By Julie Rehmeyer

MSRI
Mathematical Sciences Research Institute

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"If the teaching of mathematics is going to flourish as a profession, the fundamental engine for this will have to come from a leadership community within the teaching profession itself. Math for America would say that that professional community will reach critical mass when it comprises 10-15% of practicing teachers. Math teacher preparation programs (as joint enterprises between math departments and colleges of education) should look to identify and cultivate those student teachers who have what it takes to be future members of that teacher-leader community."

– Herb Clemens, Ohio State University

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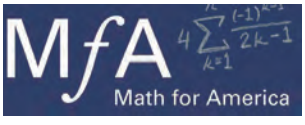
This workshop was held in March, 2014 at the Mathematical Sciences Research Institute

CIME Organizing Committee

MSRI 2014-2015 Educational Advisory Committee

DMS-0932078



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Photography: *Please see page 65 for photo credits and related information.*

The Role of the Mathematics Department in the Mathematical Preparation of Teachers

Julie Rehmeyer, author of booklet

The United States needs more, better-educated mathematics teachers. Mathematics departments are key players in developing those teachers, because future teachers learn mathematics content through their courses. Mathematics departments therefore have a major responsibility to teacher education — but often, this responsibility is little attended to.

There is thus an enormous potential for math departments to improve and to impact the quality of teachers, and therefore, ultimately, to impact children and the nation as a whole.

To facilitate that, the Mathematical Sciences Research Institute hosted the 2014 Critical Issues in Mathematics Education workshop focusing on the role of mathematics departments in preparing future teachers.

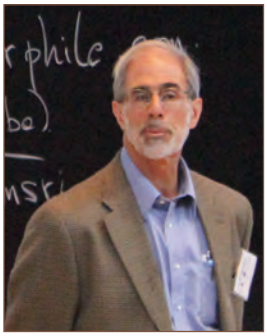
The workshop aimed first to acquaint mathematicians with basic facts about teacher education and the critical role math departments play in their development — whether that role is attended to or not. It discussed what is known about effective mathematical preparation of teachers, including curriculum, instructional approaches, and assessments. It examined the support that mathematicians and mathematics departments need to carry out this role effectively. It offered successful models along with evidence for their effectiveness. It studied persistent challenges that arise for mathematicians and math departments and found promising examples of ways of addressing these. And finally, it identified the actions math departments can take to more effectively contribute to teachers' mathematical education.

This report summarizes the presentations at the workshop. The workshop organizers intentionally sought to present a diversity of perspectives and opinions at the workshop. Opinions expressed are those of the speakers and not necessarily those of MSRI or any workshop sponsor.

Workshop Snaps

Sybilla Beckmann (University of Georgia), and William McCallum (University of Arizona) discussed the *Recommendations of the Mathematical Education of Teachers II.*

Before the opening workshop session, organizing committee chair Jim Lewis (University of Nebraska-Lincoln) and Deborah Ball (University of Michigan) with keynote speaker Howard Gobstein (Association of Public and Land-Grant Universities).



MSRI director David Eisenbud welcomes attendees to the workshop.

Scott Baldridge
(Louisiana State University)

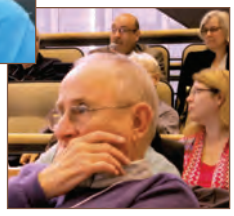
Sol Friedberg (Boston College)

Right: Lillie Albert (Boston College) and Fabio Milner (Arizona State University) discuss educating a teacher workforce that reflects the diversity of our K-12 classrooms.

Contemplating ideas at the Freidberg session

Below: exploring ideas during a break

Simons Auditorium snaps from sessions



Discussion in the Baker Board Room

Above:
Theresa Jorgensen and Silvia Saccon during a break

Contents

1 The Context

The current context of math education in the U.S. has an enormous impact on the education of mathematics teachers. Howard Gobstein, executive vice-president of the Association of Public and Land-Grant Universities, described the global situation that is creating enormous change and instabilities for universities and how that change impacts mathematics education *(page 9)*. **William McCallum characterized the newly implemented Common Core State Standards in mathematics** *(page 10)*. **Hung-Hsi Wu diagnosed current problems in the preparation of mathematics teachers** *(page 12)*. **Deborah Ball described the ad-hoc nature of our current system to educate teachers and its pernicious effects** *(page 14)*. **And Fabio Milner discussed diversity and equity in teaching** *(page 18)*.

The Context • Howard Gobstein

Change, Challenge, Opportunity

Howard Gobstein, executive vice-president of the Association of Public and Land-Grant Universities, began by describing the global situation creating enormous change and instabilities for universities, and then described the impact of that for mathematics education.

The societal forces are cacophonous. The globalization of employment makes it essential that the American workforce is educated to compete with workers from around the world. At the same time, economic inequality has risen sharply, and with it a severe gap in economic opportunity. On top of that, college costs have been rising dramatically: Twenty years ago, roughly half of students graduated with student loan debt averaging $7,000, whereas now, 70% have debt averaging about $30,000. The number of students going to college has risen, and at the same time, federal and state governments have cut financial support to universities and demanded higher accountability (particularly about the percentage of students who complete a degree). And all of this is happening while technology is revolutionizing and disrupting education, with new forms of course delivery, student assessment, and institutional use of big data.

All of this creates a critical need to transform mathematics education. High-quality mathematics education is essential to creating a competitive workforce as well as to addressing economic inequality. Mathematics courses are one of the largest gateways at most universities and are critical even to students who aren't majoring in science, technology, engineering or mathematics (STEM). Many students must succeed in introductory mathematics in order to achieve any degree. Universities are under pressure to prove their usefulness and success, and that will require showing that they can effectively teach mathematics to students with a very wide variety of backgrounds and preparedness.

Math departments have a vested interest, as their institutional funding is largely determined by teaching demands. To ensure future funding, departments must show they can succeed in retaining and inspiring students who are not self-motivated to study mathematics.

In addition, state leaders reward universities for collaborating with school systems and providing well-prepared teachers.

All of this means that taking teacher education seriously will help math departments prosper.

Job losses between 2007 and 2010

For those with no more than a high school degree:
5.6 million

For those with some college:
1.75 million

For those with a four-year degree:
187,000 *gained*

Percentage with a college degree among 24-year-olds

In the upper economic quartile:
80 percent

In the bottom economic quartile:
10 percent

The Context • William McCallum

The Common Core State Standards

At the time of the workshop in 2014, a set of common underlying standards for math curricula in 45 states were beginning to be implemented. **William McCallum** of the University of Arizona was the Math Work Team Chair for the effort. He described what mathematicians need to know about the Common Core State Standards (CCSS).

The standards were commissioned by the states and adopted voluntarily by them. This began in 2007, when the states got together through their professional organizations (initially the Council of Chief State School Officers and later the National Governor's Association) and agreed to commission the writing of common standards. This included Republican and Democratic governors alike and was entirely uncontroversial.

The standards were developed by a world-class team of about 50 mathematicians (including a dozen research mathematicians and two members of the National Academy of Sciences), math education reseachers, teachers, policymakers, and representatives from state departments of education. It probably involved more contributions from mathematicians than any previous process of writing state standards. A three-person team, including McCallum, led the writing effort itself.

The writing took place during 2009–2010 and included extensive review, with feedback coming from the states, committees of teachers, mathematicians, teachers and math educators. The standards were released for public comment in March 2010 and received ten thousand comments. Some were not actionable and others were mutually contradictory, but the team made considered decisions about each one.

The final version was released in June 2010. States started considering whether to adopt them using their usual procedures, and 45 states did so.

The standards are well aligned to the standards of high achieving countries, more so than any previous state standards. William Schmidt of Michigan State University did a study comparing the Common Core State Standards with those of high-achieving countries and found closer alignment than any previous state standards. As part of his study he developed a way of measuring "curricular coherence," the degree to which one topic builds on the previous ones and then completes.

The standards are particularly designed to lead to better outcomes in algebra. The individual state standards that preceded the Common Core typically suffered from the mile-wide inch-deep problem. Students were studying many, many topics and repeating them in every grade level. The result was that students never developed fluency with arithmetic and understanding of the core notions needed for algebra.

Unpopularly, the developers of the CCSS decided that the preparation for algebra had to start in elementary school. Since students often hit a cliff when they hit algebra, the designers aimed to build a ramp starting in kindergarten. The standards include a strong emphasis on arithmetic, focusing not just on calculation but on the understanding of operations that is the preparation for algebraic thinking.

Middle school standards continue that, with a strong focus on ratios and proportional relationships. That's a precursor to thinking about functions. Students use letters to stand for numbers, and they write algebraic expressions and equations describing relationships between quantities. The CCSS for eighth grade have much of what's typically in an Algebra 1 course, and, by some definitions of "Algebra 1," all of it.

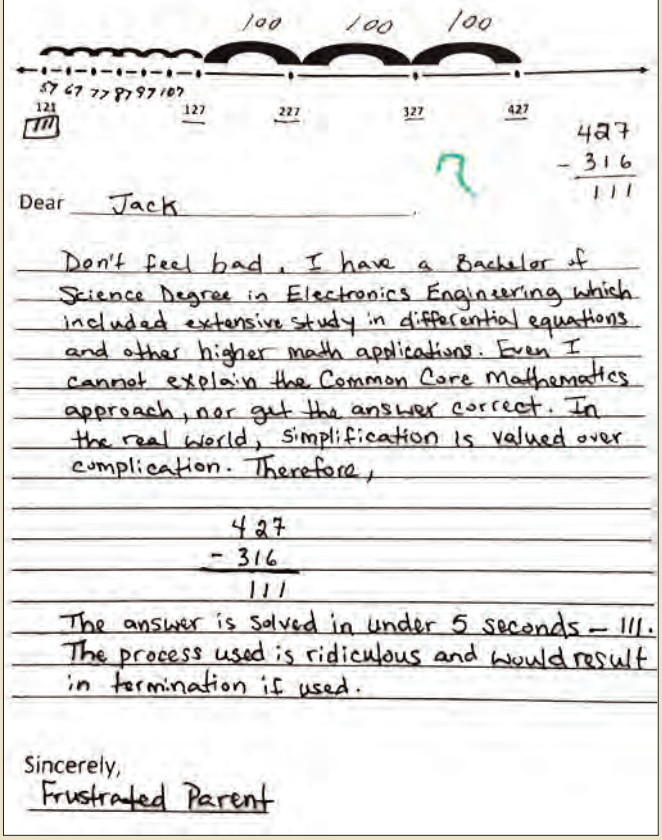
The difference between standards and curricula

As the CCSS has begun to be implemented in classrooms, it has attracted criticism. Much of it actually criticizes particular curricula whose intent is to implement the standards, not the standards themselves. The difference is that the standards describe what needs to be learned in which grade, whereas a curriculum gives the particular tasks and explanations intended to help students to learn those things.

For example, the letter on the right went viral on the internet.

McCallum pointed out that the method used in the problem (which the parent dislikes) is not in the CCSS at all, and that the standard algorithm (which the parent champions) is. It is the particular curriculum that has chosen to focus on this number-line skipping method.

Jack used the number line below to solve 427−316. Find his error. Then write a letter to Jack telling him what he did right, and what he should do to fix his mistake.



THE CONTEXT • HUNG-HSI WU

How Universities Have Failed Teachers

Hung-Hsi Wu of the University of California, Berkeley, argued that mathematics departments have been failing future teachers by not teaching them a correct version of the mathematics they will have to teach.

He once saw an ad from IBM that said, "Stop selling what you have. Start selling what they need." For math teachers, he argued, we've been selling them what we have: courses in higher math. But what preservice teachers need above all is a mathematically sound, correct understanding of school mathematics.

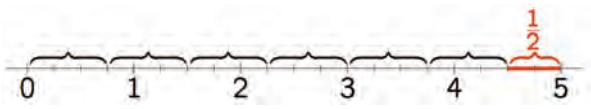
Preservice teachers certainly are unlikely to have acquired this understanding as grade school students themselves, Wu argues, because "textbook school mathematics" (TSM) doesn't make sense, and they know it. Textbooks provide almost no definitions; they fragment a single, coherent topic into pieces handled over multiple years; they blur the line between a proof and a heuristic argument; and they lack precise reasoning. The result is that mathematics is simply not learnable.

The result is a cycle of victimization: In grade school, teachers learn TSM. In college, they expect a better treatment, but our universities do not provide that; instead, universities repeat the same approach or teach them *advanced* mathematics, along with pedagogical strategies to implement TSM. So grade school teachers end up regurgitating TSM and victimize the next generation of teachers.

Presenting school mathematics in a way that is precise and understandable to children is not a trivial task, even for a highly trained mathematician. For example, try to come up with a definition of fractions. Equivalence classes of pairs of integers — the standard definition for a mathematician — won't cut it, because no child can grasp that.

To grasp the incoherence of textbook school mathematics, consider these examples:

1. Many or most high school students believe that $^{-7}/_{-3} = {}^{7}/_{3}$ because they are told that a negative times a negative is a positive, so it seems reasonable that what's good for multiplication is good for division, so it must be that a negative divided by a negative is positive. Abstract reasoning goes out the window.
2. Students are taught to reason through the problem $5 \div \frac{3}{4} = 6$ remainder $\frac{1}{2}$ by reasoning in the same way they would for $32 \div 5 = 6$, R 2, through considering this graph:



Students then *guess* that $\frac{1}{2} = \frac{2}{3} \times \frac{3}{4}$ and conclude that $5 \div \frac{3}{4} = 6\frac{2}{3}$. The problem is that this doesn't generalize to a problem like $^{2}/_{11} \div {}^{81}/_{29}$. Furthermore, how can they critique this reasoning?

3. The following table gives the number of miles Helena runs in minutes: How many miles does she run in 25 min?

Min	Mi
10	1
20	2
30	3

Students learn to model the data by proportional reasoning. The unit rate is $1/10$ mi/min. So in 25 minutes, she runs $25 \times 1/10 = 2\frac{1}{2}$ miles.

But the problem doesn't justify this conclusion, because it turns out that this is an Olympic 400 meter specialist training for a meet. Every 10 minutes, she runs ½ mile in 2 minutes and walks the next ½ mile in 8 minutes. So in 25 minutes, she covers about 2.7 miles. For the problem to be coherent and correct, it would need to include the information that Helena runs at a constant speed.

In the fall, teachers will be asked to implement the Common Core State Standards for Mathematics (CCSS), which is, to a large extent, free of TSM. However, equipped only with a knowledge of TSM, teachers have little hope of implementing the CCSS.

If a general sends soldiers to the front without any ammunition, he would be court-martialed, at least, Wu said. Yet, universities can do this to prospective teachers year after year with impunity. This is not something math departments — in fact the mathematics community — should be proud of.

A long history of criticism

Deborah Lowenberg Ball pointed out that it's helpful to remember that the criticisms of math education are not at all new, though teacher educators now often feel very beleaguered. For example, here is a quote from a recent op-ed from the New York Times:

> "How America prepares its teachers has been a subject of dismay for many years… There are 3.3 million public school teachers in America, and they probably can't all be trained by start-ups. Raising up the standards of our university programs should be an urgent priority. But one reason for the widespread mediocrity is that universities have had a cozy, lucrative monopoly. It's about time the leaders of our education schools did feel threatened."
>
> — Bill Keller, *New York Times*, October 20, 2013
> http://www.nytimes.com/2013/10/21/opinion/keller-an-industry-of-mediocrity.html

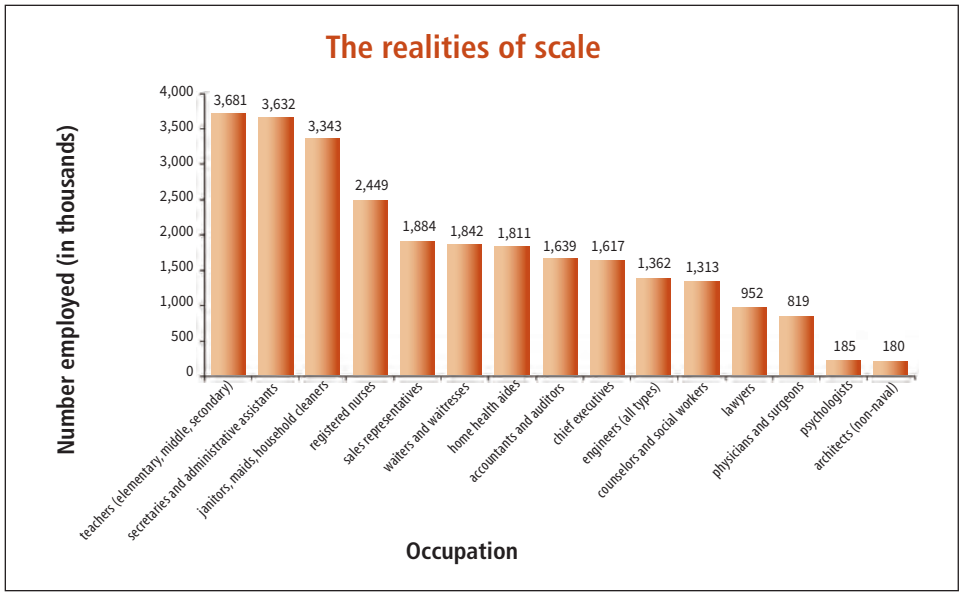
But a very well-known book, The Miseducation of Teachers, was published in 1963, and criticisms have gone all the way back to the turn of the century. The roots of this are very deep.

The Context • Deborah Lowenberg Ball

How the Lack of a True Teacher Education System has Failed Teachers

Deborah Loewenberg Ball of the University of Michigan asked: What would it look like if as a country, we collectively took responsibility to ensure that beginning teachers had the skills they needed to adequately teach children?

The first thing to consider is that teaching is the single largest occupation in America: There are more teachers than secretaries or janitors. So the question has to be answered on a scale to match that reality.



There is a true national crisis around a lack of skilled teachers. The concentration of beginning teachers is higher than we've ever seen before: There are more students with brand-new, first-year teachers than students with teachers who have any other number of years of experience. And the inequities are significant. If you're a child in a low-income community or a child of color, the chances that you have a first-year teacher are much greater than otherwise. Furthermore, not only do beginning teachers tell us they feel significantly underprepared for the work regardless of their pathway, there's substantial evidence that they are. We also know that the effect of unskillful teaching on students is compounded if it happens more than one year in a row.

This lack is difficult to address because we have a non-professional non-system of building the teacher workforce, Ball said. It is not even remotely a system. There is no other occupation, trade or profession in this country that is as non-systematic about supplying new people into the work than teaching is, and there's probably no occupation that's more important for the future of the country.

The non-systematic nature of our process is illustrated by the fact that no one even knows how many different pathways into teaching there are in this country. There are at least 3,000 independent providers of initial teacher training, but this is a conservative estimate. There's no way to count, as many aren't accredited.

There's also no common, specific curriculum for preparation for initial teaching. By contrast, plumbers, nurses, and airplane pilots must master defined capabilities before being licensed to enter the work. And these requirements are very specific. Pilots don't need to just know, "Take the plane into the air" — licensure requires being able to accomplish very specific tasks. But with teaching all we can say is something like, "Plan a lesson." We can't yet define the tasks of teaching with the level of specificity that we can for flying an airplane or installing pipes. Indeed, we're left relying on conventional academic credentials as the standard for content knowledge (such as SAT scores, GPA, course completion), though these are a very poor proxy.

Quite a few people teach without licensure at all, but even among those who are licensed, the requirements they've had to meet are only weakly linked to practice. Having a license simply doesn't prove that you know what you need to do to ensure student learning. There is no common standard of performance for eligibility to independently practice on young people. We're used to thinking that you learn how to teach on the job, but we pay for that system with very high turnover. That turnover penalizes both the people who haven't gotten the training they deserve and the students they've failed to effectively teach.

This non-professional, non-system developed as a natural result of the history of American education. A massive effort began in the mid-1800s to provide schooling to all American children, creating a very great need for teachers quickly. The career of teaching therefore needed to be easy to get into, for easy recruitment. It was also key that teaching be easy to go in and out of, because until the 1960s, women who were pregnant or had young children were often prohibited from teaching. So the rapid turnover of teachers is built into the system.

This meant that communities wanted to be able to staff classrooms and replace teachers easily when they moved on. And if you're going to have a rapid turnover workforce, you don't want classrooms to be interdependent within the school, because if teachers were working together, the school would become dysfunctional when a teacher left. This made it hard to build up codified knowledge in teaching. After all, it doesn't make sense to invest in training if you expect that teachers will soon leave the occupation.

This "common schools" movement that began in the 1840s was based on the idea that the nation-state deserved a common school system in order to build up a brand new country. But then immigrant students came in. The first wave of immigrants were Irish Catholic and didn't like the white male Protestant backbone of the common schools. They said, "Not for our students." This led to the rise of Catholic schools.

And the pattern continued. Various groups of people rightly said, "We want choice in what our students learn in public schools." The result of that was local control of schools. And there are very good reasons for cultural groups to want to have sovereignty and control over schools.

But it has other consequences, including that voters elect lay people to run our school system. It's not obvious that's the best way to do it. Another is that we have huge inequalities in educational opportunity.

> **"Mathematics teaching is an an extraordinarily complex activity involving interactions among teachers, students and the mathematics to be learned in real classrooms."**
>
> **— National Math Panel, 2008**

This is the backdrop for the challenges we have to contend with in order to develop a true professional system of preparing a teaching workforce:

- Lack of shared codified professional knowledge and shared standards for teaching quality
- Highly varied and inequitable opportunities and outcomes for students
- Lack of investment in professional training
- Cultural diversity
- Parental rights
- Democratic political authority and governance of schooling
- School management and unionization
- Dominance of individualism in education and education reform

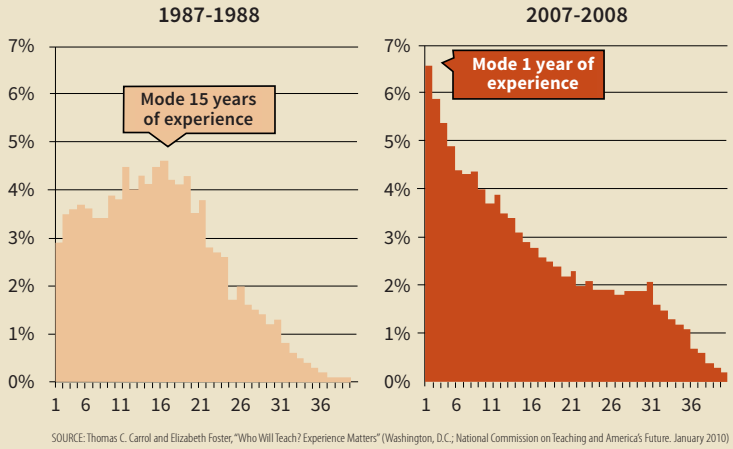
Much of this information comes from "Schoolteacher: A Sociological Study," by Dan Lortie of the University of Chicago.

An additional challenge to creating a professional teaching workforce is combatting false beliefs that have developed around teacher education. Many Americans are unconvinced of the value of teacher preparation, believing that you learn to teach on the job. If current teacher preparation is inadequate, many conclude, then we don't need it at all.

A second false belief is that university students who are preparing to teach are less capable than students in other fields. But faculty who teach mathematics content to student teachers and get frustrated with their struggles often never see their skill in pedagogy classes. Student teachers are capable at the sorts of relational and intellectual work central to helping other people learn content. While the SAT scores of student teachers may be lower than other students, the SAT score says nothing about one's ability to listen to students' explanations of mathematics and build a bridge to a correct understanding.

A third false belief is that teacher education programs are "cash cows" for universities. This is often said to suggest that change is impossible, because universities are relying on these programs for income. The reality, though, is that second to the doctoral program, teacher education is by far the most expensive program at the University of Michigan. Clinical education is very expensive.

Beginning teaching on the rise



SOURCE: Thomas C. Carrol and Elizabeth Foster, "Who Will Teach? Experience Matters" (Washington, D.C.; National Commission on Teaching and America's Future. January 2010)

Teacher experience as share of workforce

Entry-level teachers have become an increasingly large percentage of the workforce. In 2007-2008, nearly seven percent of teachers were in their first year, more than with any other number of years of experience. In 1987-1988, by contrast, more teachers were in their fifteenth year than any other. This is critical, because evidence shows that beginning teaching is on average significantly less effective. Furthermore, low-income and high-minority schools have a disproportionate number of inexperienced teachers.

Despite these challenges, we are at a moment in which we have a special opportunity to change this situation and to create a truly professional system of teacher education. For one thing, "teacher quality" is of more interest than ever. By contrast, in the 1970s, curriculum and testing were seen as the main avenue for educational improvement. In addition, there are currently teacher shortages, which make recruitment and training urgent issues. Furthermore, because 90 percent of American teachers come out of the university system, universities have the possibility to claim responsibility for creating a unified, professional system. So the moment for change is now.

The place to begin is by building on the compelling evidence that both teaching skill and content knowledge matter for teaching. Despite that understanding, we've never tried to identify the key specific practices of teaching and high-leverage content essential for responsible entry-level practice. Here are some examples of high-leverage practices:

- Leading a whole-class discussion
- Eliciting and interpreting individual students' thinking
- Explaining and modeling core content
- Establishing norms and routines for classroom discourse
- Recognizing particular common patterns of student thinking
- Setting up and managing small group work
- Selecting and using specific methods to check understanding and monitor student learning
- Composing, selecting, adapting quizzes, tests, and other methods of assessing student learning of a chunk of instruction
- Conducting a meeting about a student with a parent or guardian

Notice that these are far more fine-grained than most descriptions of teaching practices.

We also need to build on a commitment to clinical preparation. Pilot training doesn't put student pilots in the air right away, and we similarly shouldn't put student teachers in the classroom without teaching them the necessary skills first. We need to develop specific approaches to training beginners to carry out the key practices we identify.

Then we need to build on a commitment to assessing performance. We need to require individuals to demonstrate an entry-level standard of practice before beginning teaching.

None of this will be easy. The scale of the enterprise is vast and the U.S. has weak educational infrastructure. It will take a movement to build a rigorous system of professional training for the responsibility of teaching our nation's youth, in order to provide equal access to skilled teaching. This also goes against current trends by focusing on teaching not teachers, and on building capacity not finding talent and firing. And finally, it is not the "American way": it's detailed (even prescriptive) and doesn't allow individual discretion.

But it is our moment. Teaching is more broadly understood as crucial to children's life chances. There's more attention to teacher preparation than ever before. And now the teaching community is coming together to accomplish change.

> **"Professional development for preservice teachers should include explicit discussion about the fact that learning to teach mathematics for social justice is a complex, long-term process, and adequate contextualization of social issues, for example, will not occur in the course of one professional development experience."**
>
> **– Rochelle Gutiérrez (2009)**

The Context • Fabio Milner

Diversity and Equity in the Teaching Profession

Fabio Milner of Arizona State University spoke on the importance and challenges of recruiting and preparing a diverse workforce of K-12 mathematics teachers.

He pointed out that Hispanics make up about 17% of the U.S. population but just 7% of the teachers. Similarly, blacks make up 13% of the U.S. population but also about 7% of the teachers.

More than 20 states have at least 25 percentage points difference between percent of teachers and percent of students of color. Furthermore, 27 percent of African-American and 25 percent of Hispanic teachers are certified through alternative routes, whereas only 11 percent of white teachers are alternatively certified. There is some evidence that alternatively certified teachers tend to leave teaching faster and to be employed in higher-need schools. Another important statistic is that 37 percent of African-American and 46 percent of Hispanic teachers report being satisfied with their pay, compared with 52 percent of white teachers.

Social justice issues are especially key in mathematics education, because poor math skills can limit students' access to higher education and to jobs. A 2000 NCTM report noted that achieving equity in mathematics education is a fundamental challenge facing mathematics educators.

Tonya Bartell in 2011 did a study in which she had in-service teachers who were taking a graduate course engage in lesson study[1] about social justice issues. She examined how their conversations about teaching mathematics for social justice changed over time.

She suggested these thinking points:

- "What does it mean to teach mathematics for social justice throughout a school year?"
- "What might this look like within and across multiple units, or within a single lesson?"
- "What would it mean to adequately contextualize this social issue over time?"

These problems are complex, and solving them is not easy. ASU has a detailed plan to recruit more women and minorities, but the results are showing themselves only slowly: there are no Native American math faculty and just 2.1% are Hispanic, while over 5% of the resident population in Arizona is Native and over 30% Hispanic.

Milners said that the last time he taught Geometry for Teachers, a fifth-semester undergraduate class, he had a group of 30 students, three of whom were Native American. He was delighted to have them. As the semester went by, one started talking to him about the difficulties he was having; one disappeared despite efforts to contact him; and one failed. Milner struggled to figure out what to do to change this.

Awareness is a good starting place. Mathematicians need to play a major role together with mathematics educators. We all must learn to take responsibility for addressing the problem.

The diversity gap

Teacher-student demographic gap

• VT, ME, WV:	4%
• DC, MN, MD, KS:	21%
• MI, NE, AR, KS:	22%
• OK, RI, VA, FL, CO:	27%
• NM, GA, LA:	29%
• AZ:	34%
• IL:	35%
• NV:	41%
• CA:	43%

Source: Teacher Diversity Matters, A State-by--State Analysis of Teachers of Color, Center for American Progress, November 2011

This table shows the difference between the percentage of *students* of color and *teachers* of color for a selection of states. The last four states have the biggest difference of all 50 states and the District of Columbia: This discrepancy is important not only because of social justice, but because of the impact it has on the learners themselves.

[1] "Lesson Study" is a professional development process that Japanese teachers developed to systematically examine their practice, with the goal of becoming more effective. This examination centers on teachers working collaboratively on a small number of "study lessons." http://www.tc.columbia.edu/lessonstudy/lessonstudy.html

2 Perspectives

Educating mathematics teachers is a multidisciplinary effort. Teachers, mathematicians, chairs, state administrators, and math-science curriculum consultants all have their own perspectives on the challenge, and to make progress, all these perspectives need to be considered. Speakers from each of these groups shared their experiences.

Teachers Reflect on their Interactions with Mathematics Departments

Four teachers shared how their experiences with mathematics departments affected their later teaching. Breedeen Pickford-Murray teaches at the Bay School of San Francisco and graduated from Scripps College with a bachelor's degree in mathematics and classical studies. Andrea LaGala Lamb of the Waltham Public Schools used to teach high school math and is now an elementary math coach/specialist. She studied secondary education and mathematics as an undergraduate at Boston College. Allison Krasnow taught elementary school, then middle school mathematics, and is now on special assignment in instructional technologies for K-8 math and science. She's also a Math For America master teacher, and she got her degree from the University of California, Berkeley. Stefanie Hassan is a middle school math teacher in the Little Lake School District who is now working on designing curricula to implement the Common Core.

Andrea LaGala Lamb of the Waltham Public Schools used to teach high school math and is now an elementary math coach/specialist. She studied secondary education and mathematics as an undergraduate at Boston College.

At a retreat for BC students about career discernment, Lamb was asked, "What are you good at? What does the world need you to do? What brings you joy?"

She then took the typical higher math courses, like analysis, abstract algebra, and number theory, but she struggled with them and didn't enjoy them. While this taught her to persevere and take risks, it also left her wondering whether she'd made the right decision. The support of a key mentor got her through.

She did a practicum in the Boston Public Schools starting in her sophomore year and loved it. She loved asking herself, "How would I teach this? How would I assess it? How would I use my data to inform how I taught it again?" The best hour of her day was discussing classroom issues at the end of the day with her peers—but it was only an hour.

When she took a math methods course, she felt as though she had found the haven she was looking for. It connected her majors, deepening her understanding of the mathematics that was relevant to her teaching. It gave her the opportunity to explore technology, plan units and lessons, create assessments, discuss classroom management, and make mathematical connections to practice. Additionally, the professor modeled effective teaching practices with student-centered, engaging activities that served as a model for her own later teaching. But it only lasted a semester, and it left her wanting more.

When she began student teaching, she learned an enormous amount from her cooperating teacher. But the gap between her experience in the classroom and what she was learning in her math courses frustrated her.

She had three math courses left to take in the second semester of her senior year, but none of the ones available seemed like they would serve her teaching. She got special permission to

take Fundamentals of Math I, a course intended for elementary education majors that wasn't supposed to count for a high school education major. The course used two books, *Knowing and Teaching Math* by Liping Ma and *Mathematics for Elementary Teachers* by Sybilla Beckmann. The texts explored the foundations of math that were important for classroom teaching, like the number system, place value, the standard algorithms.

In that class, she finally got a deep look at how students come to learn these concepts. She read vignettes that showed what students were thinking and talked about the misconceptions and how one could address them. She analyzed student work and talked about how one could help the students better comprehend the content. She was given opportunities to reason, to make conjectures, to prove and argue, to solve problems and to find patterns. She investigated the ideas through a wide range of models, visuals and representations.

This was the most helpful course she took, though it didn't even count as part of her math major.

Breedeen Pickford-Murray, at the Bay School of San Francisco, graduated from Scripps College with a bachelor's degree in mathematics and classical studies. While working as an educator at the Museum of Flight in Seattle, she obtained her teaching credential at Seattle Pacific University.

She entered Scripps College after having had a very disheartening experience with math in high school. The teaching methods were very traditional, and she had uninspired teachers.

When she began Scripps, she took a math placement test, and, she said, "Thank goodness I didn't place out of math, or I definitely wouldn't be here today." Taking math classes there, she came to realize that math is supposed to make sense, and she discovered that mathematical ideas are all connected on a fundamental, deep level. She was inspired.

Breedeen Murray wrote a blog post about her upcoming CIME talk and asked her colleagues about what had been valuable for them from their university experiences in mathematics. Below are some of their comments.

What teachers got from their university education

- One teacher said the most valuable thing was having learned the "big picture of how 'algebra' works, the beauty of math, and cool stuff to toss out as teasers."
- Another said that the courses "made the beauty of math more apparent… That it wasn't a series of algorithms to be memorized, but a creative endeavor."
- "Some of what I learned was ways to structure a class to focus on understanding: problem sets filled with proofs, open book tests and class as work time."
- "Mount Holyoke has an awesome class called Explorations in Laboratory Mathematics (they wrote a book for it). A major focus is how to write papers in mathematics, so this is the class I learned to typeset in TeX, a skill which I use all the time. It is also an entire class dedicated to exploration where students take their own path and write up discoveries, which was awesome."
- "First, the majority of my math classes taught me how not to teach. Too much lecturing. Too many book problems. Almost no interaction. On the other hand, my later education classes taught me that too much group work is just as bad if not worse. There's a fine line somewhere between those two extremes."

Allison Krasnow earned her degree from the University of California, Berkeley. She then taught elementary school, then middle school mathematics, and is now on special assignment in instructional technologies for K-8 math and science. She's also a Math For America master teacher.

In grade school, she was always accelerated in math, and by junior year, she had exhausted all the math classes her school offered. So she took an independent study class "with a bunch of nerdy boys." She swore she'd never take another math class, and she didn't. She didn't want to be like the people she associated with being good at math.

In college, an urban education class inspired her to become a teacher. Once she started teaching, she was frustrated by how little math her colleagues knew, and she realized, "Oh, I'm actually better at math than most people." After a few years of teaching elementary school, she heard Deborah Ball speak on pedagogical content knowledge, and she realized, "*That's* what I'm good at." Around the same time, she bumped into the one amazing high school math teacher she'd had who had encouraged her to do math at the time. He'd told her, "If anyone I've ever had should be a math teacher, it's you." She was convinced that mathematics teaching should be her career.

She got her mathematics teaching credential very quickly, and then she pursued professional development to increase her content knowledge. She did several month-long projects with the Bay Area Mathematics Project; she went to the Park City Mathematics Institute; and she worked with a teacher at the Urban School, Henri Picciotto. One of the key things that came from those experiences for her was finding that math became a social endeavor. "It was no longer me sitting and learning, surrounded with people I perceived as being really different from me," she said.

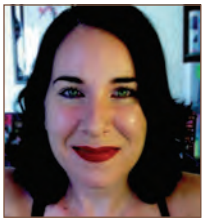
She now teaches a pedagogy class at UC Berkeley for undergraduate STEM majors who are considering becoming math teachers. She often thinks about how you get people like her, who are not traditional math students, to stick with it.

A key skill, she says, is to know formal math but also to see it from the perspective of what students should see. She offered an example from when her son was 3.5 or 4 years old. This is a picture from a Lego book:

Her son said, "This has nothing to do with Legos. Why is it in a book?" She asked him more, and he said, "Mom, I'm supposed to give you five hugs and kisses!"

This is why pedagogical content knowledge is so important. You have to understand how different people might interpret something really differently from how you might expect.

A final tip she offered is to recognize and celebrate creative insights. As a high school student, one of her teachers had a Rodin sculpture that he would place on the desk of a student who made a creative observation. She still remembers the day when she got that sculpture. She now does it in her own classroom: when students have a creative insight, she gives them a bib with a picture of Leonardo da Vinci on it to wear for the day.



Stefanie Hassan is a middle school math teacher in the Little Lake School District who is now working on designing curricula to implement the Common Core.

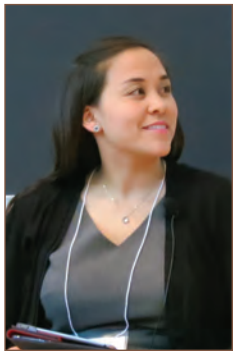
As a student, she took AP Calculus and although she couldn't solve an equation to save her life, somehow she earned an A–. But when she entered college, her placement showed that she only qualified to take a "Math for Liberal Arts Majors" class.

She became a speech pathology major, staying far away from math. But after college, while looking for a graduate program, her mother suggested substitute teaching as an easy way to make cash. She started subbing, and the principal asked her to sub for a math class for a month. The principal considered her to be a good math teacher because she could control a group of 12-year old students and she actually knew how to do the math. She didn't know it well, and she certainly couldn't explain anything, but she could do the steps. The principal offered her a full time job the very next school year, even though she had no credential and no math background.

She had to get a credential to keep her job, but the credentialing program lacked an in-depth study of the math she was teaching. It didn't even include a cross-grade-level examination of standards to learn what material was OK to skip and what wasn't.

She realized that although she was becoming a prepared math teacher, she wasn't necessarily good. Her principal said she could either teach summer school or she could attend a summer institute with Wu. She was 23 years old, and she asked, "Which way will I make more money?" Happily, the answer was attending Wu's summer institute.

She learned the importance of good definitions, such as for a fraction (a point on a number line) or for congruence (it's not same size and same shape; it's that one maps rigidly onto the other). She had never known these definitions, and suddenly with them, math began to make sense. She also learned why a negative times a negative is a positive, and why we invert and multiply in order to divide.



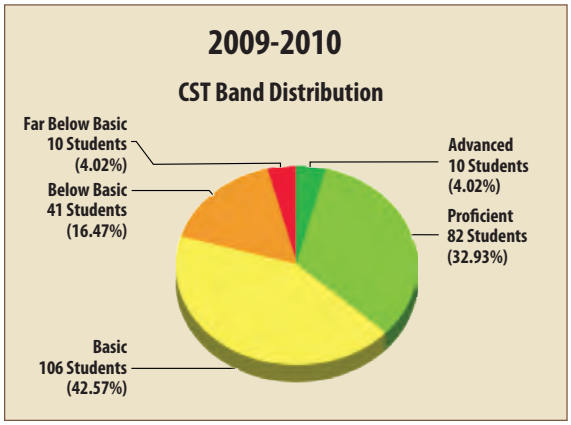
Above:
Andrea LaGala Lamb
(Waltham Public Schools)

Left: Stefanie Hassan
(Little Lake School District)

Right:
Davida Fischman
(California State University)

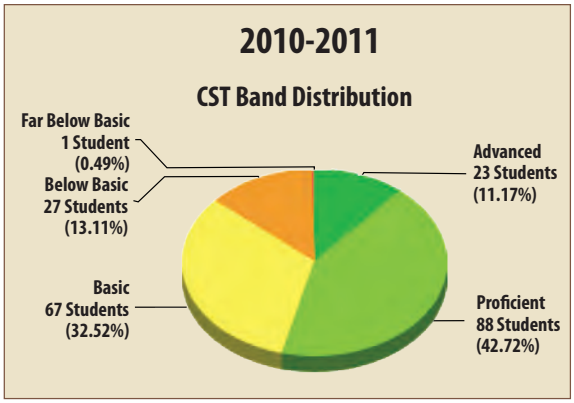
In the next semester, she saw the impact the summer institute had on her. Even though she couldn't implement everything she'd learned in those three weeks, she had at least developed a more critical view of the curriculum. She also became more critical of herself as a teacher, asking: Is what I'm saying making sense? Is it mathematically correct? Will it aid in the understanding of topics in the next grade level?

Hassan was in a low socioeconomic area with lots of English language learners and students on free lunch, and the students were pretty constant from year to year. The first year after taking Wu's summer institute, the performance of her students and those of a veteran teacher she worked with were little different from the previous year:



Only 37% of her students would have gotten a D or better.

After attending another summer institute, she worked with other teachers, preparing lessons together, and they used a curriculum she developed. These were the results for all the 7th graders, except for the top students who were doing algebra:



Just a year later, 54% would get D or better. Her principal loved it: So much green! So little red!

The next year, the other teachers did everything she suggested, and 73% of the students were proficient or advanced. The year after she left, it dropped back down to 52%.

Her conclusion is that professional development made the difference, along with her strong desire to do better. Another conclusion made was that preservice teachers need content-based courses on the math that they will teach.

> **"The young university student [was] confronted with problems that did not suggest . . . the things with which he had been concerned at school. When, after finishing his course of study, he became a teacher . . . he was scarcely able to discern any connection between his task and his university mathematics . . ."**
>
> **– *Felix Klein,* Elementary Mathematics from an Advanced Standpoint, *1933***

Mathematicians Reflect on their Interactions with Mathematics Education

Four mathematicians shared their experiences working on math education. Sunita Vatuk of the City College of New York has worked extensively on the professional development of teachers in New York City, New Jersey, and north and south India. Dev Sinha of the University of Oregon taught in the Program in Mathematics for Young Scientists (PROMYS) early in his career and is a content leader at Illustrative Mathematics and a founding partner of the Oregon Math Network. James Epperson of the University of Texas at Arlington has worked on the Texas Mathematics Standards and Assessments as well as on teacher professional development. And Yvonne Lai of the University of Nebraska-Lincoln is redesigning the university's mathematics courses for prospective secondary mathematics teachers.

Sunita Vatuk of the City College of New York is a mathematician who has worked extensively on the professional development of teachers in New York City, New Jersey, and north and south India.

The idealism that underlies Vatuk's interest in mathematics education traces back to childhood experiences with her Indian family. She had relatives who some Westerners might consider to be living in poverty, but their mental lives, which included mathematics, were anything but impoverished. This sustained them and gave them strength in a way that has nothing to do with wealth. She saw that for some people, mathematics is transcendent, and it can be sustaining regardless of whether it's a vocation or an avocation. That belief, combined with a deep concern about equity in math education, is at the heart of why she does this work.

Vatuk feels she followed a path common for mathematicians who get involved in schools. Her first extended experiences with children were in the Princeton school district as a graduate student. She wanted to work with math teachers, but the district perceived academic mathematicians as unqualified to work directly with teachers or in classrooms. So instead, she taught an enrichment class for upper elementary students, teaching them material that was peripheral to what they were doing in school.

As an example of her approach, she presented a wide variety of problems whose solutions hinged in some way on parity, in language the children could understand. To the students they initially appeared to be unrelated, but after several activities, the kids started saying, "Oh, this is about odd and even, too!" Many spontaneously saw the deeper structure—noticing, for example, that combining reflections and rotations was similar to adding odd and even integers, or multiplying negative and positive integers.

She was very excited by the success of these activities with the students. But she was unable to convince the school to let the mathematics graduate students work with teachers, and as a result, the program stopped when the grant ended. She was left feeling that her efforts had been a failure, because they didn't change the way math classes were taught in that school.

Recently, as part of a Math Science Partnership at Rutgers University, she conducted interviews of mathematicians, participants and math educators in the program. She also collaborated with the mathematicians in planning the content classes and observed all three groups in the classroom. She was struck by how people who were hardworking, committed to similar goals, well-disposed and trying hard to listen to each other, nevertheless perceived what was going on in the classes differently.

For example, mathematicians often work very hard to choose a sequence of problems that tell a compelling story. But the care behind that selection process is often invisible to the students, whether they are undergraduate students or teachers. On the other side, when working with teachers, the mathematicians didn't always see that it was commitment and concern for students that was the primary motivation for some teachers, and that this concern, if understood, could be harnessed to motivate their math learning.

The content of some of the classes was, like her own enrichment curriculum, unconnected to textbook school mathematics. Those were often the teachers' favorite classes, but they couldn't figure out how to integrate them into their lessons, because of the stress put on the state exams and the consequent need to teach particular topics.

All of these experiences have influenced Vatuk's current approach to working with teachers. She argued that mathematicians who care about teachers are well-placed to help them find and understand the mathematical connections that might be invisible to them, in the choice of representations to use in presenting particular topics, the choice of examples to use, the sequencing of problems, and so on. She supports the work of her colleagues who teach math methods courses by incorporating them into her math content classes for secondary school teachers at City College. But she also feels that she has a corresponding responsibility to explicitly connect the mathematics she introduces to teachers with the material they are required to teach.

The beautiful opportunity in defining negative exponents

When students are exposed to the idea of negative exponents, they can be shown that x^{-n} must be defined as $\frac{1}{x^n}$ if the law of exponents is to continue to hold. **Dev Sinha** of the University of Oregon pointed out that this is a more sophisticated kind of reasoning than students have been exposed to in earlier grades in the Common Core, which relies for example on pictures to reason about fractions. It is also an opportunity, Sinha argued. In early grades, students are now being taught to reason with models (a big improvement from the past), but reasoning according to properties in middle grades gets closer to what mathematicians do. What an opportunity to share!

Wason Selection Task

Dev Sinha

Effective teaching requires understanding how people think about and learn mathematics. A key experience for Dev Sinha was learning about the Wason Selection Task, a standard tool in cognitive science.

Two playing cards are chosen from a red deck and two are chosen from a blue deck. The four cards are then placed on a table, two face down (showing the cards' colors) and two face up (showing their values). Of the face-down cards, one is blue and one is red. The two face-up cards are a 2 and a king.

People are then given the claim that if a card is from the red deck, it's a face card, and they are asked which cards they have to turn over to see if the statement is true.

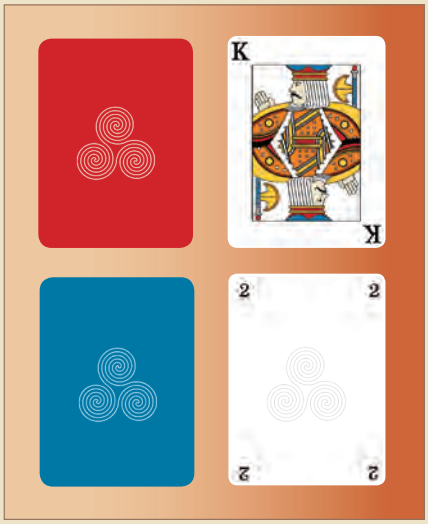
The correct answer is that they need to check the 2 (because if it's red, that will violate the claim) and the red card (because if it isn't a face card, that will violate the claim). The other two don't matter. Not surprisingly, people don't do well on this task.

But consider this logically equivalent task: There's a bouncer at a bar, and of course the rule is that you can't drink unless you're over 21. There's a 50 year old, a 12 year old, someone with a beer, and someone with a lemonade. Who do you approach to see if the rules are being followed?

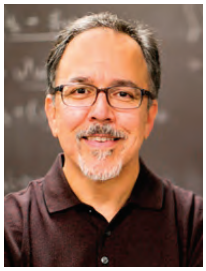
Everyone can do that (check the 12-year-old and the person with the beer) – even though logically, it's equivalent to the card task that people found difficult.

Previously, it had never occurred to Sinha that logically isomorphic problems could have such a different effect on people, and he didn't want to believe it. But the finding has been reproduced many times over 50 years.

This made him realize that classroom context can seriously affect students' ability to reason.



The claim is that if a card is red on one side then it has a letter on the other side. Which cards must you turn over to see if the claim is true?

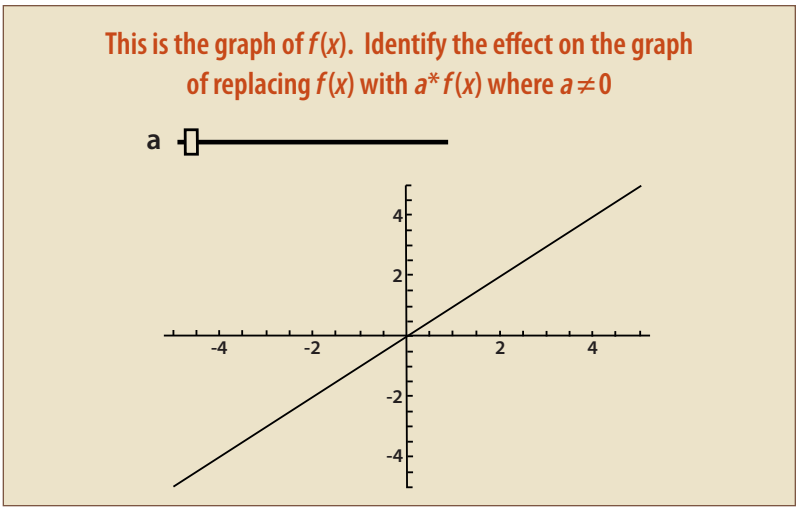


James Epperson of the University of Texas at Arlington has worked on the Texas Mathematics Standards and Assessments as well as on teacher professional development. He argued that mathematicians will only be effective in mathematics education if they can communicate their ideas to a broad audience.

His first experience in mathematics education was as a postdoc at the Dana Center where they had a grant to rewrite the Texas standards. He sat in on a committee on middle school probability and statistics standards. The committee members were using the terms "experiment" and "event" interchangeably. He tried to distinguish between the two terms during the group discussion, but as a newcomer, he sensed resistance to his clarifications. He then wrote a respectful, detailed email to the chair and explained what the differences were and gave examples. She was extremely thankful and said, "I would never have guessed that you were a mathematician, because your reaction was quite calm!"

Later, he wrote a policy brief about mathematicians working in mathematics education, designed to help mathematics departments evaluate the contributions of mathematics faculty to mathematics education when making tenure decisions. The policy brief has been used in several such cases in Texas and outside of Texas. From that work, he got to know people at the Texas Education Agency and developed trust with the director of mathematics. As a result, officials there would come to him with questions. These solid relationships are key for real communication between mathematicians and policy makers.

He has also been part of committees to evaluate state-wide mathematics assessments, checking exam questions for preciseness and clarity. These questions can be starting points for professional development work with teachers. For example, consider this problem:

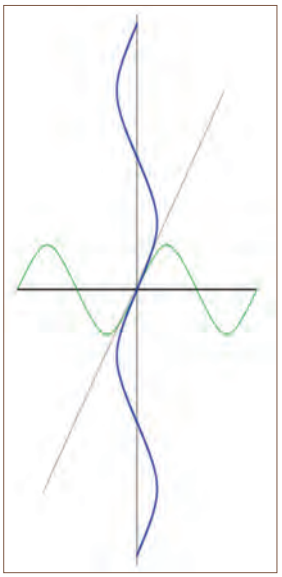


This problem comes with an animation that changes the graph as a changes. Students will naturally say that changing a rotates the graph, because the animation looks like that. Teachers have to think about what a rotation means and whether this is right.

Yvonne Lai, of the University of Nebraska-Lincoln, is redesigning mathematics courses for prospective secondary mathematics teachers.

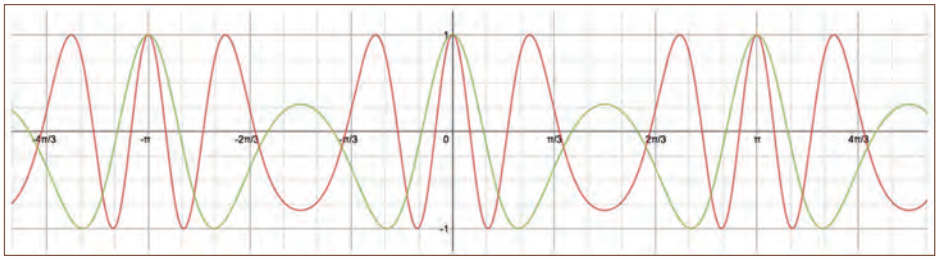
Lai did her undergraduate work in math at MIT, earned a Ph.D. in mathematics at the University of California, Davis, and then did a postdoc at the University of Michigan in 2008, in hyperbolic geometry and geometric group theory. Next, she worked as an Assistant Research Scientist in the University of Michigan School of Education and now teaches at the University of Nebraska-Lincoln. She specializes in mathematical knowledge for teaching.

Lai highlighted task design as an area where mathematics education benefits from collaboration between mathematicians and educators.



Some tasks Lai has used in capstone courses that have produced particularly interesting thinking and conversation from her students include:

1. Explain the vertical line test in terms of the definitions of coordinate, graph, and functions.
2. At right are the graphs of $y = sin(x)$ and its reflection over the line $y = x$. Explain how the construction of the reflected graph adheres to the definition of reflection while not appearing to be congruent to the original graph.
3. Explore the functions $f(x) = cos(k \sin(x))$ and $fx = (\sin kx)$for different constants k. What do you notice about the periods of these functions? What do you notice about the relationship of the periods to the constant? Explain your observations in terms of the definitions of radian, period, sine, and cosine. (Variation on task discussed in Thompson, Carlson, and Silverman (2007))



In teacher education, knowledge must be built collectively; no one party holds all the knowledge necessary to prepare teachers to teach mathematics or to manage the interactions among students, teacher, mathematics, and local system policies and idiosyncrasies. Mathematicians may understand mathematics, but probably don't know the curricular structure of mathematics, what motivates K-12 students or teachers, how to use methods and mathematics courses to complement each other, or school district policies. The mathematics tasks above resulted from collaboration with teachers in local school districts, methods instructors, and education researchers. In addition to task design, areas where mathematicians and educators can collaborate profitably potentially include:

- The close study of mathematics in the context of K-12 teaching, including the search for pattern, structure, connection, representations, and generalization
- Conveying and clarifying the nature of mathematics
- Studying the nature of mathematical argument from pre-kindergarten through graduate school
- Writing and reviewing curricular and assessment materials
- Identifying what it takes to teach mathematical knowledge for teaching, the sources for learning what it takes, and why mathematical knowledge is not the whole picture.

Chairs of Math Departments Reflect on Mathematics Education

Math department chairs shared their experiences working on math education. Brigitte Lahme is a professor at Sonoma State University, where math education is a major focus. Steven Rosenberg is at Boston University, which is primarily a research department. And Sol Friedberg is at Boston College, which has a more diverse emphasis.

Brigitte Lahme of Sonoma State University (SSU) said that math education plays an important role in her department. The entire university has between 8,000 and 9,000 students, with about 170 math and statistics majors. Of those, between a quarter and a third plan to become secondary teachers. In addition, elementary education students take classes in the math department. One faculty member in the math department is a formal math educator, plus two mathematicians work primarily in math education. The department of education also has one elementary and one secondary math educator.

In 2011-2012, the California State University system educated 49% of the newly credential teachers in California. An additional 7% were educated by the University of California, and 44% came through private/independent schools.

Undergraduate preservice elementary teachers at SSU must take nine units of math content courses (modern geometry, elementary number systems, and data, chance and algebra). In addition, in their fifth, credentialing year, they take a three-unit math methods course.

Secondary teachers typically earn a math major. They take 54 units of math courses, including six units specifically for future teachers. They also take nine units of education courses, plus a three-unit math methods class in their fifth, credentialing year. That means that mathematicians can have a strong positive influence during those 54 units, if they are careful about how they use the time.

In addition, many of the faculty at SSU also work with in-service teachers. SSU has had California Math and Science Partnership grants from 2000 to 2013. SSU faculty are also involved with the North Bay Math Project. In addition, they've worked with educators to design a master's program in math education, along with other projects in the school of education.

Mathematicians' work on math education also influences their work with math majors. The math educators have introduced lesson study to the mathematicians. The department has revised tenure criteria to reflect the value it places on work in math education. The mathematicians talk to other mathematicians about changing the experiences we give to our math students. Math faculty are also involved in professional development work and other teacher education activities, such as IllustrativeMathematics.org.

Steven Rosenberg of Boston University (BU) said that the emphasis of his department is primarily on research, and it produces many fewer teachers than Sonoma State University. In this context he has found it essential to use great care in working with colleagues in the school of education. "There are many ways of doing it wrong," he said.

Rosenberg offered the opinion that at BU, mathematicians who work with math educators should first have tenure. He encouraged educators who approach colleagues in the math department to keep that in mind. Work in math education is respected in a math department if it's funded, but still not as much as research in pure or applied math or statistics.

His department has had a powerful leader in math education, Glenn Stevens. Stevens works effectively two full-time jobs, one as an educator and one as a researcher, and as a result, his work has earned respect. Because of his work, the department has had a Math-Science Partnership, Noyce fellowships, and has established a branch of Math for America. These grants have made other chairs who aren't particularly focused on math education come to see the value of this work quite genuinely.

Part of Stevens' work has been to build a Boston-area math community consisting of middle and high school teachers, math educators and mathematicians. BU holds colloquia that are suitable for math teachers, and both mathematicians and teachers love them. Deep mathematics has come from some of these talks.

Math educators will find mathematicians useful as content experts, not pedagogy experts. If a mathematician doesn't understand this, math educators might want to avoid them. But finding the right people will substantially aid professional development, Rosenberg said.

Rosenberg recommended getting mathematicians involved in panels on education policy committees, because they are given a lot of respect. But it is key to find ones who are good listeners, not ones who have decided that math education is failing because not everyone is a mathematician.

If a math department chair isn't sympathetic to math education issues, faculty who are sympathetic have the job of educating the chair. They should point out the similarities between the missions of math research and math education, as well as opportunities for colleagues to work on good math education topics.

Peter Trapa (University of Utah) at the plenary session of mathematics department chairs. Discussion focused on how mathematicians can work in math education and collaborate with math educators; how math departments can approach tenure and advancement for individuals involved in math education; and the role of math departments in providing math-content courses to future teachers and in professional development. It also included departmental considerations about math-content courses for teachers, e.g., which courses count for a math major, pedagogy in math content courses, use of lesson study, and implications for other math department courses. Finally, the group considered the role of department chairs in encouraging mathematics faculty to get involved with future and present teachers.

Sol Friedberg of Boston College described the work he has done as a chair to support math education.

As chair, he's responsible for everything in the department, so he cares about every single student who takes a math class. He cares about the student who hates math and will take one class and then become a citizen and vote on the future of our country. He cares about the majors, those who will become mathematicians and those who will become doctors or lawyers. He cares about our future teachers. He cares about the graduate students, those who will do bang-up research and those who will find a different path. And he's responsible for managing resources to meet the needs of all those individuals.

Math for future teachers is expensive, particularly for high school. His department doesn't have enough people graduating each year as future high school math teachers to have dedicated courses solely for them, but those students have specific needs that aren't met by courses on complex variables or Galois theory. The solution in his department is to teach dual-purpose courses. For example, they have a course reading Euclid's Elements in which students TeX up the arguments from Euclid and present them to the class. There's a lot of very sophisticated mathematical reasoning in the course, and his colleagues as a result are quite comfortable with this carrying upper division credit and serving as an elective that counts towards the math major. The department also offers an upper division course in mathematical problem-solving that serves as a review of a lot of high school math from a higher level.

Sol Friedberg supports math education in many ways. He talks to his dean, communicates the values of math education to the department and supports his math education colleagues. In Fall 2014 he chose to teach Math for Elementary Teachers.

A challenge is that the coin of the realm in evaluation of faculty is publications. A professor can spend a lot of time working with future or in-service teachers but not have this work lead to any publications, and that can tie Friedberg's hands in supporting the person for salary increases. Although it's helpful to get grants, Friedberg also wishes there was a professional culture in which mathematicians could publish their experiences in math education in a way that's genuinely useful. That would both strengthen the math education community and aid mathematicians involved in math education work in getting suitable recognition for their work.

It is very important to maintain good relations with the education school, Friedberg said. At BC, the priority of the education school is English language learning, not mathematics, and Friedberg considers pushing math education at the education school to be part of his job.

He has reached out to some of his colleagues who are less active in research and encouraged them to work with future teachers. He points out, though, that you have to be thoughtful about whom you reach out to, because modeling good pedagogy is important. But he also says that people are often willing to try ways of teaching that are different than they're used to.

He has worked to build broad support for math education in the department. He has convinced a lot of his colleagues to visit high schools, and that experience has opened their eyes to the complexities of teaching. Even those who don't get involved in math education themselves are more supportive of their colleagues who do. Teachers also come to campus. These connections can change the culture of the department.

The School of Education and the Mathematics Department jointly run a math education colloquium. They choose speakers with a variety of experiences and perspectives. They've had mathematicians, such as Sybilla Beckmann, Jim Lewis, Hung-Hsi Wu, and Bill McCallum, along with math educators, such as Alan Schoenfeld, Karen King, and Diane Briars. When Friedberg invites young math faculty who are doing cutting-edge research to the colloquium, they come and they're excited about it. They understand that there's a wider world out there than just math research. They were especially energized by Bill Schmidt's talk about social justice.

In communicating with administrators, Friedberg often uses the language of social justice because that's part of the Boston College mission statement. He recommends explicitly referring to the institution's mission statement to get administrators to understand and value your efforts.

Discussion during a break, from left:

Andrea LaGala Lamb
(Waltham Public Schools)

Sunita Vatuk
(City College of New York)

James Epperson
(University of Texas at Arlington)

Dev Sinha (University of Oregon)

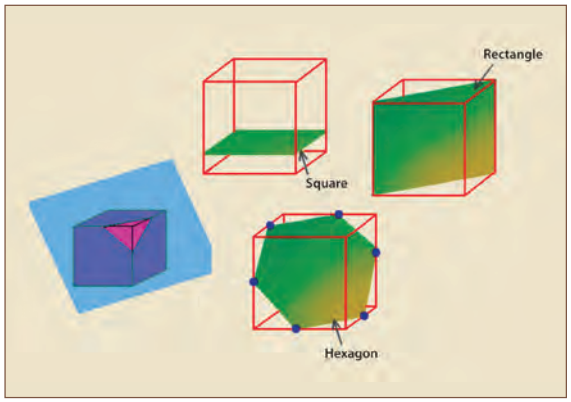
Brigitte Lahme (Sonoma State University)

Math/Science Consultant Reflects on Mathematics Teacher Education

Prior to working at the University of Utah and the Center for Science and Mathematics Education, **Maggie Cummings** was the math and science consultant for the Jordan School District in Utah, a very large district with 87,000 students. There she oversaw mathematics and science instruction and professional development for grades K–12. She realized that the content knowledge of many teachers was weak, and she felt that working with an institution of higher education was key to improving the quality of education for all students. She turned to Hugo Rossi at the University of Utah for help, and together they developed a cohort-based Masters of Science for Secondary School Teachers (MSSST) for practicing teachers. MSSST teachers are now actively sought for various leadership positions and professional developments across the state.

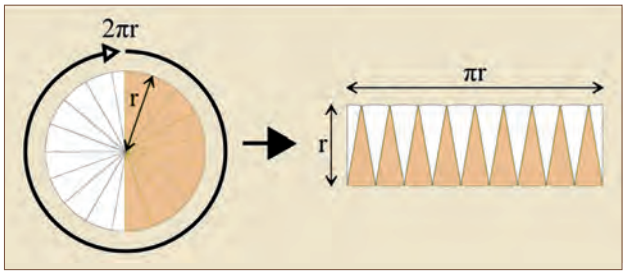
When teachers in the southeastern portion of Utah were having trouble implementing the new Common Core State Standards, their director of curriculum turned to Cummings for help. A 40-hour professional development program was organized for math teachers and Cummings asked four of the master teachers to lead various grade level discussions.

Here are two illustrations of what these master teachers worked on with them.

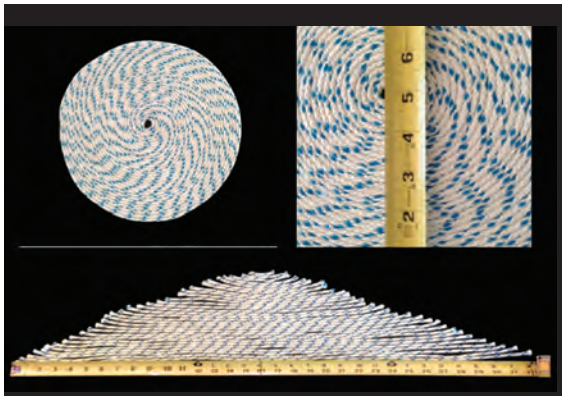


Question 1: What two-dimensional figures result from slicing a cube with a plane? This is related to the seventh grade standard, "Describe the two-dimensional figures that result from slicing three dimensional figures, as in plane sections of right rectangular prisms and right rectangular pyramids." The master teachers brought play-dough and the teachers made little cubes and sliced them with dental floss. The teachers explored questions like: What would happen if you made slices parallel to some face? Perpendicular? Can you make a square, pentagon, hexagon, a seven-sided figure? The discussions were both mathematically rich and pedagogically driven.

Question 2: What is the relationship between circumference and area of a circle? Or, explain why πr^2 gives the area of a circle. This connects with the Common Core Standard, "Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle."



Most teachers were familiar with this representation of how the circle is put together.



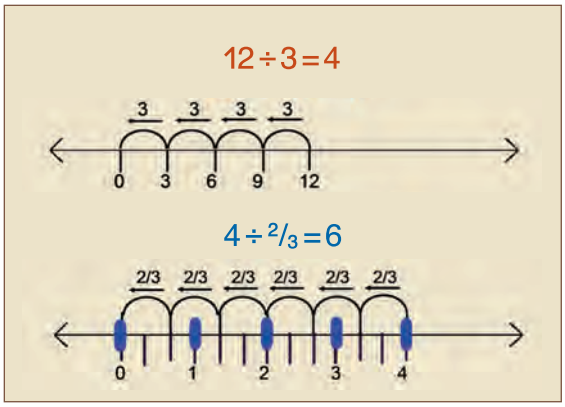
However, most were not familiar with other historical methods like the Egyptian argument illustrated on the right.

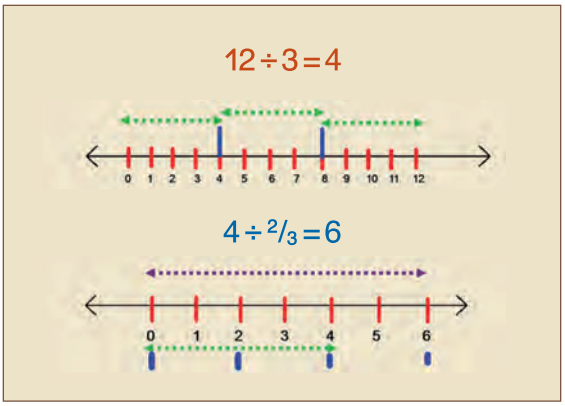
On another occasion, Cummings took a group of elementary and secondary preservice teachers to Monument Valley for a week-long professional development designed to benefit both the preservice teachers she took with her and the in-service teachers at the schools. The preservice teachers spent the whole day with assigned practicing teachers learning about issues teachers face in remote schools. In-service teachers received grade band professional development each day.

In-service teachers wanted help that week in developing better strategies for ensuring that students connected ideas from one grade level to the next. Of particular concern was division. Teachers learned that division with fractions works the same way as it does with whole numbers, and it's key that students come to see that the two are consistent with each other. Division can be thought of in two ways. One is as measurement, so that $12 \div 3 = 4$ can be seen as, "I have twelve, I pull out groups of three and I end up with four groups." Similarly, $4 \div \frac{2}{3}$ can be seen as, "I have four, I pull out groups of $\frac{2}{3}$, and I end up with six groups."

The second way to view it would be to see $12 \div 3 = 4$ as meaning, "I have twelve, I split it into three equal groups, and I have four in each group." Similarly, $4 \div \frac{2}{3}$ can be seen as, "I have four, it's two-thirds of a group, and the entire group would have six."

This discussion was eye-opening for all the teachers.





State Administrator Reflects on the Role of Mathematicians and Mathematics Departments in Teacher Education

Diana Suddreth, STEM Coordinator at the Utah State Office of Education, stressed the importance of mathematicians forming connections both with state administrators and with the public.

In Utah, she reported, they work closely and well with universities on projects including licensure, development of new curricula, and designing professional development opportunities for teachers — but she's heard that this is an anomaly. She credits the close relationship with the University of Utah to Hugo Rossi. Seven years ago, shortly after she started the position, Rossi came into her office and said, "I've got some ideas." They discussed them, and he ended up getting professors around the state interested in working on them. This laid the groundwork for close relationships.

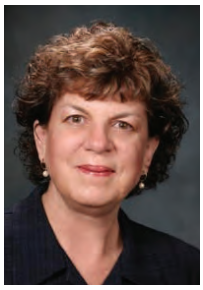
These relationships were especially helpful in implementing the Common Core in Utah, where there was some backlash, just as in many states. Suddreth had a file folder full of letters from mathematicians in Utah explaining why they support the Common Core. She could pull those out to explain to parents her reasons for supporting it and to show that it wasn't just mathematicians nationwide who support the Common Core, but mathematicians in Utah.

She pointed to the following key ways that mathematics departments can contribute:

- An important starting place is to recognize that everyone involved in mathematics education has the same goal: students' mathematical understanding. This makes it far easier to work with all the other parties involved.
- Mathematicians can build connections in their teaching both among areas of mathematics and to other fields. Her discussions with leaders in industry have particularly stressed this, explaining that they need workers who know how to apply their mathematical skills to real-world problems.
- A key role for mathematicians is to help teachers understand and remember what mathematics is really about, including problem solving.
- Mathematicians can intentionally build relationships with public education and with colleges of education.
- Mathematicians can participate in professional learning experiences with practicing teachers as well as in designing mathematics courses targeted at future teachers.

Speakers described a few key reports and programs that are relevant to math education. Linda Gojak, outgoing president of the National Council of Teachers of Mathematics, described the accreditation process for universities for teacher education *(page 38)*. **Joan Ferrini-Mundy, Assistant Director of the National Science Foundation's (NSF) Directorate for Education and Human Resources, reported on the programs the NSF offers to support teacher education** *(page 39)*. **Anna Bargagliotti discussed the report "The Statistical Education of Teachers"** *(page 42)*. **Sybilla Beckmann and William McCallum, both leads for developing the report, described the 2012 Mathematical Education of Teachers II report** *(page 44)*. **Beckmann also described the National Resource Council report, *Mathematics Learning and Early Childhood*, that describes how to effectively teach young children.**

The Council for the Accreditation of Educator Preparation



Linda Gojak of John Carroll University, outgoing president of the National Council of Teachers of Mathematics, described the accreditation of universities for teacher education in mathematics.

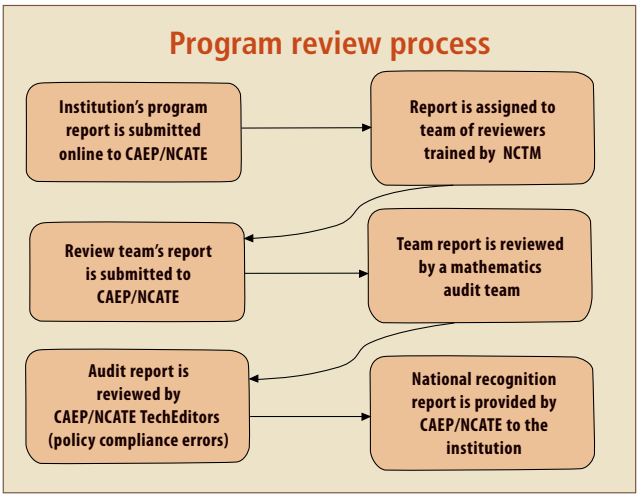
Universities providing teacher education get national accreditation through The Council for the Accreditation of Educator Preparation (CAEP), which is the merger of the National Council for the Accreditation of Teacher Education (NCATE) and the Teacher Education Accreditation Council. This is a voluntary peer review process that involves a comprehensive evaluation of the professional education unit that is primarily responsible for the preparation of teachers and other professional school personnel.

The review is based on the NCATE Unit Standards, a set of research-based national standards developed by all sectors of the teaching profession. Accreditation requires an on-site review of the unit, and a review of the individual programs within the unit, which is conducted by the state or electronically through NCATE and the specialized professional organizations.

Regrettably, even with the recent creation of CAEP, fewer than half the nation's teacher education programs are formally accredited. Some programs choose not to pursue accreditation, some would not meet the standards (particularly because they use too many adjuncts), and some complain about cost.

Current Accreditation Assessments include:

- Mathematical content knowledge (typically Praxis or a state's assessment)
- Mathematical content knowledge (GPA's, portfolio)
- Pedagogical and professional knowledge (Unit Plan)
- Pedagogical and professional knowledge (Student Teaching)
- Impact on Student Achievement (comparing pre-test to post-test)
- Pedagogical and professional knowledge and skills

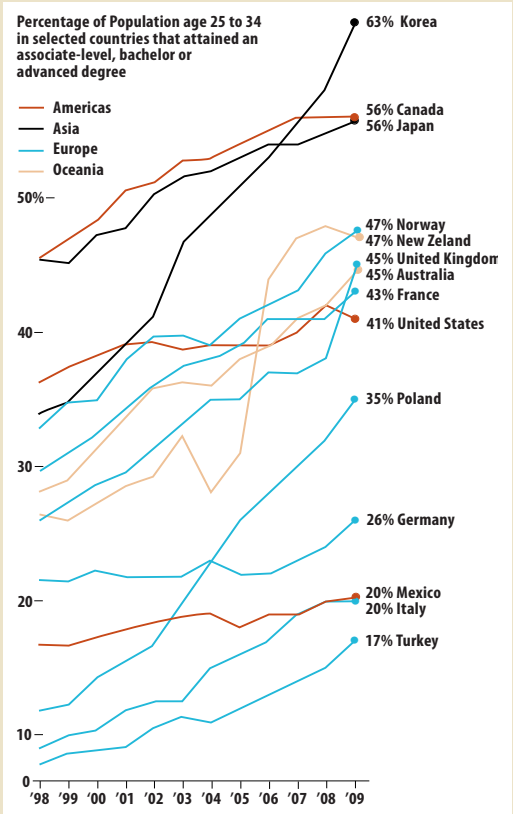


The National Science Foundation

Joan Ferrini-Mundy, Assistant Director of the National Science Foundation's (NSF) Directorate for Education and Human Resources, reported on the programs the NSF offers to support teacher education.

Given that Science, Technology, Engineering and Mathematics (STEM) education is a national priority, there is a critical need to provide quality education and training for teachers. A well-prepared cadre of STEM graduates would strengthen the nation's ability to compete in a global economy. Improving support for teachers from preservice to mature career development opportunities is a key strategy in building the STEM student pipeline. STEM teachers can also play a role in bolstering interest in the STEM fields and increasing preparedness for postsecondary STEM courses, which may lead to a rise in STEM degree retention and completion.

The U.S. trails much of the developed world in college attainment among young adults, a key measure of global competitiveness.

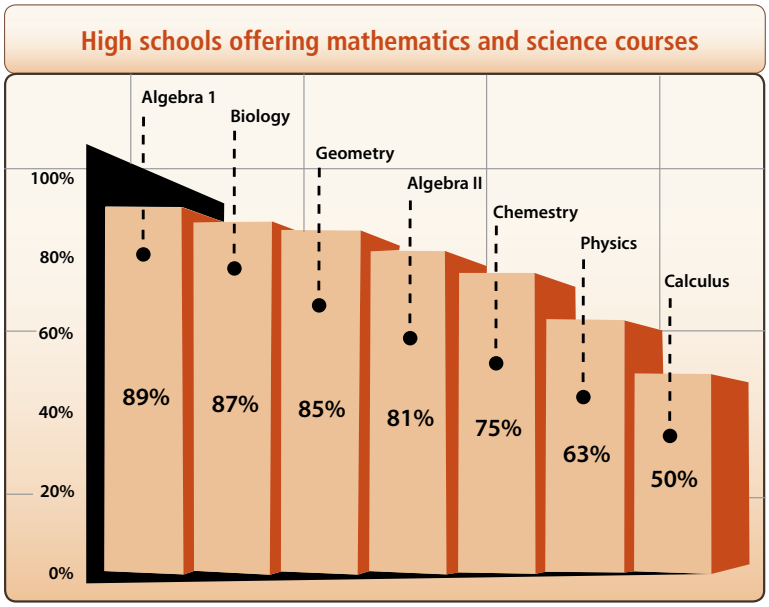


Source: Washington Post
September 12, 2011

http://washingtonpost.com/education/playing-catch-up-in-college-completion/2011/09/12/glQAegt6NK_graphic.html

One of the strategic objectives within the Obama administration's focus on STEM education is preparing 100,000 effective STEM teachers over the next decade, as laid out in the National Science and Technology Committee on STEM Education's (CoSTEM) Five-Year Strategic Plan. NSF is committed to the CoSTEM plan and future decision making will aim to meet this goal.

Mathematics is in the spotlight in mostly good ways. However, the President's Council of Advisors on Science and Technology released a report in 2012 focused on undergraduate STEM education [*Engage to Excel: Producing One Million Additional College Graduates with Degrees in Science, Technology, Engineering, and Mathematics*]. It recommended that post-secondary mathematics teaching and curricula be developed and taught by faculty from math-intensive disciplines other than mathematics, including physics, engineering and computer science. The conversation is now more focused on the Math 2025 report from the National Research Council [*The Mathematical Sciences in 2025*], particularly thinking about the future of mathematics and the role of mathematics in interdisciplinary research.



NOTE: "High Schools" is defined as public schools offering grades 10 or 11. Data in this chart represents 99% of high schools in the CRDC universe (25,030 high schools). *Source: U.S. Department of Education, Office of Civil Rights, Civil Rights Data Collection, 2011-12.*

Equity is another key concern of the NSF. For example, the Civil Rights Data Collection [Office of Civil Rights, U.S. Department of Education] shows that only half of all high schools offer calculus. If you dig further into the data, you will find that the high schools that offer calculus are not the ones serving the neediest students. We are a nation in need of an approach that will provide good learning opportunities for all students, and we're not very close to that.

The NSF does not have programs specifically focused on teacher education, although the Noyce program and the Math-Science Partnerships do include support in part of the broader efforts. Here are opportunities, however, to be creative in which program one turns to. The following areas are relevant for mathematics departments working to strengthen their capacity to prepare teachers.

1. Partnerships

- 100Kin10: This effort is run by the Carnegie Corporation in New York with strong governmental support. It is a coalition of organizations working toward preparing 100,000 new STEM teachers in 10 years.
- The Science & Mathematics Teacher Imperative by the Association of Public and Land-Grant Universities: This group includes 135 public research universities, with 14 university systems in 45 states. Collectively, they prepare more than 8,200 teachers annually.
- STEM-C Partnerships: This is the new name for Math-Science Partnerships. The C stands for computing, and it was brought into the foreground because there is a strong interest in moving computing into secondary schools as a serious subject. The basic notion is the same as MSP: supporting partnerships that promote effective K-12 STEM education.
- Robert Noyce Teacher Scholarship program (Noyce): It is focused on encouraging talented STEM majors and professionals to become K-12 mathematics and science teachers by supporting students, developing

NSF master teaching fellows, and increasing the capacity of institutions to provide innovative teacher preparation programs.

- Louis Stokes Alliances for Minority Participation (LSAMP): The LSAMP program assists universities and colleges in diversifying the STEM workforce through their efforts at significantly increasing the numbers of students successfully completing high quality degree programs in STEM disciplines.

2. Curriculum development

Note that NSF is going to be most interested in proposals that include evaluation and research, in line with its research mission. Partnering with educators is a good way for mathematicians to add this component to their projects. A great idea about a course that might work, or a summer institute, can be recast as a question or hypothesis. For example: Why would doing a five-week summer program in robotics be productive?

- Improving Undergraduate STEM Education (IUSE:EHR): This program is about increasing student retention in STEM, preparing students well to participate in science for tomorrow, and improving students' learning outcomes.
- Cyberlearning and future learning technologies: EHR collaborates with the Directorate for Computer and Information Science and Engineering on this program, which is focused on inventing and improving a next-generation of learning technologies; advancing understanding of how people learn in technology-rich environments; and promoting broad use and transferability of new genres of technology-based learning.

3. Research on teaching and learning

- EHR Core Research (ECR): This is about foundational research. This program focuses on synthesizing, building, and/or expanding fundamental research in STEM learning, STEM learning environments, workforce development, and broadening participation in STEM. In the Mathematics and Physical Sciences Directorate, you may be familiar with the core research programs in the Division of Mathematical Sciences in such areas as algebra or number theory. Such funding opportunities are consistently available year after year. The Education Directorate at NSF has never had anything exactly like that, but the ECR program aims to provide such an open opportunity.
- Discovery Research K-12 (DRK-12): This program funds the development of teacher development materials. It aims to enhance the learning of STEM by PreK-12 students and teachers through research and development of innovative resources, models and tools.

4. Faculty, staff and leadership expertise and commitment

The NSF doesn't offer much specific support here. A longshot possibility is Advancing Informal STEM Learning (AISL), which has supported TV shows and museums as avenues for STEM research on learning. AISL's aim is to advance new approaches to and evidence-based understanding of the design and development of STEM learning in informal environments.

5. Students

- Noyce is good for funding students.
- NSF Graduate Research Fellowship Program (GRFP): This is very competitive, with 14,000 applications and 2,000 fellowships. Three years ago, STEM education research was explicitly made an eligible field for this.
- Research Experiences for Undergraduates Program (REU): People don't always think of education as an REU-eligible field, but it is.

The Statistical Education of Teachers

The Common Core State Standards place a large emphasis on statistics in the middle- and high-school grades. **Anna Bargagliotti** of Loyola Marymount University says this is welcome and needed, particularly given the increasing importance of statistics in society. The shift was emblemized when *Harvard Business Review* said that "Data Scientist" was the sexiest job of the 21st century. And typically, teachers and students enjoy statistics, making it an easy "sell" in the classroom.

Statistics has key differences from mathematics. Statistical problem solving and decision-making depend on understanding, explaining, and quantifying the variability in data. Statistics uses probability to quantify variability and to build sampling distributions—but it does not study it. And statistics relies on context, whereas in mathematics, the context is often stripped away.

The Common Core makes it a very exciting time for statistics education, with many opportunities as well as many challenges.

The CCSS places some statistics in elementary school, but a limited amount. Even this small amount places significant demands on teachers, however, because they need to be able to reason statistically to lay the groundwork for their students to move successfully to the middle grades.

In middle school, the standards put a big emphasis on statistics. The main topics are statistical variability, distributions, drawing inference about populations using samples, simulations, and bivariate data analysis. This is a lot of content, and teachers need a huge amount of preparation for it. And elementary school teachers need to know it to help prepare their students for it.

In high school, topics include categorical and quantitative data analysis, inference, conditional probability and probability rules, and probability for decision making.

> **"Statistics requires a different kind of thinking, because data are not just numbers, they are numbers with a context. In mathematics, context [sometimes] obscures structure. In data analysis, context provides meaning."**
>
> *— Moore & Cobb, 1997*

The Mathematical Education of Teachers II (MET II) report recommends that middle school teachers take two statistics courses, an introduction to statistics plus data analysis. For high school teachers, it recommends an introductory sequence of courses lasting one or two semesters. These courses should all emphasize concepts, and they should use activities and technology.

The Statistical Education of Teachers (SET) is slated to come out in December 2014. It was commissioned by the American Statistical Association to be a companion piece to the MET II report. Bargagliotti is one of the writers. The report emphasizes that teachers of all grade levels need to understand the "statistical process": formulating a statistical question, producing data, exploring and modeling the data, and making inferences based on it.

Here's an example of a problem that can be approached by elementary, middle, or high school students, at different levels. A newspaper article recently stated that bottled water consumption is on the rise.

- In elementary school, a teacher could ask, "In our class, what type of water (bottled or tap) do students prefer to drink?"
- In middle school, the question could be, "Are people more likely than not to correctly identify the cup with bottled water?"
- And in high school, a question to ask would be "Is there evidence that people can tell the difference between bottled water and tap water by taste alone?"

> **"Statistics is a methodological discipline. It exists not for itself, but rather to offer to other fields of study a coherent set of ideas and tools for dealing with data. The need for such a discipline arises from the omnipresence of variability."**
>
> *— Moore & Cobb, 1997*

SET recommends that elementary teachers take either a special section of an introductory course, or an entire course in statistics content for teachers, or a reconfiguration of an existing content course for teachers to include at least six weeks of study of statistics and related ideas in probability.

It recommends that middle school teachers take a special section of an introductory course as well as a course focused on the statistical content they will be teaching using the framework provided by the Guidelines for Assessment in Statistics Education (GAISE) report. This course should also help teachers understand connections across grade bands as well as between statistics and other areas of study in middle grades (mathematics, science, social science, etc.).

It recommends that high school teachers take an introductory course that emphasizes modern data analysis with simulation approaches to inference using the appropriate technologies. In addition, they should take a statistical modeling course based on multiple regression, including the use of both categorical and numerical explanatory variables and the fitting of exponential and power models. Finally, they should take a course exposing them to the theory of statistical inference through one- and two-sample classical inference procedure and an introduction to analysis of variance.

The standard college level introductory statistics course is not adequate for teacher preparation. SET recommends a modern day introductory course emphasizing (1) data analysis and simulation approaches to understanding inference, (2) more on modeling, and (3) statistical theory that ties in with probability theory (e.g., the use of conditional probability in contingency tables and the chi-squared test). The current typical standard course tends to be formula-based, going through the list of hypothesis tests and probability theory somewhat removed from the statistical concepts being covered.

Math departments should carefully examine the introductory course to see how it can be reconfigured to meet the needs of teacher preparation, possibly opening up a specific section of the course for all K-12 teachers. Additional teacher coursework can be tailored to the grade levels a teacher will be teaching. Statisticians with experience in statistics education should help design the courses.

Mathematical Education of Teachers II

In 2012, the Conference Board of the Mathematical Sciences released the report *Mathematical Education of Teachers II (*MET II), an update of the initial version from 2001. Like the original, MET II recommends the mathematics that teachers should know and how they should come to know it, and it urges greater involvement of mathematicians and statisticians in teacher education.

In the intervening years, mathematicians and statisticians have taken increasingly active roles in teacher preparation and content-based professional development for current teachers, to the benefit of teachers' mathematical educations.

The MET II report updates the original report based on the experience and knowledge of the past decade and the development of the Common Core. Along with updating the recommendations about the mathematical preparation of teachers, the new document now addresses the professional development needs of math teachers and discusses the mathematical needs of teachers with special responsibilities such as elementary mathematics specialists and special education teachers.

The core themes, however, are unchanged. The report asserts that there is intellectual substance in school mathematics; that proficiency with school mathematics is necessary but not sufficient mathematical knowledge for a teacher; that the mathematical knowledge needed for teaching differs from that of other professions; and that mathematical knowledge for teaching can and should grow throughout a teacher's career.

> **"A major advance in teacher education is the realization that teachers should study the mathematics they teach in depth, and from the perspective of a teacher.**
>
> **Thus, this report recommends that before beginning to teach, an elementary teacher should study in depth, and from a teacher's perspective, the vast majority of K–5 mathematics, its connections to prekindergarten mathematics, and its connections to grades 5–8 mathematics."**
>
> ***— Mathematical Education of Teachers II***

The report has six main recommendations:

Recommendation 1.

Prospective teachers need mathematics courses that develop a solid understanding of the mathematics they will teach.

Recommendation 2.

Coursework that allows time to engage in reasoning, explaining, and making sense of the mathematics that prospective teachers will teach is needed to produce well-started beginning teachers. Although the quality of mathematical preparation is more important than the quantity, the following recommendations are made for the amount of mathematics coursework for prospective teachers.

i. Prospective elementary teachers should be required to complete at least 12 semester-hours on fundamental ideas of elementary mathematics, their early childhood precursors, and middle school successors.

ii. Prospective middle grades (5-8) teachers of mathematics should be required to complete

at least 24 semester-hours of mathematics that includes at least 15 semester-hours on fundamental ideas of school mathematics appropriate for middle grades teachers.

iii. Prospective high school teachers of mathematics should be required to complete the equivalent of an undergraduate major in mathematics that includes three courses with a primary focus on high school mathematics from an advanced viewpoint.

Recommendation 3.

Throughout their careers, teachers need opportunities for continued professional growth in their mathematical knowledge.

Recommendation 4.

All courses and professional development experiences for mathematics teachers should develop the habits of mind of a mathematical thinker and problem-solver, such as reasoning and explaining, modeling, seeing structure, and generalizing. Courses should also use the flexible, interactive styles of teaching that will enable teachers to develop these habits of mind in their students.

Recommendation 5.

At institutions that prepare teachers or offer professional development, teacher education must be recognized as an important part of a mathematics department's mission and should be undertaken in collaboration with mathematics education faculty. More mathematics faculty need to become deeply involved in PreK–12 mathematics education by participating in preparation and professional development for teachers and becoming involved with local schools or districts.

Technical knowledge for elementary school teachers

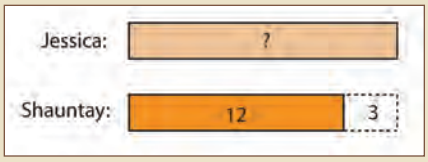
The National Research Council issued a report, *Mathematics Learning in Early Childhood*, that describes how to effectively teach young children. Sybilla Beckmann of the University of Georgia gave one example of the kind of technical knowledge elementary school teachers need to have that is described in that report:

These two problems are fundamentally the same, but can be presented in two different ways:

(a) Shauntay has 12 cards. Jessica has 3 more cards than Shauntay. How many cards does Jessica have?

(b) Shauntay has 12 cards. That is 3 fewer than Jessica has. How many cards does Jessica have?

The first is the "more" version, the second is the "fewer" version. Students who rely only on keywords may mistakenly *subtract* 3 from 12 in the "fewer" version. Students can be helped to reason through this using a drawing like this:



Every teacher going into the classroom for the first time should understand this.

Recommendation 6.

Mathematicians should recognize the need for improving mathematics teaching at all levels. Mathematics education, including the mathematical education of teachers, can be greatly strengthened by the growth of a mathematics education community that includes mathematicians as one of many constituencies committed to working together to improve mathematics instruction at all levels and to raise professional standards in teaching.

The writers of the report recognized that teacher preparation programs can't possibly teach prospective secondary teachers all the content that would be useful for the classroom, so they divided their content recommendations into three categories: topics and experiences that are essential for a teacher to know before entering the classroom; those that are desirable for teachers to have before entering, but that are essential for them to acquire early in their careers; and those mid- to late-career teachers should develop.

Essential topics to learn before teaching are single and multivariable variable calculus; vectors and matrices; analytic geometry; abstract linear algebra; statistics and probability; geometry and transformations; the real number system; the complex number system; experience with reasoning and proof; algebraic structures in high school mathematics; modeling; trigonometry; history of mathematics; experience with technology; and research experience.

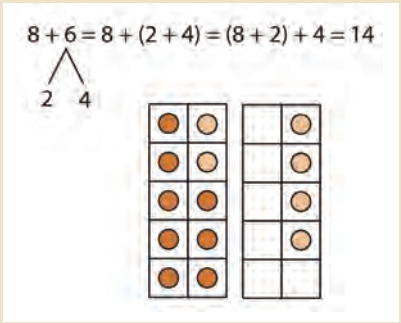
Topics teachers should learn either before teaching or early in their careers are further statistics; discrete mathematics and computer science; further geometry; and further experience with algebra.

Topics for mid- to late-career teachers are the mathematics of high school; differential equations; group theory; number theory; advanced calculus; and further history of mathematics. Activities that might lead to greater understanding of mathematics are math circles, immersion experiences and lesson study.

> **"The mathematics of elementary school is full of deep and interesting ideas, which can be studied repeatedly, with increasing depth and attention to detail and nuance. Therefore, although prospective teachers will undertake an initial study of elementary mathematics from a teacher's perspective in their preparation program, practicing teachers will benefit from delving more deeply into the very same topics."**
>
> *— Mathematical Education of Teachers II*

Technical knowledge for elementary school teachers

Elementary school teachers should all understand and teach their students "making a ten" methods of adding numbers. Beckmann gave the example of adding 8 and 6 by making a ten:



This method is particularly important because it is a way to engage students in grades 1 or 2 in algebraic thinking, thereby building the beginning of a ramp that leads to algebra in Grade 8.

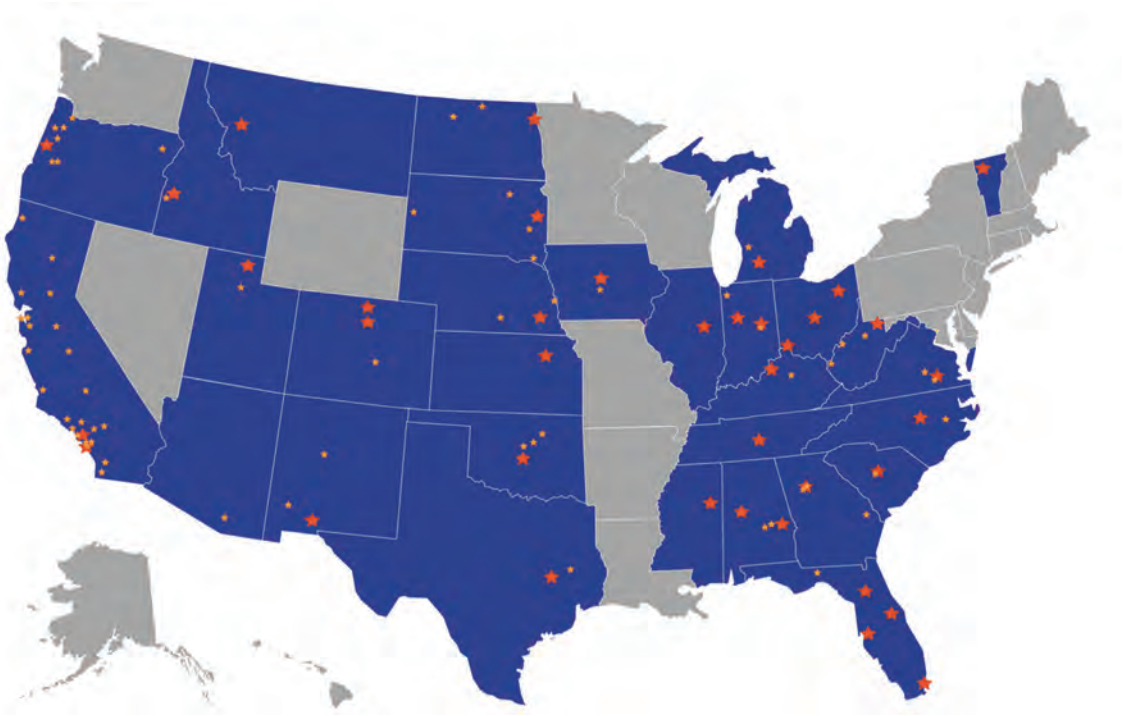
Many innovative approaches to educating mathematics teachers are being developed. W. Gary Martin, a co-director, described the Mathematics Teacher Education Partnerships, an effort organized by the Association of Public and Land-grant Universities to collaboratively research and transform the preparation of secondary mathematics teachers *(page 48)*. **Michael Marder, co-director of UTeach, related how UTeach is providing a model and support for mathematics and education departments around the country** *(page 51).* **Lillie Albert presented the efforts at Boston College to educate a diverse teacher workforce** *(page 53)*. **Hyman Bass described his course at the University of Michigan on cross-domain problem solving** *(page 55)*. **And Patrick Thompson commented on Arizona State University's bachelor's program for mathematics majors who also want to teach, a program that emphasizes mathematical meanings for teaching math** *(page 57)*.

The MTE-Partnerships

W. Gary Martin of Auburn University described a program called Mathematics Teacher Education Partnership (MTE-Partnership) that aims to transform the preparation of secondary mathematics. The Association of Public and Land-Grant Universities (APLU) organizes the program as a part of its Science and Mathematics Teacher Imperative, and Martin is a co-director for it.

MTE-Partnership teams include an APLU institution as the lead, at least one K-12 district, and at least one other organization (such as a collaborating university, college, or community college; an additional K-12 district; a regional inservice center; or a state department of education). Each team must have mathematics educators, mathematicians and K-12 educators actively involved.

A small group conceived of this partnership at the Science and Mathematics Teacher Imperative National Conference in June 2011. The first annual MTE-Partnership conference was held



MTE-Partnership Teams as of the summer of 2015. The Partnership's membership includes 39 teams across 31 states comprised of 103 universities, university systems, and community colleges; 142 K-12 schools and school districts; and several state departments of education, education consortia and other education-focused organizations.

in March 2012, and in September 2013, a grant from the Helmsley Charitable Trust launched the action phase.

Beyond individual partnerships, partnership teams collaborate with one another in "Research Action Clusters" (RACs) to address specific issues in secondary math teacher preparation.

Here are some current RACs:

- **MATH: Marketing for Attracting Teacher Hopefuls**

  MATH is working on the critical problem of increasing the recruitment and retention of highly qualified teachers of mathematics through creating a purposeful marketing plan to attract a diverse population of teacher candidates.

  Teacher education programs at higher education institutions are not enrolling or graduating nearly enough secondary mathematics teachers to satisfy the needs of U.S. middle and high schools. To get a sense of the size of the problem, consider that a typical high school hires 2 new mathematics teachers annually, and there are over 27,000 secondary schools in the U.S. That means that about 50,000 mathematics teachers need to be hired every single year. Meanwhile, 31% of secondary mathematics classes are currently taught by teachers without mathematics-related backgrounds. Given that the average program graduates 10 new mathematics teachers per year, there is a significant challenge in meeting the demand for well-prepared secondary mathematics teachers.

  Some specific approaches taken by participating institutions are these:

  - The University of South Carolina will pilot a professionally created marketing plan, which will then become a resource to share
  - Florida International University and Arizona State University will develop recruiting materials and will recruit STEM majors, other undergraduates, and high school students
  - Boise State University will use survey data on interest and experience to identify high-impact recruitment strategies

- **Actively Learning Mathematics**

  Too many students with the potential to enter STEM programs don't succeed in introductory mathematics courses (particularly precalculus and calculus). This blocks them from entering STEM careers, including secondary mathematics teaching. These courses often have many students who get Ds, fail or withdraw. They don't teach students to persist and don't expose them adequately to a wide range of mathematical practices.

  For a solution, this RAC is finding ways to increase students' active engagement in learning mathematics. They're encouraging more group work in class and more group assignments outside of class; the use of technology to increase practice outside class; an emphasis on concepts, problem solving, and motivating examples; the development of habits of mind (aka "mathematical practices"); and the development of communication skills.

  To give an example of their work, the University of Nebraska-Lincoln team is focusing on transforming instruction in college algebra and precalculus, two courses that serve 1,000 students in the fall and 500 in the spring. Their five year average for student success was 62% for college algebra and 68% for precalculus — not good.

  In fall 2012, they began work on their college algebra course. The initial results were poor, with only a 59% success rate. So they chose a new textbook (Connally, Hughes-Hallett, et al.), and in the summer of 2013, they made a major effort to write better lesson plans. They incorporated an "entry" Gateway Exam, WeBWorK homework, and Team Quizzes into both courses. They created a professional

development workshop for graduate teaching assistants (GTAs) the week before classes, and they also invested a substantial amount of faculty time to provide leadership and mentor GTAs.

In fall 2013, the results were much better, with an 81% success rate in college algebra. Similar efforts led to a 76% success rate in precalculus.

In the future, UNL plans on improving GTA professional development through a week-long professional development workshop and through ongoing teaching mathematics seminars for GTAs.

Other Research Action Clusters include one on developing effective clinical experiences, one on developing course materials specifically aimed at the mathematical knowledge needed by secondary mathematics teachers, and one on assessing mathematical knowledge for teaching. These clusters began piloting interventions and collecting data in the spring of 2014.

Additional teams are invited to apply to join the Partnership (see http://www.x.co/mtepappl/). For more information about the Partnership, visit http://MTE-Partnership.org.

Clockwise from top left:

Elizabeth Burroughs
(Montana State University)

Sybilla Beckmann
(University of Georgia)

W. Gary Martin
(Auburn University

Cody Patterson
(University of Arizona)

UTeach

Michael Marder of the University of Texas, Austin, is the executive director of the UTeach Science Program and co-director of UTeach, a teacher education program that began at UT Austin in 1997. The model has now spread to 40 institutions around the country.

The core principles driving the UTeach program are these:

- Students complete both a degree in a STEM field and earn a secondary teaching certification.
- Students begin working in the classroom right away to help them figure out whether they enjoy teaching quickly.
- Master teachers teach UTeach courses and supervise students' field experience. Students can complete all the requirements in four years. Many students simply can't afford five years, no matter how valuable the extra material.
- UTeach actively recruits to attract and retain the greatest possible number of STEM majors, both in the program and in the teaching career.
- Classes emphasize inquiry, hands-on investigation, problem-solving, and reasoning, not rote memorization.

About 820 UT Austin students have graduated from UTeach so far. Remarkably, 90% go into teaching, and many of the remaining 10% go to graduate school. About half teach in low-income schools. Five years after entering teaching, 80% of the students are still doing so, as classroom teachers or curriculum coaches. About 25% of all UT Austin math majors are in UTeach.

The UTeach Institute has developed a collection of resources and support materials to aid institutions in replicating UTeach. Each university replicating UTeach is assigned a site coordinator, who provides technical assistance and serves as a primary point of contact. Institutions are provided grants to help them make the transition into UTeach, through funding from the National Math and Science Initiative and others. Institutions have to commit to hiring former secondary teachers of math and science permanently and continuing to pay them after the grant ends.

Evaluation is another key component. Once or twice a year, the UTeach Institute conducts site visits. They facilitate online student surveys, observe instruction and review course materials, and prepare both quantitative and qualitative reports. These reports are made available to the universities to facilitate their own planning.

In addition, there is a nationwide UTeach conference every June, allowing UTeach institutions to learn from one another's experiences. UTeach institutions are now diverse enough that when a problem arises for one institution, another institution in the network has probably already faced it. The conference also allows institutions that are considering joining UTeach to learn about the program.

Early teaching experiences are key for recruiting students, but they have to be very carefully supported, otherwise schools can't afford the time required to deal with the student teachers. So UTeach has developed this incremental process:

1. In their very first semester, students take a one-hour course that gives them field experience in elementary schools coordinated by master teachers. They teach research-based science classes.
2. In their second semester, they do the same thing in a middle school. Students who ultimately drop out of the program typically do by this point, usually saying that they decided teaching wasn't for them. UTeach thinks this is an appropriate time for them to make that decision.
3. Students then take a three-course sequence in the college of education, taught by research faculty in science and math education. The first focuses on current theories of learning and conceptual development. It includes no field experience. The second includes a field experience in high schools, in which students are videotaped. The course teaches different modes of instruction, group work, jigsaw, differentiation, equity, and whole-class instruction. In the third course, students teach project-based STEM lessons to high school students using new technologies.
4. After finishing the UTeach coursework, students begin apprentice teaching, supported by a very big team. The classroom teacher, facilitators, and course instructors all observe and provide feedback and support. Students say that this is the best experience of all.

When students enter their first year of teaching, the same team continues to support them.

Marder has learned that collaboration is absolutely crucial. This is why UTeach requires that master teachers must be included. The math and science education faculty help raise the level of mastery of content knowledge and bring new ideas of how things might be done. The teachers have the real-world experience needed to evaluate those ideas.

Marder has also learned that it is critical to consider from the very beginning what happens when the starting grant terminates, because many reforms simply end at that point. The key, Marder has found, is for the provost to put it on the permanent budget and for donors to give money for an endowment. The UTeach endowment at the time of the conference was $15 million, and that has saved its life many times. Practical knowledge like this is part of what is conveyed as the program spreads.

Teacher shortages

National: school and staffing survey

	Number	Percent either without major or certification
Subject Area		
Biology/life Sciences	**245,000**	**43**
Physical science	**289,300**	**65**
Chemistry	**106,900**	**65**
Earth sciences	**53,100**	**82**
Physics	**43,200**	**69**
Mathematics	**676,900**	**38**
Science	**562,700**	**29**

The U.S. faces severe teacher shortages. Universities are the only entities that can match the scale of the problem, and UTeach offers a model that can help universities to quickly and dramatically improve their education programs.

Educating a Teacher Workforce that Reflects the Diversity of our K-12 Classrooms

On March 10, 2009, Obama remarked to the Hispanic Chamber of Commerce: "From the moment our children step into a classroom, the single most important factor in determining their achievement is not the color of their skin or where they come from. It's not who their parents are or how much money they have. It's who their teacher is. America's future depends on its teachers. Then how do we recruit, prepare and retain a diverse teaching workforce to provide quality as well as transformative instruction that nurtures the creativity of our students?"

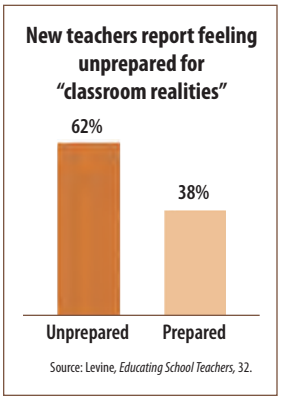
Lillie Albert of Boston College applauded President Obama for his vision, but she also remarked that we simply cannot carry out any of these intentions in isolation. Our recruitment efforts will falter if our teacher preparation programs are weak. If we do not support programs that include strong mentoring and effective professional development, then new teachers will continue to leave the teaching workforce.

Research indicates that the underserved population benefits both academically and socially by having teachers with whom they can identify. They don't need to have the same ethnicity, cultural background or language. A white teacher can provide this academic and social benefit for students of color. One of Albert's best mentors has been Professor Ken Travers at the University of Illinois Urbana-Champaign, whom she describes as "a burly white guy."

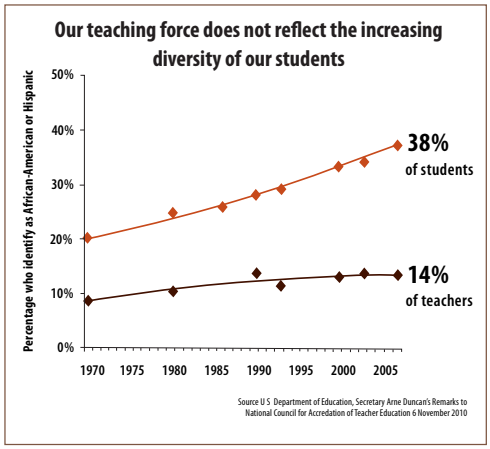
The recruitment of teachers can be difficult and time-consuming. States and districts are managing to bring in teachers of color, but three to five years later, these teachers are leaving the workforce as quickly as white math teachers. How do we keep them there?

First, we have to work on preparation.

Here's the problem:



These data reveal that feeling unprepared is a problem for most math teachers, not just teachers of color. If teachers continue to arrive in the classroom feeling unprepared, teacher retention rate will continue to remain low, meaning that we're going to continue seeing the trends as shown in the figure at right.



Boston College has an endowed fund, the Donovan Urban Teaching Scholars Program, which recruits and supports a diverse cohort of up to 30 graduate students for an intensive one-year Masters in Education, covering at least half of their tuition costs. About one-half are teachers of color, while the remaining one-half are white, all of whom want to work in urban schools. Teachers receive the strong mentoring and preparation they need to succeed in the classroom. Teachers who participate in this program are staying in the workforce at higher numbers than our general population.

Promoting social justice is a strong feature of all Boston College programs. Boston College Lynch School of Education Teacher Education Program carried out a 2.5-year-long self-study of what it means to teach for social justice. On the first day of classes, each professor introduces the idea of teaching for social justice and informs students how the course will meet the goal:

> "We see teaching as an activity with political dimensions, and we see all educators as responsible for challenging inequities in the social order and working with others to establish a more just society."

Given the shifting demographics of U.S. schools, all candidates have the opportunity to obtain a Teaching English Language Learners (TELL) Certificate. This experience prepares teachers to work with English Language Learners in mainstream classrooms.

In addition, Boston College Lynch School of Education Office of Induction offers strong professional development to support their graduates during their first few years of teaching. For example, it offers a summer institute that all graduates can attend and where they can learn alongside experienced teachers.

Another way in which they have supported their students studying mathematics education is through the involvement of mathematicians in high school mathematics classrooms and as content mentors. This aspect was an advantage when they applied for a NSF Noyce Grant. Although clinical supervisors for the student teachers in the program are former high school teachers, they may or may not have a math background. Therefore, they may not be able to provide student teachers the feedback they may require regarding the content of their lessons. Mathematicians from the Arts & Science Mathematics Department volunteer to be a content mentor to a student teacher.

Preparing the math faculty for this is important. The Lynch School of Education started this preparation about fifteen years ago with a Carnegie grant, *Teachers for a New Era* (TNE) for about $10 million with which they could work across departments and schools. Their proposal was to work more collaboratively with Arts & Science faculty. To do so, they had "content mentoring circles," where once a month mathematicians and mathematics educators would sit around the table with student teachers. The student teachers would bring issues and dilemmas, and the mathematicians and mathematics educators would work together to help resolve them.

As the TNE Grant came to an end, they planned ahead for the next round of funding so that programs such as the content mentoring circles might continue. Through the NSF Robert Noyce Teacher Scholarship Program, the Lynch School in collaboration with the Mathematics Department was funded for $1.6 million, with $800,000 matching funds from the university. NSF funded grants and TNE initiatives have enhanced LSOE collaboration with BC Mathematics Department faculty. Central to this University-wide initiative are efforts to enhance teacher candidates' academic background in the content area of specialization they would teach and to support graduates during their first years of teaching.

Boston College also has an active student service program. In August 2011, Albert led a successful two-week Immersion Trip to Cape Town, South Africa, with eleven students to work in schools supported by the Amy Biehl Foundation, including the Linawo Children's Home in Pinelands. Boston College students have traveled to Guatemala, Jamaica and Ghana. Service trips also take place across the U.S., such as Mississippi, Georgia and Alabama.

At Boston College, Albert's department has about 20 faculty, and she is the only African American. In the entire school of education, she's one of three. At the university, she's one of about 30. If we want to increase the presence of teachers of color in K-12 schools, she argues, we really must do it at the college level as well. She points out that diversity isn't just her business, it is everyone's business.

Cross-Domain Problem-Solving

Hyman Bass of the University of Michigan School of Education and mathematics department described a course in the mathematics department he developed that is oriented toward teachers. It is problem-based and interactive (thereby fostering the mathematical practices), and it especially emphasizes making mathematical connections. The topics are primarily focused around the arithmetic, algebra and geometry of the number line.

Bass had noted that the school (and even college) mathematics curriculum efficiently organizes mathematics into distinct subjects, and students, even when academically successful, often lose awareness of the unity and connectivity of mathematical ideas across domains. The cognitive literature on learning suggests that networks of connections are characteristic of deep understanding and of high problem solving skills. The connected, networked thinking of expert problem solvers enables them to recognize a common structure across diverse mathematical contexts. Networked knowledge is also deeper, more efficient, and high leverage in holding complex bodies of information.

He thought that perhaps students failed to see cross-domain mathematical connections because these skills and habits are not deliberately taught. This course was designed to help students learn to see and use such connections.

One technique was to design "cross domain problems" whose resolution needs to draw resources from different mathematical topic areas (arithmetic, algebra, geometry, combinatorics, etc.) He wanted students to make connections between mathematics and "real world" situations and science (e.g., modeling), but he especially wanted them to make connections between different mathematical concepts or topics, and also among different mathematical problems that have a common structure.

Here are some examples of cross-domain problems Bass has developed.

1. Find all (real valued) functions $f(x)$ of a real variable x that satisfy the condition:

$$|f(x)-f(y)| = |x-y|$$

for all real numbers x and y.

Students started by finding some linear functions that worked. Then they tried quadratics, and none of them worked. Then they concluded that the linear examples they found were the only ones.

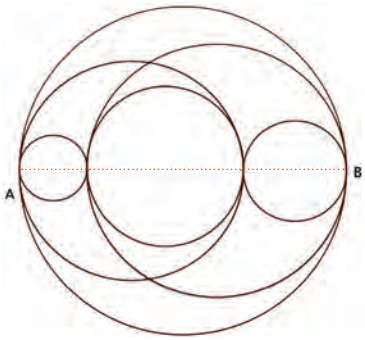
That was true, but how do you prove it? No one had an idea. This is actually a geometry problem.

The isometries of the real line are of the form $f(x) = \pm x + c$, that is they are translations, possibly combined with a reflection. If you know the distance of a point from two fixed points on the line, then you know the point exactly. So if you have a function that fixes two points, it is the identity. You can bring any isometry, by a translation and then possibly a reflection, back to the place where it fixes 0 and 1, for example. That's a geometric argument.

The students wouldn't recognize this as a geometry problem, because the notational environment is foreign to them.

Students have difficulty recognizing this as a geometry problem, because they are not in the habit of thinking of $|x - y|$ as a distance function.

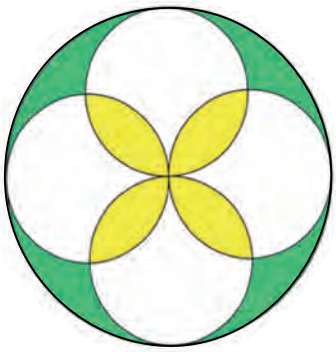
2. Traveling in circles:



Circle Park has a network of circular trails for cyclists to use (see Figure at right). The trails have bridges so that they meet only along the diameter AB. Which is the shortest way to travel from A to B using this network of circular paths?

All the paths have the same length, because the semi-circumference is proportional to the diameter. So no matter how you travel, without backtracking, it's the same. This problem uses geometry, a bit of linear functions, and proportionality.

3. Which area is more, green or yellow? Explain why.



There are assumptions they have to take, particularly that each small circle has ½ the diameter of the big one. Then the areas of the four small circles add up to the area of the big circle. So the area covered by the white circles is the area of the big circle minus the yellow area. But it is also the area of the big circle minus the green area. So the yellow and green areas are equal.

4. Question: where is $f(x)$ continuous, where $f(x)$ is defined as:

- $f(x) = 0$ for x irrational
- $f(x) = 1/q$ for $x = p/q$, rational, in reduced form

This is a problem where intuition is not much help, since it is impossible to visualize the graph. After some reflection you can see that it will be discontinuous at any rational number x, because you have a non-zero value but you can approach x with irrational numbers.

To see that it is continuous at irrational numbers, a key lemma is that if a sequence of rational numbers approaches an irrational number, their denominators approach infinity. The proof of this uses the fact that the rational numbers with a bounded denominator are uniformly discrete.

So f is continuous exactly at the irrational numbers.

This is a much more sophisticated problem, but it illustrates the crossover of mathematical ideas that go into solving it.

"The essential competencies of an effective teacher are command of subject, preparation in effective pedagogical practice, and high overall academic performance."

***To Touch the Future*, American Council on Education**

A Bachelor's of Science in Mathematics that Emphasizes Mathematical Meanings for Teaching Mathematics

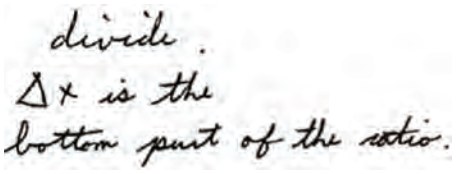
Patrick Thompson, a professor of math education at Arizona State University (ASU), described a bachelor's program for mathematics majors who also want to teach. The program is designed to address a fundamental problem in teaching: Students leave high school with poorly formed meanings for ideas of the secondary mathematics curriculum. They then take mathematics courses that presume they have meanings they in fact do not have. They apply coping mechanisms in college math that allowed them to succeed in high school mathematics, such as memorization. They then return to high schools to teach ideas they've understood poorly, have rarely revisited, and for which they still have poorly-formed meanings. This then perpetuates the problem in the next generation of students.

The bachelor's degree program drew on the findings of Project Aspire: Defining and Assessing Mathematical Knowledge for Teaching Secondary Mathematics, which was a project to understand the mathematical meanings held by 260 high school mathematics teachers. It used an evaluation instrument called MMTsm, for Mathematical Meanings for Teaching secondary mathematics. Rather than simply getting a score, teachers are given a profile that gives them information about ways they understand things that are either problematic or productive. The MMTsm was administered to 160 high school teachers in the summer of 2012 and 100 high school teachers in the summer of 2013.

Here's one of the problems on this test:

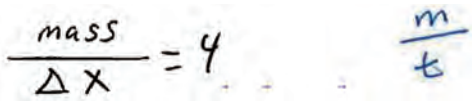
> A college science textbook contains this statement about a function f that gives a bacterial culture's mass at moments in time:
>
> *The change in the culture's mass over the time period Δx is 4 grams.*
>
> **Part A.** What does the word "over" mean in this context?
>
> **Part B.** Express the textbook's statement in mathematical notation.

For Part A, "during" is a reasonable answer. But here is an answer from a teacher with a bachelor's in math education:



A teacher with a B.S. in math wrote:

For Part B, the correct answer is: $f(x+\Delta x)-f(x)=4$, or something similar. Here are some other answers teachers gave:



Results showed that 62% of these working teachers recognized that "during" was at least a possible meaning of "over." But of those, 87% answered part B with either nonsense or a quotient.

While the definition of the word "over" is not inherently so important, it reveals whether the teacher has an orientation to computational interpretations. This is important, because teachers convey that orientation to their students.

The teachers performed poorly on the majority of items in the MMTsm, Thompson said. He explained this by pointing out that, very early in primary school, kids get off track and math becomes meaningless. The problems then accumulate. Students learn how to cope with meaningless mathematics by trying to memorize procedures. Either they manage to memorize it and teachers think they understand, or they don't, and teachers think they're not smart. In either case, students leave high school with weak meanings for the mathematical ideas of elementary and secondary mathematics.

Thompson also noted that the teachers his team assessed took substantial mathematics *from mathematics departments*, and it's hard to imagine how they succeeded in undergraduate mathematics programs with such weak mathematical meanings.

Thompson then described ASU's response to the problems uncovered by the MMTsm. They designed a Bachelor of Science in mathematics with a concentration in math education that grants graduates automatic licensure from Arizona to teach secondary math. The program focuses "like a laser" on future teachers' mathematical meanings for teaching secondary math and on high school students' mathematics.

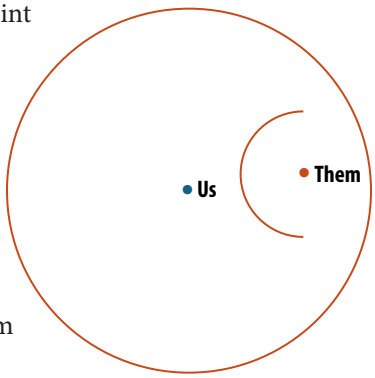
The ASU B.S. math/math ed program includes six specialized courses in math education. The first course, Algebra and Geometry in High School, requires that they tutor other students, which is exceptionally important because it provides students with an occasion to begin building images of others' conceptions of mathematics. It also helps them to rethink the math they learned in high school so that it becomes a coherent body of ideas. Students' tutoring experiences are exceptionally important because it helps them to become aware of the ways that other students struggle to understand mathematical ideas and to imagine themselves as managers of other people's learning.

Students take calculus, developed according to Harel's Necessity Principle, in their first to third semesters.

In their third semester, they take Technology and Mathematical Visualization. In it, students create "didactic objects," artifacts that are designed to support reflective conversations about important mathematical ideas and ways of thinking. For example, consider this word problem:

> The speed of sound in water is 1503 meters per second. Define a function *d* that takes as input the number of seconds for a ping's echo to return and that gives the distance from Them to Us as output.

The students created a mathematical visualization, with a point for Us and a point for Them. A sonar ping goes out from Us in an ever-widening circle. When the ping hits Them, it creates an echo wave in the form of an ever-widening semi-circle. The two points can be moved anywhere on the screen. The students' purpose for this animation is simply to help students visualize the situation described in the problem.



Students in this course must use mathematics to create these objects. Their work focuses simultaneously on their mathematics and their future students' mathematics. The class also emphasizes lesson design and how to hold classroom mathematical conversations based on the objects the students create.

The remaining three specialized courses (Curriculum and Assessment, Development of Mathematical Thinking, and Mathematical Methods) build from this foundation while extending the program's focus on understanding mathematics of the secondary curriculum deeply and on conceptualizing the mathematics curriculum in terms of students' learning.

5 Professional Development

Professional development is a key aspect of teacher education. Darryl Yong described Math For America, an effort to develop leaders in math education *(page 60)*. **And Gail Burrill and Darryl Yong discussed the Park City Math Program** *(page 62)*.

Math for America

Darryl Yong of Harvey Mudd College is a member of the steering committee for Math for America Los Angeles, and he described the Math for America (M*f*A) program.

M*f*A makes teaching a viable, rewarding, and respected career choice for the best minds in science and mathematics. It was started in 2004 by James Simons, a mathematician and hedge fund manager. It has traditionally operated four programs: a teaching fellowship program for brand-new teachers; early career fellowships for beginning teachers; master teacher fellowships; and school leadership fellowships. The initial program was in New York, and then programs were started in Los Angeles and San Diego. It has since expanded to Berkeley, Boston, Washington, D.C., and Salt Lake City, with variations in the programs in different cities.

While it's widely known that Obama aims to have 100,000 new STEM teachers over the next decade, it's less well known that he also aims to have 10,000 master teachers in that same time. The Math for America program is contributing to that. At the time of the workshop, there were 349 M*f*A master teachers in New York City, with a goal of 750.

The New York master teacher fellowship offers:

- A $15,000 stipend per year for four years, on top of the regular salary
- Professional development and mentoring obligations
- National Board Certification support
- Community / connections / support

Math for America is not trying to "fix" teachers. M*f*A Master Teachers are already experts in content, pedagogy, and knowledge of students. M*f*A connects them to a community of professional peers and colleagues who keep them inspired and growing together.

Meetings take one of three forms. Professional Learning Teams are monthly meetings organized by grade or subject. Minicourses last for six weeks and are designed to help teachers delve into a particular content area. And Leadership Courses are focused on helping teachers become community leaders, with conversations about being a department chair, for instance, or speaking at a conference.

The Master Teacher Fellowship Program helps both to retain expert teachers and attract new candidates to teaching. Master Teachers strengthen the overall work of their school departments and have a positive impact on school culture.

New York State has launched its own version of the M*f*A Master Teacher Fellowship across the state, and it now has three hundred Master Teachers. The selection process and program design for the State Fellowship mirror Math for America's.

Math for America in Los Angeles is a separate organization with strong connections to the New York Math for America. It was organized by Harvey Mudd College, the University of Southern California, and Claremont Graduate University.

It has been supported by two Noyce grants, private philanthropy, and some support from the New York Math for America. Math for America in Los Angeles currently has 80 teaching fellows and master teaching fellows.

Los Angeles is a large metropolitan area with high income disparity and a high cost of living. The Los Angeles Unified School District (LAUSD) is the second-largest school district in California, with 690,000 students and 45,000 teachers. It is dysfunctional. They are rolling out iPads for every student, but they're also rolling out breakfast. Every student gets breakfast, and teachers are wait staff.

Yong himself taught at a LAUSD high school a few years ago, and he found the experience in the classrooms to be chaotic. In his school, 80% of the students qualified for free or reduced meals, 85% identified as Hispanic or Latino, and only 3% were proficient in mathematics on year-end California State Tests. This is typical of many LAUSD high schools.

In March 2014, for the first time, whites are no longer in the majority in California. M*f*A LA has a relatively diverse set of teaching fellows and master teachers.

The Los Angeles Master Teachers program has some key differences with New York. A pair of teachers from a single high-need school must apply together. They receive a $10,000 stipend per year for five years. They must design and implement an improvement plan that fits into needs of the school and has measurable outcomes. The Math for America program pays the school so that the teachers can get a one-course reduction per year. Teachers have monthly professional development and mentoring obligations. They must attend the Park City Mathematics Institute Summer School Teachers' Program at least once. They receive conference travel support, and by the end, they should be presenting about their own work. They get a $5,000 school grant for classroom supplies. They receive support for getting national board certification. And they receive community connections, and support.

One pair of teachers worked on implementing the Common Core version of Algebra I, with an extra support class for the students. They showed positive results on the California state tests, though now the problem is that those tests have ended, so they no longer have a way of measuring their results.

Another pair worked with CPM (College Preparatory Mathematics) to pilot a new Common Core version of Integrated Mathematics 1. They've been working with the writers giving feedback on the curriculum.

The teaching fellowship is being phased out.

Three of several Math Teacher Circles participants

From left:

Brianna Donaldson
(American Institute of Mathematics)

Kristin Umland
(University of New Mexico)

Brian Conrey
(American Institute of Mathematics and consulting professor, Stanford University)

The Institute for Advanced Studies / Park City Mathematics Institute Summer School Teachers Program

Gail Burrill of Michigan State University and Darryl Yong of Harvey Mudd College described the Institute for Advanced Studies / Park City Mathematics Institute (IAS/PCMI) Summer School Teachers Program.

There is little evidence that most professional development programs work (Gerstein et al, 2013). Furthermore, the premises of most professional development programs are questionable. They often have a goal to change teacher practices or beliefs; are developed around content that teachers teach; are dependent on "professional" professional development leaders; are localized to a school, region or state; and don't involve mathematicians.

The IAS/PCMI Summer School Teachers Program goes against all those things, and it follows this MET II recommendation for in-service professional development:

"All courses and professional development experiences for mathematics teachers should develop the habits of mind of a mathematical thinker and problem-solver, such as reasoning and explaining, modeling, seeing structure, and generalizing. Courses should also use the flexible, interactive styles of teaching that will enable teachers to develop these habits of mind in their students." (CBMS, 2013)

IAS / Park City Math Institute is a three-week playground for mathematics, with lots of different things going on at the same time in the same space. There is a residential Summer Session for school teachers, undergraduate college faculty, undergraduate students, graduate students, and mathematics researchers. The five groups meet simultaneously, pursuing individualized courses of study. Each year, a central focus is chosen, which, in 2014, is materials science.

The Summer School Teachers' Program (SSTP) includes 55-60 grade 2-12 teachers from across the U.S., including some from Math for America. They have diverse backgrounds with respect to mathematical content knowledge, experience, schools, and cultures. "E-tables" allow teachers from around the country to spend a morning doing math with those in residence.

The goals are to deepen content knowledge, to reflect on practice, and to become a resource to colleagues.

The mathematics course is the core of the program. Teachers meet for two hours each day. The course is different every year, and is focused around the year's theme. In 2014, the focus is on fractions, tiling and geometry, with questions that connect to materials science, the theme of the year. The course has both second grade teachers and Ph.D. mathematicians who are teaching high school, so the questions have to be accessible on multiple levels, which they denote as "important stuff," "fun stuff," and "tough stuff." "Tough stuff" occasionally includes open problems.

The program is designed using these principles:

- Exposure before closure, where the learner uses examples to build intuition
- Multiple entry points into the mathematics
- Emphasis on connections and relationships among problems
- Problems repeated in different contexts and forms (e.g., geometry-algebra)
- Key ideas foreshadowed but not formalized until late in course
- Learning about mathematics and teaching mathematics by doing mathematics

The instructors hand out problem sets at the beginning of class and then wander around, helping and asking questions but giving very little direct instruction. The problems drive the experience. Teachers work on problem sets in groups of six, with a "table leader" who has been to the summer school at least once who acts as a facilitator for the discussion, though the leader doesn't necessarily have any more math knowledge than the others. The leader's role is simply to make sure everyone's ideas are heard.

Here's an example of a problem from day 2:

> Can perfect shuffles restore a deck with 9 cards to its original state? If so, how many perfect shuffles does it take? If not, why not?

The teachers were able to get to some pretty sophisticated mathematics, with very little explicit instruction. For example, they looked at the representation of fractions as decimals in different bases. That turns out to be connected to the way that a particular card gets shuffled. Other things that came up were modular arithmetic, Euler's theorem, Cayley graphs, generators of groups, and units.

The problem sets are all available online, from every year starting in 2001.

The teachers really find this fascinating. They tend to start off worried, and then they gain confidence as they go. And it's hard to get them to stop at the end of each class period.

Teachers are given a set of norms, including:

- **Don't worry about answering all the questions.** If you're doing that, we haven't written the problem sets correctly.
- **Don't worry about getting to a certain problem number.** Some participants have been known to spend the entire session working on one problem (and perhaps a few of its extensions or consequences).
- **Stop and smell the roses.** Getting the correct answer to a question is not a be-all and end-all in this course. How does the question relate to others you've encountered? How do others think about this question?
- **Respect everyone's views.** Remember that you have something to learn from everyone else. Remember that everyone works at a different pace.

The second part of the program is Reflecting on Practice. Six teachers, who have usually gone to PCMI for at least three years, act as staff. They help design and deliver a Reflecting on Practice course. A specific aspect of teaching practice is chosen each year to focus on. In 2014, for example, the focus was on productive discussions:

> Classroom conversations are central in developing student understanding of mathematical concepts. What should discussions look and sound like to make reasoning and making sense of mathematics the norm? What are some strategies for managing discussions to enable students to be partners in the

learning process? Participants will consider research related to teaching and learning mathematics with a particular focus on facilitating discussions. The work will be grounded in the study of discourse in lessons and classroom practice in the United States and in other countries.

The courses are meticulously planned. The staff teachers think through what they think participants will say, how to respond, what questions to pose, and then after the class, they debrief, discussing what worked and what didn't. For example, they might pose the question "What is an equation?" and then ask how to handle it if no one responds. One approach in that situation would be to give examples and then ask, "Is this an equation? What about 0=4?"

They use videos, student work, math projects, relevant articles, and they bring the lesson study procedure to it. They divide into three rooms of about 20 participants, rotated weekly.

The third part of the program is Working Groups, which is where the teachers become a resource for their colleagues. They meet for two hours, four days a week, creating some kind of product. In the past two years, they've focused implementing some particular topic in the Common Core State Standards. One example is a group that worked on how to assess the standards for mathematical practice. The result is put online at www.illustrativemathematics.org. The groups (typically consisting of three teachers) are closely monitored to make sure that the work will be productive. Every couple of days, the groups debrief with one another for feedback.

Finally, they have Cross Programs between the groups (teachers, undergraduates, graduate students, researchers). There are Cross Program talks, research mathematician discussion groups, pizza and problem solving (with people from all the groups working on a common set of problems), and Clay Institute lectures by leading mathematicians that will engage all the different groups.

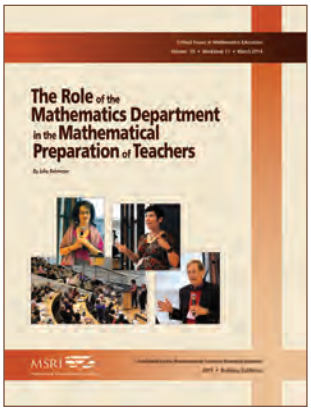
Summer school teacher's program

Some feedback teachers have given

"PCMI is simply the best professional development experience I have ever had."
(Participant/Exit Survey)

"The focus on being intentional really hit home for me. I kept thinking about how I did things on the fly so much in my class and how that may have affected their learning…"
(Participant Report)

"I have received so many rich, valuable morsels from the SSTP program that I have integrated over the past few years and continue to find new ways to be a better teacher. One way is having classroom discourse and how to scaffold it to make it safe for students to share, to be "wrong," to learn from each other and learn from their mistakes and others'."
(Participant Exit Survey)



On the cover *(page 1)*

Top row from left:

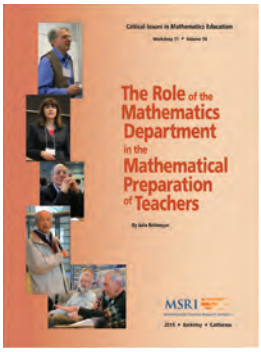
Deborah Ball (University of Michigan) at a Friday session

Allison Krasnow (University of California, Berkeley) shares observations about teachers' interactions with mathematics departments

Bottom row from left:

Joan Ferrini-Mundy begins a session on Friday in Simons Auditorium

Jim Lewis (University of Nebraska-Lincoln), workshop organizing chair, at the closing session on Friday



On the title page *(page 3)*

Photos from Simons Auditorium, from the top:

Robert Perlis (Louisiana State University)

Diana Suddreth (Utah State Office of Education)

Denny Gulick (University of Maryland)

Hung-Hsi Wu (University of California, Berkeley)

At left: Robert Megginson (University of Michigan)
Seated to his right: Steven Rosenberg (Boston University)

Photographers: Wendy Smith and Tom Clark

Dr. Smith, with the Center for Science, Mathematics and Computer Education at the University of Nebraska-Lincoln, began her career as a middle level mathematics teacher. Her research interests include K-16 mathematics education, rural education, teacher change, teacher professional development, teacher leadership, action research, and estimating teacher professional development effects on student achievement.

At the time of the workshop Tom Clark was a graduate student at the University of Nebraska-Lincoln working on applied functional and numerical analysis of a 3-D fluid-structure interactive PDE. Currently he is an assistant professor at Dordt College.

Author: Julie Rehmeyer holds a master's degree in mathematics from MIT and has taught math, science, philosophy, literature, and music at St. John's College. She is a freelance math and science journalist and contributing editor at *Discover* magazine. Her work has appeared in the *New York Times*, the *Washington Post, Discover, Science News, Aeon, Wired, High Country News* and many other publications.

Workshop booklet coordinators: Dr. Hélène Barcelo is the Deputy Director at MSRI and serves on its Educational Advisory Committee. Claude Ibrahimoff is MSRI's International Scholar Consultant and Executive Assistant and oversaw production and distribution of this booklet.

The Mathematical Sciences Research Institute (MSRI), located in Berkeley, California, fosters mathematical research by bringing together the foremost mathematical scientists from around the world in an environment that promotes creative and effective collaboration. MSRI's research extends through pure mathematics into computer science, statistics, and applications to other disciplines, including engineering, physics, biology, chemistry, medicine, and finance. Primarily supported by the U.S. National Science Foundation, the Institute is an independent nonprofit corporation that enjoys academic affiliation with more than 100 leading universities as well as support from individuals, corporations, foundations, and other government and private organizations.

MSRI's major programs, its postdoctoral training program, and its workshops draw together the strongest mathematical scientists, with approximately 1,800 visits over the course of a year. At any time, about eighty mathematicians are in residence for extended stays. Public outreach programs and VMath, the largest mathematical streaming video archive in the world, ensure that many others interact with MSRI throughout the year.

MSRI created the Critical Issues in Mathematics Education Workshop Series in 2004. This series of workshops addresses key problems in education today and is designed to engage mathematicians, mathematics education researchers, and K-12 teachers. The workshops provide participants a unique opportunity to learn about research and development efforts in this area. In addition participants develop ideas about methods for working on these problems and get to analyze and evaluate current or proposed programs. These workshops offer a space to make connections and exchange ideas with others concerned with the same issues in their fields.

Most workshops are held at MSRI and last for a few intensely secluded days. Each workshop attracts approximately 200 participants. Workshop organizers make sure to ensure diversity and relevant expertise by reaching out to mathematicians from a broad cross-section of colleges and universities.

For more information visit www.msri.org

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"If math teacher education is going to flourish in a mathematics department, there must be a teacher education group just like there is a research group in an area of mathematics research represented in the department. Given current political realities, in most cases that group needs to be coordinated and protected by a tenured department member with mathematics research standing – this is possible even when that tenured member is not particularly skilled or celebrated in the area of math teacher education."

– Herb Clemens, Ohio State University


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