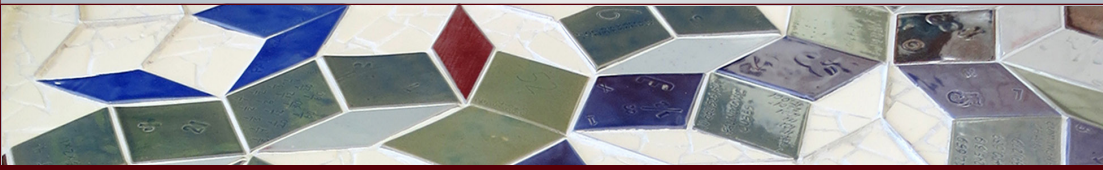
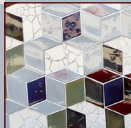
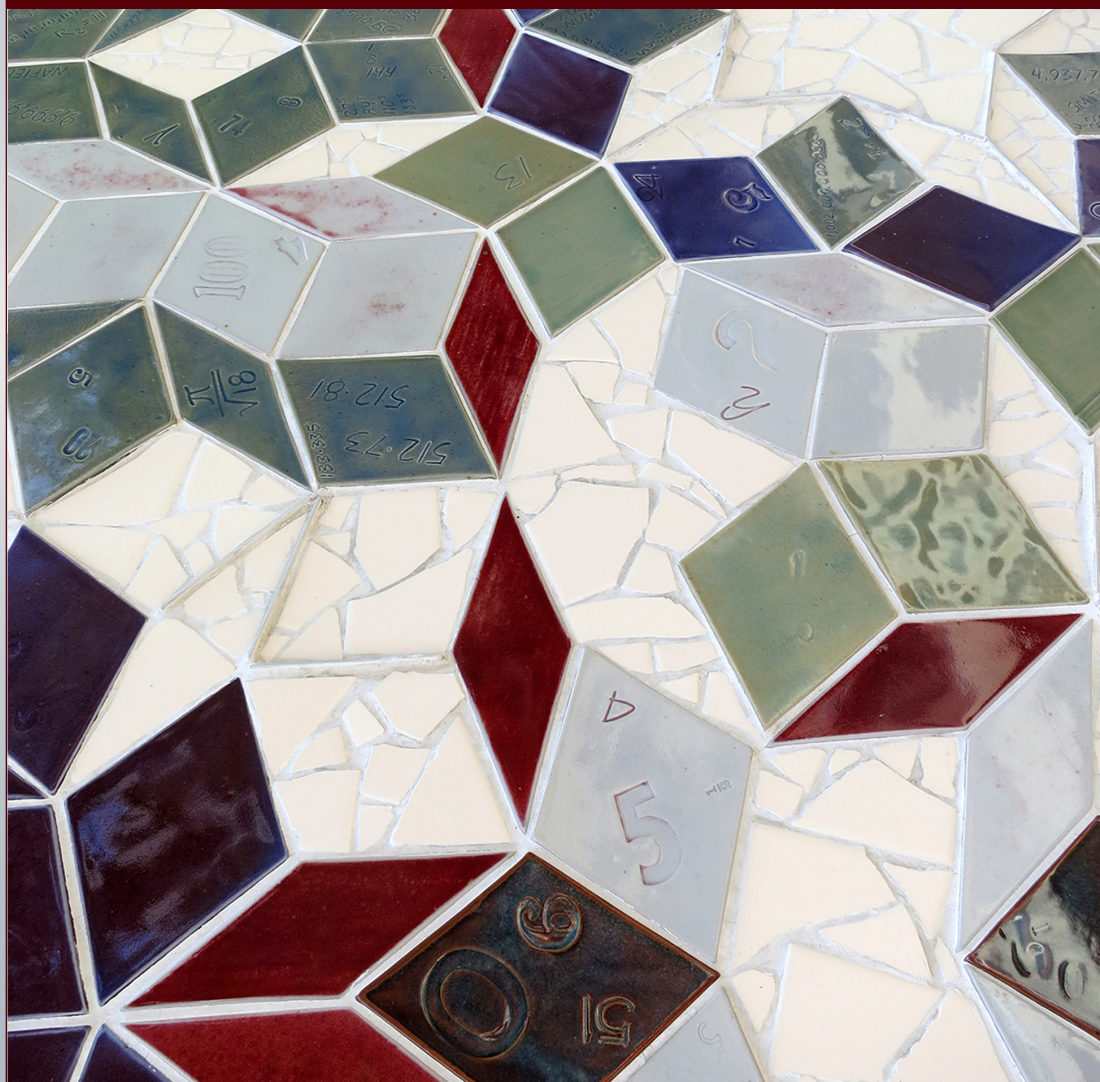


Critical Issues in Mathematics Education 2016 Workshop
Volume 12 • Workshop 13
February 2016



**Learning to Observe and Observing to Learn:
Using In-Person and Video Observation to Study
Mathematics Teaching and Learning**

Cody L. Patterson



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Critical Issues in Mathematics Education • Volume 12 • Workshop 13

Learning to Observe and Observing to Learn: Using In-Person and Video Observation to Study Mathematics Teaching and Learning

February 2016 • Mathematical Sciences Research Institute • Berkeley

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LEARNING TO OBSERVE AND OBSERVING TO LEARN:

Using In-Person and Video Observation to Study Mathematics Teaching and Learning

INTRODUCTION

The 2016 Critical Issues in Mathematics Education Workshop, hosted by the Mathematical Sciences Research Institute, focused on observation of mathematics teaching, and on the use of video to aid and enhance observation in particular. Speakers at the workshop addressed the wide variety of purposes for which we observe mathematics teaching, the complications and challenges inherent in the process of observing teaching, and ways in which we can enhance this process — before, during, and after observation — so as to increase the likelihood that observations achieve their intended purpose. A common thread among many presentations was that presenters used video clips of mathematics classrooms to highlight key points of their presentations. In some cases, the inclusion of these clips allowed participants to try using a specified framework for observation, or to notice firsthand a facet of teaching or learning that is made visible through the careful viewing of a classroom video.

Clarifying variables in observing and viewing classroom videos about teaching and learning mathematics

Workshop speakers came from a variety of settings and backgrounds, reflecting the diversity of perspectives and goals with which professionals approach the observation of mathematics teaching. Speakers included mathematics faculty from institutions of higher education who use observation as a basis for evaluating and improving undergraduate mathematics instruction, faculty in mathematics and education who use observation and video as tools for research on the teaching and learning of mathematics, and K-12 educators who use classroom videos to stimulate discussion of student thinking about mathematics in professional development settings. These speakers offered numerous insights into the potential of classroom observation to help us better understand emergent phenomena in mathematics teaching and learning. As a collective, they also painted a clear picture of some of the challenges and potential pitfalls that may hinder efforts to study and improve teaching through observation. As a field, we have an integral role to play in helping people in all facets of the educational enterprise —

teachers, researchers, school and university administrators — use observation and video recordings of classroom teaching in ways that advance our understanding of mathematics teaching and learning, and abstain from applying these tools in ways that oversimplify teaching or lower the professional status of teachers. As a whole, the 2016 CIME workshop helped participants develop some shared understandings of the affordances and challenges of observation, and orient ourselves toward approaches to observation and the viewing of classroom videos that highlight, rather than obscure, salient aspects of teaching and learning.

What might we gain in the use of in-person and video observation of mathematics teaching?

The purpose of this electronic “booklet” is to synthesize some of these shared understandings into an organized whole, both for the benefit of CIME attendees who might want a concise summary of the wealth of information presented at the workshop, and to provide access for professionals in mathematics education who were unable to attend but wish to learn more about how educators and scholars from different backgrounds approach observation of mathematics teaching and learning. Rather than simply present a summary of each of the more than ten plenary talks and discussions presented at the workshop, I have endeavored to present key points from these presentations in a way that allows these points to build upon one another and coalesce into an organized picture of how we might exercise care and professional judgment in the use of in-person and video observation of mathematics teaching, and what we might gain from doing so. I hope to convey to the reader some of what we learned about how we can learn to observe mathematics teaching in ways that turn our attention to features of teaching that are most predictive of student learning and success, and at the same time observe to learn how students assimilate unfamiliar mathematical ideas and engage as a community in problem-solving, and how teachers facilitate or hinder this process through their presentation of mathematics content and their organization of classroom activity.

Accordingly, I have organized this essay into four sections:

1. Goals, Applications, and Potential Challenges of Observation

In this section, I summarize some of the purposes for which people observe mathematics teaching, and present some comments by CIME speakers about how they use observations to evaluate and improve mathematics instruction. I also discuss some of the challenges entailed by the use of observations to inform the improvement of teaching.

2. Learning to Observe: Developing and Using Frameworks for Observation

I briefly describe some frameworks for observing mathematics teaching that have seen widespread use prior to the 2016 CIME workshop. I then present some frameworks or lenses for observation or viewing of classroom video shared by speakers at the workshop.

3. Observing to Learn: Observation and Video as Sources of Insight and Professional Growth

I present a few examples of prior research in mathematics education that makes extensive use of in-person or video classroom observation. I then summarize some insights that workshop presenters gained through careful use of in-person and video observation, organized roughly into “insights about learning” and “insights about teaching.”

4. Next Steps: Directions for Future Study, Action, and Advocacy

Drawing from the plenary lectures at the workshop, I suggest some directions for further work that have the potential to enhance the utility of classroom observation as a tool for research or for the improvement of mathematics teaching. I close by relaying some cautionary notes from speakers at the workshop about potential misuses of observation.



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GOALS, APPLICATIONS, AND POTENTIAL CHALLENGES OF OBSERVATION

The complexity of observation

Teaching is a complex endeavor; as a result, the observation of mathematics teaching inherits many of the complexities of teaching, especially for observers who are experienced teachers and are attuned to these complexities. Observing a mathematics classroom in person or viewing a video segment of a mathematics class presents an observer with a wealth of visual and auditory stimuli, of which the observer will likely focus on a select few. One key message of the 2016 CIME workshop was that the way a person observes mathematics teaching depends on at least five factors:

1. *The overall purpose for which observation is used, and the role of the observer*

An instructional coach who observes a mathematics class with the goal of providing formative feedback for the teacher is likely to use a different analytic lens, and notice different phenomena, from a researcher who observes a class as part of a study on ***how teachers' uses of language focus students' attention on mathematical concepts or procedures.***

2. *The specific goal of the observation*

A student teacher supervisor from a preservice teacher preparation program will generally approach observations of a student teacher with the overarching purpose of assisting the student teacher's growth and providing formative and evaluative feedback. However, the supervisor may, depending on programmatic goals or on benchmarks for individual growth set by the supervisor, the mentor teacher, and the student teacher, focus a specific observation on a particular aspect of instruction, such as how the teacher organizes and facilitates students' problem solving, how the teacher introduces a new mathematical concept, or how the teacher works to ensure participation and access for all students. While no specific one of these foci precludes attention to the others, approaching an observation with a particular lens will naturally focus the observer's attention on certain aspects of teaching while obscuring others.

3. *The social identities and backgrounds of the observers*

Individuals bring racial, ethnic, linguistic, gender, socioeconomic, and professional identities to the task of observing teaching; these identities may allow one observer to notice phenomena that are invisible to another observer. For example, observers from varying backgrounds may be more likely to notice if students from marginalized groups have equal access to the mathematics being co-constructed by the teacher and students.

4. *The prior experience of the observer*

Each observer brings to the table knowledge and experience that add context to, and may create expectations for, the teaching and learning under observation. Some aspects of teaching, especially at the undergraduate level, may be visible only to observers with an advanced mathematics background; the way a calculus teacher introduces the Riemann integral is likely to appear different to an observer well-versed in real analysis than to one whose mathematical background does not extend far beyond calculus. Prior teaching experience may assist an observer in noticing specific aspects of teaching that may be invisible to the untrained eye. For example, an observer who is familiar with cognitively guided instruction (Carpenter et al., 1999) may be especially likely to notice that a second grade teacher, when teaching a lesson on word problems involving addition and subtraction, does not include any problems that illustrate the use of these operations in contexts involving comparisons. Exposure to a professional community of educators may encourage attention to issues previously left unexamined; for example, an observer who has received professional development on issues of equity and access may be more likely to notice opportunities to use diverse learners' funds of knowledge to enrich classroom discussions of mathematical concepts.

5. *The framework used for the observation*

In many cases, an observer uses a specific framework to guide the observation; when the goal of the observation is to provide formative feedback for the teacher, the same or a related framework may be used to guide the subsequent debriefing session. The framework may be informal and may be developed locally for the purpose of focusing the observer's attention on a specific aspect of teaching; for example, a mathematics teacher coach might categorize the questions a teacher asks during a whole-class discussion into several types (closed-ended numerical-answer questions, open-ended "why" questions, etc.), and observe which types of questions are asked during a specific

lesson. The framework may be based on extensive theoretical and experimental work, and may even be formalized in a clearly specified observation instrument, such as the Reformed Teaching Observation Protocol (Sawada et al., 2002). The specificity and rigidity of the framework may help focus the observer’s attention on facets of teaching that have been identified as possible targets for inquiry or professional development, but may also limit attention to aspects of teaching that are relevant but outside the scope of the framework.

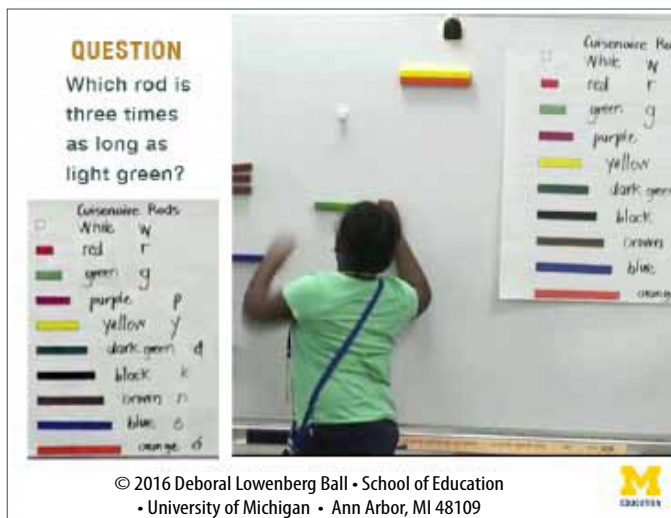
In her presentation “(How) Can we “see” the work of teaching mathematics?”, Deborah Ball, University of Michigan, highlighted the natural variation in what observers notice about teaching in the absence of a specified framework. She asked workshop participants to view

Seeing the work of teaching mathematics

and comment on a brief video of fifth-grade students in a summer mathematics program.

In the video, students share their work on an activity in which they use Cuisenaire rods to explore the idea of

First Viewing



multiplicative comparison, answering questions such as, “What rod is three times as long as the light green rod?”

After the video concluded, Ball asked participants to record what they saw and felt in different fields of a live Google document.

As expected, responses to this exercise were varied. Six responses, selected at random from the document, are shown in the following table:

<i>First viewing responses to students' use of Cuisenaire rods</i>
Children eager to try mathematics tasks
Students made their thinking visible by using the rods and orally explaining how the rods are related. Teacher gave explicit directions on social norms that supported respectful, productive participation.
Two different answers were posed by the students. Students have different understandings of "3 times."
Two answers proposed by students and each was treated as equally valid.
The question reveals a critical issue in teaching "times as much."
I heard the discussion of the problem being repeated so that everyone could understand.

From this small sample of responses, we can see that observations focused on different aspects of teaching and learning that were visible in the video. Some participants made observations about the learners in the room, while others focused on the teacher. Some observations were aimed at the mathematics in the segment, namely the meaning of the phrase "three times as much"; others described norms and teaching practices meant to build a classroom culture in which all student ideas and contributions are valued.

The variation in the responses notwithstanding, one participant noted that most of the comments in the Google document were free of evaluative language; they instead focused on reporting what they saw and noticed about the classroom in the video. This may have owed in part to Ball's initial framing of the activity: at the beginning of the discussion, she described several reasons, aside from evaluation and improvement of teaching, why we might observe classrooms; for example, we might wish to learn about the practice or teaching, or learn about the resources and supports necessitated by the work of teaching done at the enormous scale required by our education system. She asked the group to set aside the goals of evaluation and improvement for the duration of this discussion. Those who would like to use video to analyze the work of teaching rather than evaluating teaching, such as professional learning communities and video clubs, may benefit from setting norms similar to Ball's that focus observers' attention on the work of teaching and learning rather than on the perceived efficacy of the teacher. This idea was explored in further detail in Miriam Sherin's talk on video clubs for teachers (see the discussion of her presentation in the section *Observing to Learn: Observation and Video as Sources of Insight and Professional Growth*).

Second Viewing

After a brief discussion of the results of this broadly framed observation exercise, Ball asked participants to watch the clip again, this time without sound, and observe the classroom in the video through the following lens:

- What do you see about the space?*
- How are the space and physical features being used?*
- How do people use movement, gestures?*



Again, participants entered their comments in a Google document. Six responses, selected at random from the document, are shown as follows:

Second viewing responses to students use of Cuisenaire rods
The arrangement of the tables focuses the students' attention on the board and the center of the room. Students used several gestures to indicate the length of the rods — pointed to each rod and then used a sweeping motion to describe length; students used hand raising to indicate engagement.
There is a place that is a focal point for the examination of ideas. Students at front or at tables in u-shape arrangement. Teacher is in the back of the room or behind the students at times.
Space is too cold, maybe due to the white all over the walls, ceiling, and tables.
The teacher sat in one of the students' seats.
Board space for explanations, table space for exploration. Both spaces primarily for students (not teacher) to interact with mathematics. Teacher rarely standing at the front of the room.
Classroom is set up so that all students can see each other. DB moves around the outside of the horseshoe when the students are presenting, often standing at the back. This allows for the focus to be on the student who is presenting.

Again, one notices some variation in the comments: some observers highlighted the physical setup of the room, while others focused on how the teacher navigates around the classroom space. Some observers focused on the students' use of gestures to describe their mathematical thinking and show engagement. While some participants speculated about the purpose of a specific move by the teacher (suggesting, for example, that the teacher's presence at the back of the room encourages students to focus on a peer who is presenting), others simply reported what was directly observable in the video. However, the comments strongly suggest that the lens Ball suggested for this observation, together with the lack of audio, helped to focus participants' attention on physical aspects of the classroom and of the teachers' and students' use of it.

Participants' comments during the ensuing discussion support this hypothesis. One participant stated that he focused this time on what he could see rather than what he could hear, and that this led him to notice students' active engagement with the Cuisenaire rods. Another stated that much of the mathematics "disappeared" when the sound was muted; however, another claimed that the themes of proportional reasoning and geometric measurement stood out prominently in the visuals available in the muted video. I can personally vouch that my own viewing experience was significantly altered by the absence of sound: at the end of the clip, a student presents her solution to the question, "Which rod is three times as long as the light green rod?" On the first viewing, I focused on the student's explanation of her solution, the subsequent exchange between her and her classmates, and the teacher's facilitation of that exchange. On the second viewing, without the benefit of being able to hear the explanation, I noticed that a classmate of the presenter left his seat during her explanation and stepped out in front of her to receive a pencil. My focus on the student's explanation during the first viewing made this subtle detail invisible to me, and I was surprised to see something so different on the second iteration.

The lens suggested for this observation, together with the lack of audio, helped to focus participants' attention on physical aspects of the classroom and of the teacher's and students' use of it.

Third Viewing

Ball then asked us to select one of two frames for a third viewing of the video:

Students' Strengths: Select two of these five students: Langston, Madison, Michio, Larayne, Jerone. What does each one know and know how to do? What is your evidence?

Mathematics and Mathematical Practice: What mathematics do you see in this segment? Who is doing mathematics and what math are they doing? Identify specific examples and why you would label that “mathematics.”

Where do we focus our attention?

Each lens gave rise to a quite different set of observations. Taken as a whole, the viewing exercises in Ball's presentation serve as a vivid illustration of the inherent complexity of observing a classroom. There are so many aspects of learning, of teaching, and of the environment in which these take place that it is impossible for a single observer to take full notice of all of them at once. If we hope to use observation as a tool to learn about teaching — either to understand the work of teaching or to make improvements to the process of teaching — we must be intentional about where we focus our attention.

However, intentionality and focus are not sufficient. We must overcome some other challenges in order to obtain the full benefits that observation may afford us.

Analytic Challenges

In her presentation “When words get in your eyes: On challenges of investigating mathematics-in-teaching and on the importance of paying attention to words,” Anna Sfard, The University of Haifa, stated that “we are ... prisoners of our own vocabularies” when we attempt to describe the mathematics that unfolds in a classroom we are observing. She claimed that the words we use to describe teaching often cause us to see less and with less definition, rather than more and with greater definition. An example of the ambiguity and lack of specificity inherent in the language we use emerges as soon as we attempt to unpack the word “mathematics” when we say that we want to observe the mathematics in a classroom. Do we mean the mathematical *content* of a lesson, or the conceptions taught in the lesson, or the ideas observable in student work and discourse?

According to Sfard, the language we use to describe teaching suffers from three defects:

1. **Observers' words are too vague.** We often describe the teaching that we see, or that we hope to see, using words that are not sufficiently well defined and operationalized for purposes of observation and reporting. Our language often reflects a duality between the *form* of mathematics teaching — the ways in which teacher and students interact — and the *content* being taught. We frequently have difficulty describing classroom interactions in a way that faithfully represents their mathematical substance and accounts for how students develop desired mathematical understandings.
2. **Observers' words are too broad.** Experts often find it difficult to notice distinctions among different representations or expressions of the same mathematical object or idea. These distinctions, however, are often quite noticeable to students, who may not yet know that these different representations refer to the same object. Similarly, in the work of observing and analyzing teaching, our words often entice us into regarding teaching practices as equivalent rather than noticing subtle but important differences.
3. **Observers' words blind them to the unnamed.** We tend not to see things that we do not have language to describe. When we look for a specific element of teaching or learning, we often notice its presence or absence, but do not have language to describe it or the surrounding context in greater detail.

Deborah Ball suggested several analytic obstacles, aside from the lack of a sufficiently precise technical language for the analysis of mathematics teaching, that may limit what and how much we can learn from observation of mathematics classrooms:



Anna Sfard asked, "What do you see in this picture?" She pointed out that while we have names for tree, dog, and person in the picture, we don't have names for the other shapes and forms — and thus we don't "see" them as we see the things that we can name.

1. Much of the work of teaching is invisible, hidden by individual cognition and social practices that "disappear" aspects of the work that are not formally articulated as parts of teaching but are essential to the functioning of a classroom (Lewis, 2007).

2. Student perspectives and experiences are difficult to take into account in the observation of teaching, and often are not included in analyses of teaching.
3. The natural inclination to judge and evaluate what we see in an observation often prevents us from seeing what actually happens in a classroom in adequate detail and describing it with fidelity.

Systemic challenges

If we hope to use classroom observation as a tool for the improvement of teaching, we must develop articulations of the work of teaching that help us to identify potential areas for systemic and individual improvement. The lack of a common professional language that allows us to describe the work in adequate detail can be viewed as an analytic challenge, one that the field of mathematics education research can lessen through ongoing effort. However, the improvement of teaching is also constrained by many systemic and practical limitations. Efforts to improve instruction often take place in settings where individual and organizational bandwidth is occupied, first and foremost, by the ongoing work of teaching. Furthermore, certain social and cultural obstacles conspire to hinder improvement and reform efforts. These obstacles can be daunting even under the best of circumstances; they worsen considerably when improvement efforts do not take care to protect the dignity and professional status of the stakeholders involved.

Teacher isolation

Mara Landers, Los Medanos College, pointed out that most higher education institutions, like many K-12 schools, follow an “egg crate” model of teaching, in which classrooms follow a cellular organization and teachers spend most of their time at work in physical isolation from other professionals (Lortie, 1975). This physical separation creates a cultural expectation that teaching is an activity to be performed in isolation, and hinders the growth of collegial relationships that may lead to collaboration in teaching. The increasing use of adjunct faculty at institutions of higher education threatens to increase this professional isolation, since many adjunct instructors are on campus for only a few hours each week.

Arguably the most challenging cultural obstacle, as Ball points out, is that the teaching profession does not have norms that encourage the routine observation and discussion of teaching. While educators in some countries regularly use observation to test hypotheses regarding student responses to lessons and make adjustments to improve the lessons' effectiveness (Stigler & Hiebert, 2009), professional norms and the structure of the work of teaching discourage such activity in the United States.

On the first day of the CIME workshop, a panel of postsecondary mathematics faculty discussed the use of observation, including video, in the specific institutional settings within which they work. This discussion shed light on several practical and systemic factors that may inhibit the use of observation to learn about and improve the process of teaching.

High stakes of observation

In many educational settings, classroom observation is used exclusively as an evaluation tool; therefore, observations are high-stakes events that may influence promotion and retention decisions. Because these observations are high-stakes, they are in many places subject to constraints imposed by faculty unions. At some institutions, unions require that observations used to evaluate faculty follow a uniform rubric that is applied across all departments. In such cases, evaluation criteria tend to be generic and ask whether instructors have met certain basic standards of professionalism, such as treating students with respect, stating learning objectives, and starting class on time. Because these evaluations cannot focus on characteristics of effective teaching that are specific to mathematics, they may lend little insight into why students are meeting or falling short of specific learning goals.

Reluctance to observe and be observed

Possibly due to the fact that most teaching observations are high-stakes, most higher education faculty members are initially reluctant to be observed. As Rob Indik, University of Arizona, noted, many mathematics faculty are suspicious of college and university level administrators who attempt to impose reforms or regulations on teaching at the departmental level; this sometimes translates into a reluctance to use institutional resources that provide for observation and videotaping of teaching.

Difficulty of mapping observations to student learning outcomes

Indik stated that one of the greatest frustrations he faces as an evaluator of teaching is that he is unable to predict, based on his observations of instructors, whose students are most likely to succeed on common exams and in future courses. He has

used his university's analytics to track students' progress within and across courses, and has found that some instructors who seem to perform poorly during observations have excellent student learning outcomes, while others who appear to be excellent in the classroom have average to mediocre student outcomes. Indik's experiences are representative of those of many faculty who see their colleagues attempt to implement research-based practices in teaching, or attempt to implement these themselves, and do not see discernible gains in student learning. When faculty see little correlation between what they observe during classroom visits and learning gains that can be quantified and measured, they may lose faith in observation as a tool that can lead to meaningful improvements in teaching.

Limited individual and institutional bandwidth

Exacerbating all of the aforementioned obstacles is the fact that almost all who teach have limited time and energy to devote to practices that may result in long-term improvements in teaching. As an example, Scott Peterson, Oregon State University, noted that the graduate students who teach under his leadership are expected to keep the amount of time they devote to teaching within strict limits; this is typical for graduate student instructors at many universities. Due to constraints on time and organizational effort, initiatives to observe and study teaching practices may have difficulty getting off the ground, or may fizzle shortly after their inception.

Efforts to overcome systemic challenges

In addition to describing some practical obstacles that have slowed the growth of teaching improvement efforts, the panel of higher education faculty discussed some initiatives at their institutions that have helped to make teaching public and an object of systematic study.

Mara Landers, Los Mendanos College, described a "teaching communities" program in the mathematics department at her community college, modeled after the Japanese lesson study model. This model, in which adjunct faculty participate voluntarily, involves observation of different community members' classes. Teaching communities work to build collegiality among instructors and allow for sharing of instructional innovations that encourage greater student participation in mathematical thinking in class. The department markets the program as an opportunity to learn more about teaching and compensates participants for their time, but Landers notes that faculty are still reluctant to join if they view teaching as an individualistic enterprise.

Peterson is responsible for the professional development of graduate teaching assistants in his department. Each fall, the department hosts a workshop for new TAs in which participants teach seven- to ten-minute lectures, and subsequently watch videos of their presentations. While some TAs dread watching these recordings, the videos help them notice tendencies in their teaching mechanics, such as favoring one side of the classroom over the other. In addition, experienced instructors present sample lessons that model ways to foster active student engagement. Thus teaching assistants in the program have the opportunity to benefit both from observing themselves and analyzing their own presentation skills, and from observing veteran teachers who model successful practices.

Indik stated that while most of his colleagues are reluctant to be observed, some efforts to improve instruction within specific contexts have germinated within his department. In one such effort, aimed at developing uniform standards and curriculum for the department's introduction-to-proofs course, faculty observed each other's classes and participated in weekly small-group discussions to build shared understanding of instructional practices that support student success in the course. With guidance and assistance from colleagues not teaching the course, the instructors also conducted a focus group of students in order to ensure that student perspectives were adequately represented in improvement efforts. The effort resulted in the creation of a departmental syllabus for the course and indirectly led to the development of a supplemental instruction course focused on building students' logical reasoning and writing skills.

These efforts differ from one another in that they use self-observation, peer observation, and observation of experienced instructors to varying degrees and for different purposes. However, a common thread among all of these efforts is that they aim to make the work of teaching more visible and allow for discussion of practice in non-evaluative contexts. This serves to make the professional practice of teaching more public and elevates instructional improvement to the status of a collaborative endeavor rather than a flurry of ephemeral innovations in isolated silos.

SECTION 2

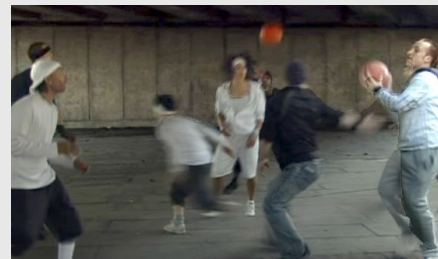
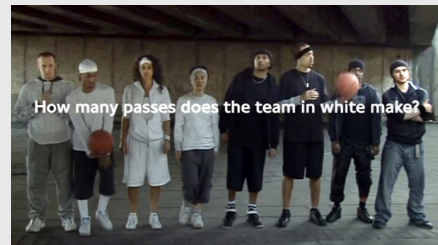
LEARNING TO OBSERVE: DEVELOPING AND USING FRAMEWORKS FOR OBSERVATION

The interactive observation exercises posed by Deborah Ball at the CIME workshop illustrated the natural variation in what different observers may notice while watching a lesson unfold in a classroom. While it can be beneficial to view the same lesson through a variety of different lenses, asking observers to view a lesson from one specific perspective can allow for more in-depth discussion of specific aspects of teaching.

In this section, I describe several frameworks for the observation of teaching that have seen widespread use and reflect different perspectives on the work of teaching. I then share some frameworks offered by speakers at the 2016 CIME workshop. In this section I do not imply any particular degree of formality or specificity when we use the word “framework”; a framework may be as nonspecific as asking observers to focus on the mathematical content of a lesson, and may be as rigid as presenting a list of possible descriptions of a teaching episode and asking the observer to rate the degree to which these descriptions were valid of the episode under observation.

When we choose a framework for observation, we make a choice to focus our attention on a particular set of phenomena. This choice is consequential, for it influences what we are likely to see in a classroom episode — and what we are likely to miss.

Given our framework for observation, what are we likely to miss?



At the beginning of this 20-second video the narrator asks, “How many passes does the team in white make?” A few frames of the video are shown here.

In their oft-cited study on “inattention blindness,” Simons and Chabris (1999) demonstrated that events can occur in an observer’s field of vision but remain unnoticed if the observer is focused on a monitoring task. Lawrence Clark and Imani Goffney, both with

the University of Maryland, illustrated this phenomenon in their CIME workshop presentation with a video produced by the 2008 THINK! Campaign of Transport for London, and adapted from a video used in Simons’ and Chabris’ study (Transport for London, 2010).

In the video, there are two teams, one in white shirts and one in black shirts. Each team performs a series of basketball passes for approximately twenty seconds. Prior to watching the video, viewers are asked to count the number of passes performed by the team in white. Because viewers are focused on this task, many miss the fact that during the series of passes, a person in a dark-colored bear costume moonwalks through the midst of the action. Simons and Chabris demonstrated that the frequency with which people miss such an unusual stimulus varies with the transparency of the stimulus and the difficulty of the monitoring task in which viewers are engaged. This effect serves as a compelling metaphor, if not as direct evidence, for one of the most essential complications associated with observing using a specified framework: the tendency to miss phenomena not directly tied to the framework, even when they may bear upon the construct that the framework is intended to highlight.

For example, an observer using a framework meant to call attention to the frequency and types of student engagement during a lesson may be asked to count the number of episodes of student talk during a lecture, and note whether the teacher responds directly to the student or directs other students to respond in each case. However, this process of counting and recording may focus attention away from other pertinent issues, such as whether the student talk is focused on mathematics, whether the student talk episodes revolve around tasks having appropriate cognitive demand, or whether opportunities for interaction are shared among all students or only a few.

Because the selection of a lens for observation necessarily draws attention away from phenomena unrelated or tangentially related to the chosen lens, I have endeavored to include in the discussion of the frameworks below some commentary on aspects of teaching and learning that may become “invisible” due to inattention blindness.



**Measuring
alignment
to inquiry-
oriented
practice: the
Reformed
Teaching
Observation
Protocol**

In the early 2000s, the Arizona Collaborative for Excellence in the Preparation of Teachers (ACEPT), a project funded by the National Science Foundation, developed a twenty-five-item observation instrument meant to help researchers and evaluators measure the extent to which classrooms have adopted practices aligned with reform principles in mathematics and science education (Sawada et al., 2002). This instrument, the Reformed Teaching Observation Protocol (RTOP), draws its operationalization of the notion of “reform” in mathematics and science education from policy documents such as the National Council for Teachers of Mathematics (NCTM) *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) and *Principles and Standards for School Mathematics* (NCTM, 2000), and from existing research

studies and observation instruments grounded in the 1989-2000 mathematics and science standards. Items on the RTOP assess whether classroom practices support a culture of student-centered exploration and inquiry and foster the development of critical thinking and problem solving. The following items from the instrument highlight this orientation:

1. *The lesson was designed to engage students as members of a learning community*
2. *In this lesson, student exploration preceded formal presentation.*
3. *Students were actively engaged in thought-provoking activity that often involved the critical assessment of procedures.*
4. *Student questions and comments often determined the focus and direction of classroom discourse (Sawada et al., 2002, p. 253)*

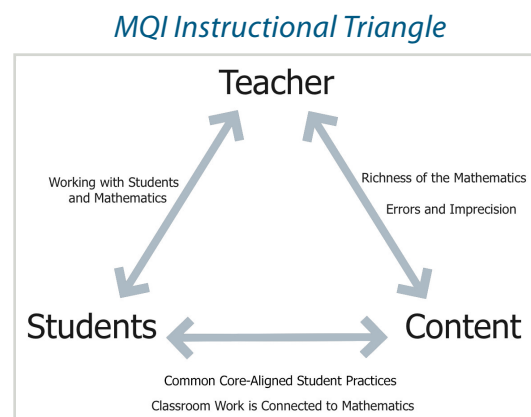
The authors found, in a study of secondary and postsecondary mathematics and science classes, that instructors’ RTOP scores were predictive of normalized gains in various measures of student learning. The instrument remains in wide use as both a tool for educational research (see, for example, Ebert-May, et al., 2011; Wilson, et al., 2010), and as an evaluation tool for professional development programs for mathematics and science teachers.

The design of the RTOP reflects a focus on the form of mathematics and science teaching: how the teacher and classroom culture set the stage for inquiry and exploration, how ideas from students are elicited and addressed, and how the structure of classroom activities fosters the development of students’ intellectual agency.

Observers using this instrument in mathematics classrooms, therefore, would likely find their attention drawn to the structure of interactions among students and between students and the teacher, to the types of activities in which students engage, and to opportunities for students to explore and make sense of mathematical ideas before being expected to develop procedural fluency with them. Because the instrument is designed for use in both mathematics and science classrooms, the items on the RTOP do not, for the most part, focus on the content of instruction nor suggest specific features of mathematical content that would indicate alignment with standards-based practice.

Measuring integrity and richness of content: the mathematical quality of instruction

In 2010, the Learning Mathematics for Teaching Project of the University of Michigan developed a framework and observation instrument for measuring the mathematical quality of mathematics instruction (Hill et al., 2008; Hill, 2010). In discussing the development of the instrument, Hill notes that one can observe, in some classrooms, practices that are structurally suggestive of reform-oriented teaching but that do not bring students into contact with key mathematical ideas. As an example, she recounts a classroom episode in which a teacher reads a book about the number π to her students, and then students spend 40 minutes cutting circular “pies” out of paper and measuring their circumference. Although an observer might characterize this activity as a student-centered exploration, much of the lesson time is spent on activities (cutting and measuring) that do not elicit engagement with the idea that the value of the ratio of the circumference of a circle to its diameter is the same regardless of the size of the circle. Hill suggests that we may gain additional insight into the effectiveness of teaching by understanding *mathematical quality of instruction* as a dimension of teaching distinct from alignment to reform-oriented practice. To view mathematics teaching through this lens is to sharpen focus on the content of instruction, though not necessarily to neglect the form of instruction altogether.



Source: http://sites.harvard.edu/icb/icb.do?keyword=mqi_training&pageid=icb.page394700

The *mathematical quality of instruction* (MQI) observation protocol asks observers to assess, according to specified criteria, dimensions of the mathematics in instruction such as the following:

- **Connecting classroom practice to mathematics.** Observers assess whether classroom activities connect to mathematical ideas or procedures, and identify segments of lesson time that are spent on activities that are not mathematically productive, such as cutting and pasting, or dealing with disciplinary issues.
- **Richness and development of the mathematics.** Observers rate whether instruction links different representations of the same mathematical idea, whether mathematical explanations are present and correct, and whether instruction contains reasoning and justification for the ideas in a lesson.
- **Responding to students.** Observers rate whether the teacher correctly interprets student productions such as written work and verbal responses to questions and whether the teacher uses student errors in instruction in mathematically meaningful ways.
- **Language.** Observers rate whether the teacher uses language and notation appropriately, and whether there is explicit discussion of the meanings of the language and notation used.
- **Mathematical errors.** Observers note whether instruction contains significant mathematical errors, such as misuses of language, incorrect mathematical explanations, or incorrect execution of procedures, that are not corrected or resolved through class discussion.

The *Learning Mathematics for Teaching* project compared teachers' MQI scores to their scores on paper-and-pencil assessments of mathematical knowledge for teaching (MKT), and found a statistically significant positive correlation between MQI scores and MKT scores (Hill, 2010). In particular, they found a highly significant negative correlation between MKT scores and the presence of significant mathematical errors. These findings suggest that mathematical knowledge for teaching supports instruction at a high level of mathematical quality. However, Hill et al. point out that the influence of MKT on the mathematical quality of instruction is mediated by many factors, such as teachers' beliefs about mathematics teaching and learning, their use of textbooks and other curriculum materials, and their general pedagogical and classroom management skills (Hill et al., 2008).

Observers using the MQI protocol to analyze lessons would likely find their attention drawn to the content of lessons: whether the mathematics is correct, whether it is developed in a meaningful way, and whether reasoning and explanations for claims made during the lessons are correct and complete. Observers' attention may, on the other hand, be drawn away from certain features of the way in which mathematical content is presented to students: for example, how the teacher motivates the mathematical content for students, or the degree to which students are invited to "discover" the key mathematical ideas of a lesson.

Seeing the mathematics in teaching

If we hope to evaluate the integrity of the mathematics in a lesson, it is desirable to be able to identify clearly the main mathematical ideas embedded in the lesson as enacted by teachers and students. In their presentation "Seeing the mathematics in teaching," Lindsey Mann and Roger Howe suggested that we may better attune ourselves to the mathematics in teaching by viewing a lesson through a non-evaluative lens, with the primary goal of identifying the mathematical ideas present in a lesson. Mann pointed out that this may be challenging for a variety of reasons:

- The mathematics taught in grades K-12, especially in the early grades, includes many ideas that are automatic — and therefore invisible — for those experienced in mathematics. As Sfard pointed out in her presentation, experts have difficulty noticing differences among ideas that we have learned to regard as the same.
- We do not always have the language necessary to describe mathematical ideas in teaching with the precision and refinement that we desire. For example, elementary students learn that any positive integer can be decomposed as a sum of numbers of the form $d \times 10^n$, where d is a digit and n is a nonnegative integer.

$$7402 = 7000 + 400 + 2 = 7 \times 10^3 + 4 \times 10^2 + 0 \times 10^1 + 2 \times 10^0$$

- However, there is no standard terminology in the literature or in elementary mathematics curricula for numbers of the form $d \times 10^n$ (Howe & Epp, 2006). (The authors have proposed the term *place value number* or *single-place number*.)
- We often would like to know more about the context in which a classroom episode is situated, so that we can better understand the mathematical foundation on which a lesson is assumed to rest. However, most observations of mathematics classrooms do not provide this context.

- There is a lack of consensus on what constitutes mathematics.
- We tend to be professionally and personally invested in student learning and in the subject of mathematics; this makes it more difficult to view the mathematics in a lesson through an objective and non-evaluative lens.
- The distinction between the mathematics actually developed in a lesson, and the mathematics that we expect or would hope to see in a lesson, is sometimes blurry.

To allow workshop participants to practice observing the mathematics in teaching, Mann and Howe showed a video of a lesson on place value in a third-grade classroom. In this video, a teacher leads students as they practice pronouncing place-value names (“ones,” “tens,” “hundreds,” “thousands,” “ten-thousands”), and explains to students that a digit in a specific place represents a certain number of the place value. For example, a 3 in the ten-thousands place represents three ten-thousands, or “thirty thousand.”

Workshop participants’ interpretations of the mathematics in the brief video varied considerably. One participant observed that the ten-thousands place was new for at least some of the children in the video, and suggested that this episode gave students the opportunity to notice a key aspect of the structure of the base-ten system: that each place value is a power of ten. Other participants argued that while the opportunity to notice structure was present, all that the lesson required of students was that they learn to pronounce the names of place values and state what a given digit in a given place represents. This led to a discussion of whose mathematics we talk about when we talk about the mathematics in a lesson: do we mean the mathematics intended by the teacher, or the mathematics enacted by the students during the lesson?

One participant noted another aspect of the structure of the base-ten system that was not made explicit by the teacher in the video: that there is an implicit “base thousand” system underlying the way we write base-ten numbers, suggested by the three-digit blocks (or “periods”) formed when we write commas in a multi-digit numeral. Howe responded that it would be impossible for the teacher at this point to make that structure explicit, since the

Whose mathematics do we talk about when we talk about the mathematics in a lesson: do we mean the mathematics intended by the teacher, or the mathematics enacted by the students during the lesson?

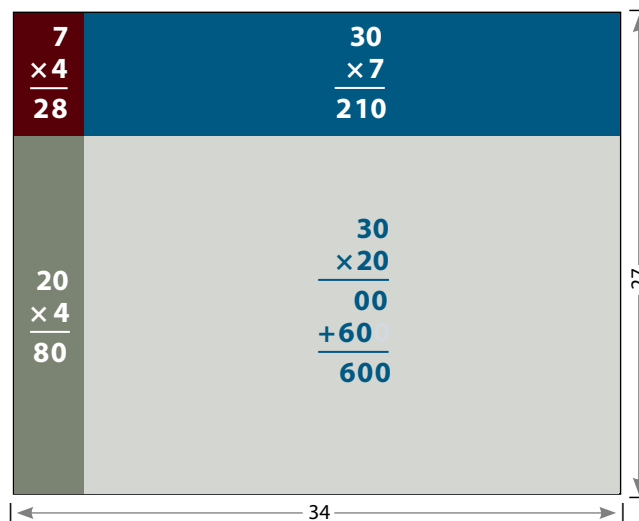
students have only learned place values up to the ten-thousands place. Another participant added that most third-graders do not have a meaningful idea of the sizes of numbers like ten thousand and a hundred thousand, and therefore may not be prepared to make sense of this periodic structure. This exchange highlighted another challenge associated with identifying the mathematics in a classroom episode: an observer often needs considerable knowledge of learning progressions and of the development of children's mathematical thinking in order to make sense of a teacher's decisions about what structure to highlight and what structure to leave hidden.

The lack of consensus on what constitutes mathematics came into view when two participants stated that the lesson seemed more like a lesson on how to pronounce the names of numbers than on mathematics. Another participant replied that the lesson did deal with the mathematical idea that there are different ways to decompose numbers based on their base-ten representations.

Mann and Howe then showed a clip from a third-grade classroom in which students learned about the area representation of multiplication of multi-digit whole numbers. After the video concluded, the presenters again asked workshop participants to describe the mathematics they saw in the video.

In the ensuing discussion, participants generally agreed that one of the major mathematical ideas of the lesson segment was that we can decompose a product of two two-digit numbers into partial products, with each partial product having two single-place number factors; for example, we can decompose:

$$34 \times 27 \text{ as } 30 \times 20 + 30 \times 7 + 4 \times 20 + 4 \times 7$$



The area representation of multiplication, when supplemented with appropriate explanation, helps students justify this idea. However, the discussion also revealed some ambiguity about whether and how the area representation actually supported students' understanding of multiplication in the lesson segment shown.

One participant pointed out that the teacher's verbal description of the area model was more in line with the way a teacher might describe an array model for multiplication, in which the product $m \times n$ is represented as an array of objects (possibly unit squares) with m rows and n columns. Another participant expressed that the work of connecting the visual representation to the symbolic representation, which is a necessary part of helping students build meaning for the standard algorithm for multiplying two-digit numbers, did not seem to be present in the clip. This participant noted, and Howe reminded the group, that this connection could have been made after the part of the lesson shown in the video. This reinforced the idea that while it may be useful to make note of important mathematical ideas that are not visible in a classroom observation or video, we should focus on taking inventory of the ideas that we do see, keeping in mind that pieces we consider "missing" may be part of the context of what happened prior to our observation, or may occur after our observation is over.

**“What
mathematics
do we see in
this lesson?”**

A participant asked, at the conclusion of the presentation, what goal might be achieved by asking, “What mathematics do we see in this lesson?”, and how answering this question helps us achieve that goal. Mann and Howe both stated that if we want to be able to describe the access students have to important mathematical ideas in a lesson, we must be able to identify those ideas with some precision; answering this question helps train our attention on the specific issue of what mathematical ideas a lesson is trying to develop.

Another participant suggested that it is impossible to disentangle this question from other issues, such as how the mathematical ideas are taught, what representations are used, and how the classroom discourse develops language for these ideas. Mann agreed, and emphasized that when we pay attention to the mathematics in a lesson, we do not necessarily block out these other issues.

Traditional and emerging foci: mathematical knowledge and mathematical dispositions

Traditional foci

In their presentation, Lawrence Clark and Imani Goffney began by presenting a list of “traditional” and frequently referenced foci in the observation of mathematics teaching and learning for which frameworks have already been developed and vetted; these foci include:

- The use of mathematical language
- Discourse practices (e.g., Moschkovich, 2007)
- Maintaining cognitive demand (Stein & Smith, 1998)
- Wait time (e.g., Tobin, 1986)

Emerging Foci

Goffney noted that we often view teaching and learning through these lenses not only because we believe they are important, but also because we generally believe we know how to measure and analyze these particular aspects of instruction. She suggested, however, that the mathematics education research field should continue to invest effort and resources in learning how to view instruction through other lenses, such as:

- Equitable teaching practice
- Positioning
- The formation of mathematical identity
- The development of students’ mathematical dispositions

Some of these foci are influenced by the “social turn” and the subsequent “sociopolitical turn” in mathematics education research (Lerman, 2000; Gutiérrez, 2013).

There is a growing body of evidence that these emerging foci can enhance our understanding of factors that influence the effectiveness of mathematics instruction. As an example, Clark shared results from a study on the relationship between teachers’ mathematical content knowledge and awareness of students’ mathematical dispositions and student achievement (Campbell et al., 2014). In this study, the authors asked teachers to complete an assessment of mathematical and pedagogical content knowledge and respond to a survey of their beliefs about mathematics learning and students’ mathematical dispositions. By examining students’ state mathematics achievement scores, the authors found that students of teachers with high mathematical content knowledge and high self-reported awareness of mathematical dispositions, on average, had achievement scores 0.24 standard deviations higher than those of teachers with high mathematical content knowledge but low awareness of mathematical dispositions.

Clark pointed out that while this difference in achievement is significant and revealing, further work is required to discover how teachers' awareness of students' mathematical dispositions helps them develop instruction that promotes student achievement. The mathematics education research field, therefore, may benefit from learning to observe instruction through emerging lenses, such as how teachers express awareness of mathematical dispositions in instructional contexts. However, this presents a challenge: in order to conduct valid, replicable research in this vein, the field must learn to observe phenomena such as the cultivation of mathematical dispositions in ways that are reliable and not dependent upon the unique perspectives of individual observers.

To illustrate some of the nuances of this challenge, Goffney and Clark presented a video of a teacher leading a lesson on multiplication of two-digit numbers (first of two-digit whole numbers, then of numbers written as decimals in ones and tenths), and asked workshop participants to rate the video according to two different rubrics: attention to precision in the use of mathematical language, and communication of high expectations. Participants were asked to rate each dimension's presence in the video as "substantial," "some," or "none."

Precision in use of mathematical language

Ratings of the teacher's attention to precision in language were somewhat consistent among workshop participants, though not wholly so. Many participants noticed that the teacher stated that in a multi-digit multiplication problem, introducing decimal points into the two factors does not change the "numbers" being multiplied or the "numbers" in the product. On the other hand, some participants also noted that the teacher pronounced the numeral 4.2 "four and two tenths" rather than "four point two," and viewed this as an instance of deliberate attention to precision. Most participants rated the episode as showing "some" or "none" when asked about attention to precision in language.

Communicates high expectations

Workshop participants' responses to the question of whether the teacher communicated high expectations were considerably more varied, and hinged largely on the language of the rubric presented:

None: There is no evidence that the teacher has high expectations that all students have the capability to engage in the mathematical activities of the lesson. Also select "none" if there is evidence that the teacher positions some students as "smart" and others as "not as smart". In your short answer response, provide examples that support your rating choice.

Convey confidence to one, all, or some?

Participants who rated the communication of high expectations in the video as “none” pointed to the fact that the teacher, when introducing multiplication problems involving decimals, explicitly referenced a group of students in an enrichment program as having already received instruction on how to multiply decimal numbers, and thus being “experts” who could support their classmates. Some participants interpreted this as an instance of positioning certain students as “smarter” than their classmates and thus selected the rating “none”, although other participants pointed to the fact that the teacher accepted and praised a nonstandard correct solution from a student, and the fact that the teacher conveyed confidence that her students would handle the introduction of decimals successfully. Ratings of the episode on this dimension ranged from “substantial” to “none.”

Clark pointed out that the greater degree of variability in ratings on the “expectations” rubric may be attributable to the fact that as researchers, we are still learning how to “see” high expectations in classroom settings, while we are generally more comfortable discerning whether the mathematics and language in a lesson are correct. This discrepancy may have been accentuated by the video the presenters selected: the most notable error in mathematical language in the video is consistent with a common misconception about decimals and place value that is familiar for most K-12 mathematics education researchers, and thus was likely to be “flagged” by a majority of participants at the workshop. On the other hand, the ways in which different teachers convey high or low expectations are more subtle, and are likely to vary considerably from teacher to teacher, making it more difficult to standardize ways of observing and assessing teachers’ expectations so as to elevate interrater reliability to an acceptable level. Efforts to develop mechanistic explanations for how awareness of mathematical dispositions leads to more effective instruction may encounter similar challenges: in order to explain why this awareness is useful for teachers, we must begin to account for the various ways in which teachers attend to students’ mathematical dispositions in the day-to-day work of teaching, both in classroom practice and in extramural activity such as lesson planning and family outreach.

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Weaving frameworks together: Observing student collaborative work in mathematics

In his presentation “Attending to student thinking and their interactions when working in small groups,” Chris Rasmussen, San Diego State University, used an episode from a college differential equations class to illustrate how we can weave together several different frameworks for observing mathematics learning to build a better understanding of how students individually and collectively develop an important mathematical idea. In his presentation, Rasmussen played a brief video segment (available in the video of his presentation at <https://www.msri.org/workshops/793/schedules/20601>) of a ten-minute classroom episode in which four students

work together on a problem designed to lead students to discover Euler’s method for approximating the solution to a first-order ordinary differential equation without having received prior formal instruction on the algorithm. He then shared the task in which the students were engaged with workshop participants:

Consider the following rate of change equation, where $P(t)$ is the number of rabbits at time t (in years): $dP/dt = 3P(t)$ or in shorthand notation $dP/dt = 3P$. Suppose that at time $t = 0$ we have 10 rabbits (think of this as scaled, so we might actually have 1000 or 10,000 rabbits). Figure out a way to use this rate of change equation to approximate the future number of rabbits at $t = 0.5$ and $t = 1$.

He then discussed how the perspectives of *disciplinary practices, collective mathematical progress, participation in mathematical activity, and mathematical meanings* can be integrated together to build a rich description of the group’s trajectory toward a solution of this problem and a formulation of Euler’s method. Rasmussen’s approach is based on the emergent perspective and interpretive framework of Cobb and Yackel (1996).

Emergent perspective and the (classroom) interpretive framework

(Cobb & Yackel, 1996)

Social Perspective	Individual Perspective
Social norms	Beliefs about one’s own role, others’ role, and the general nature of mathematical activity
Sociomathematical norms	Mathematical beliefs and values
Collective mathematical progress	Mathematical conceptions and activity

The framework (Rasmussen, Wawro, & Zandieh, 2015) expands the part of the interpretive framework of Cobb and Yackel that pertains to classroom mathematical practices and

individual mathematical conceptions, taking into account the growing work in the cognitive tradition on individual mathematical meanings (e.g., Thompson, 2013) and the essential role that disciplinary norms play in postsecondary mathematics classrooms.

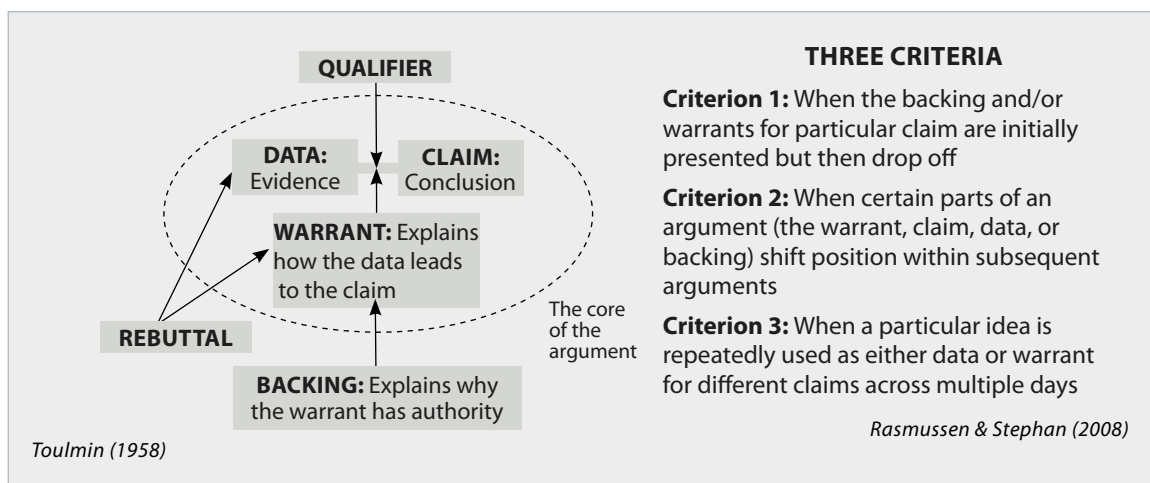
In his presentation, Rasmussen used the Euler's method episode to illustrate ways of operationalizing the four perspectives underlying the modified interpretive framework:

Disciplinary practices

Students in inquiry-oriented mathematics classes have opportunities to engage in activities similar to those of professional mathematicians, such as *defining*, *symbolizing*, *algorithmizing*, and *theoremizing* (Rasmussen, Wawro, & Zandieh, 2015, p. 264). Rather than attempting to give precise definitions for these processes, which are situated in the historical and cultural practices of local communities, Rasmussen et al. use a grounded approach to characterize the activities of students in the broader context of disciplinary practices. In their work on Euler's method, students engage in the activity of algorithmizing, developing an increasingly systematized understanding of the relationship between a quantity and its rate of change specified by a differential equation, using this relationship to build an iterative process for approximating a solution to the differential equation, and using symbols to record the steps of the algorithm.

Collective mathematical progress

As students discuss a problem in a collaborative setting, they make and evaluate arguments that guide the group's progress toward a resolution of the problem. Rasmussen et al. use Toulmin's (1958) analysis of the core of an argument, in which an argument consists of a *claim*, *data* or evidence in support of the claim, and *warrants* which explain how the data support the claim.



A student making an argument may also suggest some backing to explain why the warrant has authority. By making and evaluating arguments, students build ways of reasoning that function as if shared by the group; for example, mathematical statements that initially act as claims and require evidence may later act as evidence for more advanced claims. In their study of the Euler's method episode, Rasmussen *et al.* decompose the students' discussion into a sequence of arguments, and identify certain statements within the discussion as claims, evidence, warrants, or backing for those arguments.

Individual participation in mathematical activity

On an initial viewing of a brief video segment of the Euler's method episode, one workshop participant observed that a white female student in the group depicted appears to provide a mathematical explanation of her thinking for a white male student, while a female student of color and another male student in the group are excluded from the discussion. Rasmussen *et al.* use Krummheuer's (2007, 2011) constructs of production design and recipient design to characterize ways in which students participate in mathematical discussion. Production design categorizes students who speak in a group discussion into several roles. Recipient design categorizes students who listen to an utterance in terms of whether their participation as listeners is essential or peripheral to the group.

Production design

- An author is a speaker who is responsible for both the content and the formulation of an utterance. (Rasmussen *et al.* expand this role to accommodate the notion of co-authors, speakers who co-construct the content or formulation of an idea.)
- A *relayer* is a speaker who is responsible for neither the content nor the formulation of an utterance.
- A *ghostee* is a speaker who attempts to take part of the content of a previous utterance and express a new idea.
- A *spokesman* is a speaker who attempts to reformulate the content of a previous utterance in his/her own words.

Recipient design

- A conversation partner is the listener to whom the speaker seems to allocate the subsequent speaking turn.
- A co-hearer is a listener who does not appear to be the intended recipient of the next speaking turn but to whom the speaker's comment is addressed.
- An over-hearer is a listener who seems to be tolerated by the speaker but is not included in the discussion.
- An eavesdropper is a listener who is deliberately excluded by the speaker from the conversation.

Participation in these roles during the course of a group discussion is fluid, with students at various turns assuming one role or another. In the brief clip shown during Rasmussen's presentation, Liz, the female student who explains her thinking, spends most speaking turns

as author, while Jeff, the male student to whom she appears to be speaking, is a conversation partner. Deb and Joe, the other two students, appear to be over-hearers. However, over the course of the entire ten-minute episode, the students' participation appears to become more even according to the production/recipient design analysis, with the four group members participating as co-authors roughly the same number of times. Rasmussen hastened to point out, however, that this does not imply that all of the students' mathematical contributions were created equal: Jeff submits two mathematically incorrect arguments, while Joe participates as relayer or spokesman disproportionately often. Liz and Deb make the majority of the contributions that move the group toward a successful formulation of Euler's method.

Individual mathematical meanings

In a collaborative setting, each student brings his/her own individual meanings for relevant mathematical ideas to bear on the group's work. This is particularly evident in the ways in which group members in the Euler's method episode talk about the rate of change dP/dt ; these ways reflect some common ways of thinking about rates among students and teachers (Weber & Dorko, 2014). These ways of thinking include:

- ***Rate as "steepness"*** — that is, the rate of change as the steepness of a graph, or as an index of how sharply a quantity changes with respect to changes in an independent variable.
- ***Rate as a tool*** — for determining how much population change should occur over a time interval.
- ***Rate as "population length"*** — the rate of change of population with respect to time as the amount of population change that occurs as one unit of time elapses.
- ***Proportional reasoning*** — to determine how much change should be expected over an interval of duration less than one unit time.
- ***Rate as function*** — understanding that the rate of change increases as the population increases, and therefore is an increasing function of the amount of time elapsed.

The interpretive framework as a lens for observation

Unlike most presenters at CIME, Rasmussen focused on the collaborative mathematical work that students do in an inquiry-oriented setting, where students are expected to develop significant mathematical ideas with minimal intervention from an instructor. It is natural, then, that Rasmussen's frame for observing such a setting differs sharply from those of most CIME presenters, and indeed from those of most people charged with the task of evaluating teaching.

In observing the mathematical work of students in a small-group context, Rasmussen attends to both the *content* of the mathematical work — both the ways of thinking that students bring to the task and develop as they work on the task, and the manner in which this work fits into the larger picture of disciplinary practices such as symbolizing and algorithmatizing — and the *form* of the work.

In analyzing the form of the work, he deals with both the structure of the students' argumentation and the ways in which they participate in forming and vetting mathematical arguments. This opens the door for discussion not only of the mathematical content of students' work in an inquiry-oriented classroom, but also of issues of equity and access: while it is clear that the students are building understanding of important mathematical ideas in the Euler's method episode, we may also wish to consider whether all students have substantial involvement in and access to this process.

**Bridging the
form-content
divide:
ritualized and
explorative
mathematics**

Anna Sfard's presentation highlighted the fact that the mathematics education field currently does not have sufficiently precise language to describe the mathematics we see in teaching. Additionally, the language we currently use frequently reflects a duality between the *form* of instruction (including the structure of teacher-student and student-student interactions, the types of questions asked, and the ways in which a teacher periodically assesses student understanding) and the mathematical *content*

of a lesson. To give an example of language that might help us describe the mathematics in a classroom in a way that integrates form and content — and to illustrate by analogy how language can sharpen or distort the focus of a certain discourse — Sfard introduced the terms *ritualized mathematics* and *explorative mathematics* (Sfard, 2016). Ritualized mathematics is mathematical activity without explicit attention to mathematical objects, and tends to have the following characteristics:

- The mathematics is undertaken for social reasons and is not driven by genuine curiosity.
- The mathematics is developed as a discourse-for-others, governed by rules specified by an external authority.
- The mathematical activity is performed mainly through imitation: a teacher or other mathematical authority demonstrates a procedure for solving a certain type of problem, and then students apply this procedure to solve problems of the same type.
- The activity is usually scaffolded by others: the structure of the activity is designed to guide students sequentially through the steps needed to solve a particular problem.

Explorative mathematics, on the other hand, encourages students to build understanding of mathematical objects, and tends to have the following characteristics:

- Students do mathematics in order to know more about mathematical objects and about the world around them.
- The mathematics is done as a discourse-for-oneself: learners develop a discourse that allows them to explore.
- The mathematics is performed mainly through asking one's own questions.
- Learners engage in unscaffolded mathematical activity in which they are free to explore and discover important ideas with minimal external guidance.

When observing a lesson, we might ask two related but distinct questions:

- 1) Was the teacher's own mathematics exploratory?
- 2) Did the teacher encourage students to engage in exploratory or ritualized mathematics?

To help us answer these questions, we can analyze the discourse of the lesson, paying attention to the objects to which the mathematical discussion refers and to the locus of intellectual authority in the lesson:

<i>Locus of authority</i>	<i>Ritualized discourse</i>	<i>Explorative discourse</i>
<i>What is mathematics all about?</i>	One's actions with signifiers	Properties of mathematical objects
<i>Where do mathematical claims come from?</i>	Another person and her approval; memory	Logical derivation; exploring objects; one's own argument
<i>What are the goals of classroom activity?</i>	To become able to act according to social norms	To turn mathematical discourse into one's own

To illustrate the distinction between ritualized discourse and explorative discourse, Sfard supplied workshop participants with transcripts of episodes from two mathematics classes: a seventh grade class in Montreal, Canada, and an eleventh grade class in Johannesburg, South Africa. The transcript from the Johannesburg classroom contains the following quotations:

We want to solve [the inequality $x^2 > 4$ for x .

We are saying any of these brackets is equal to zero.

And then we transpose them [2, -2].

These quotations exemplify a discourse that focuses on actions on mathematical signifiers, such as algebraic symbols, rather than on properties of mathematical objects. Sfard suggested that a discourse more focused on mathematical objects might, for example, make explicit the fact that to solve the inequality $x^2 > 4$ means to find all real numbers x whose squares are greater than 4, or the fact that when we know that the product of two real-number factors is equal to zero, at least one of the factors must be equal to zero. Sfard noted that statements about actions on signifiers do not in themselves imply that the mathematics in a classroom is ritualized; we make this inference only when such statements are given exclusivity or dominance.

By contrast, the transcript from the Montreal classroom contains the following quotations:

"For what number of days would renting a pump from this be a better deal than renting from that?"; I'm asking you...

When is $C(x)$, which is Cheap Tools, less than $T(x)$, which is Tools 4 U...

Or I'm also asking you, "when is $50 + 3x$, this function, less than $230 + 10x$?" It's the same thing. When x is less than 9 this happens.

These quotations refer to actions on, and properties of, mathematical objects such as functions and their values. Such a discourse is, according to Sfard, more likely to support explorative mathematics.

The terms *ritualized mathematics* and *explorative mathematics* allow us to describe a teacher's mathematical activity. However, we can also use this language to analyze the ways in which a teacher invites and mediates students' participation in the development of mathematical ideas. Sfard suggested that we can classify teacher's overtures — utterances meant to elicit responses from students — into *mathematizing* overtures, which refer primarily to mathematics, and *subjectifying* overtures, which refer primarily to participants. Among mathematizing overtures, we can identify closed questions and open questions, which create varying amounts of space for students to choose, construct, or substantiate the mathematical narrative developed by the class. Sfard suggests that a preponderance of overtures that only ask students to confirm the narrative under development, rather than provide direction and substantiation for the narrative, may be indicative of a ritualized discourse. On the other hand, if a classroom culture routinely encourages students to participate actively in the co-construction of mathematical narratives, students are more likely to experience explorative mathematics.

Sfard's language offers us an opportunity to transcend the content-form duality found in many descriptions of mathematics classrooms. In many accounts of mathematics teaching one notices an implicit separation between description of the mathematical content in which a class engages, and analysis of the way a teacher manages' students' participation in the mathematical work. The language of ritualized and explorative mathematics bridges this separation and allows us to describe the structure of classroom interaction in a way that pays due attention to the profound influence this structure can have on the mathematics that students learn.

Toward a unified theory of powerful teaching: the TRU Math framework

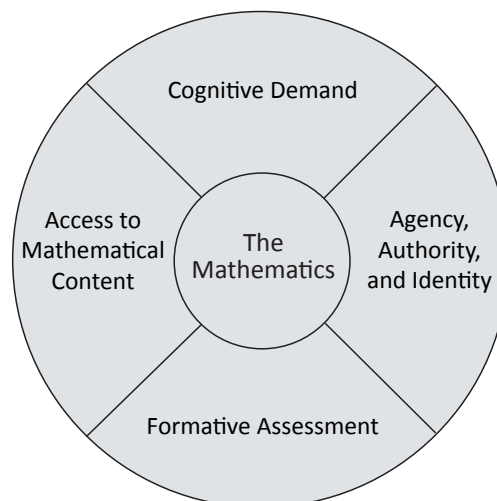
In his presentation, Alan Schoenfeld, UC Berkeley, presented a perspective on mathematics teaching that attempts to enumerate and describe characteristics of teaching that consistently produces powerful mathematical thinkers. This perspective, the *Teaching for Robust Understanding of Mathematics*

(*TRU Math*) framework, has the goal of consolidating elements of effective teaching into five essential categories, or "dimensions," so that the five dimensions together encompass everything that is required for powerful teaching, and so that each category can be improved with appropriate professional development, collaboration, and sustained effort. Schoenfeld

noted that he selected the number five in recognition of George Miller’s oft-cited finding that most people can remember and process up to seven (plus or minus two) interacting elements simultaneously (1956). He shared that by the end of the first day of professional development on the TRU Math framework, most participants in his workshops can recite the five dimensions from heart. The five dimensions of the TRU Math framework are described below and the expanded rubric is located at the end of this section.

1. **The Mathematics:** Is the mathematics discussed in class focused and coherent? Does the lesson present students with important mathematical ideas? Are connections made between concepts, procedures, and contexts?
2. **Cognitive Demand:** Do students have the opportunity to make sense of the key ideas of the lesson through activities that are appropriately challenging? Does the lesson strike a balance between spoon-feeding the mathematics in small chunks and presenting tasks that are so challenging that students are lost?
3. **Access to Mathematical Content:** Do classroom activities and structures support the engagement of all students? Is each student’s participation welcomed and actively encouraged, or can students “hide” from the mathematical activity taking place?
4. **Agency, Authority, and Identity:** Do classroom activities and structures support students in building productive identities as doers and learners of mathematics? Do students have the opportunity to make and critique arguments and build on the mathematical thinking of their peers?
5. **Formative Assessment:** Do classroom activities draw out students’ mathematical thinking, and is this thinking used as the basis for instructional decisions? Do teachers build on students’ productive thinking and address emerging misconceptions? (Schoenfeld, 2015, p. 407)

Schoenfeld suggested that while text is necessarily linear, the five dimensions of the TRU Math framework might be better represented using a visual that represents their interconnectivity and conveys the essential idea that all dimensions of effective mathematics teaching are mediated through the mathematical content:



To illustrate the dimensions enumerated in the TRU Math framework, Schoenfeld presented a brief video (available as part of the video of his presentation at <https://www.msri.org/workshops/793/schedules/20608> of a class of sixth-grade students in a predominantly low-income Chicago school working on a task in which they match different representations of rational numbers, which are printed on small cards, and sort these numbers from least to greatest. This Formative Assessment Lesson, titled “Translating between Fractions, Decimals, and Percents,” is available at the Mathematics Assessment Resource Service at <http://map.mathshell.org/lessons.php?unit=6120&collection=8>.

Although no teacher is visible in the brief video, students can be seen making mathematical arguments and challenging one another’s claims using language suggested by the authors of the lesson. (One student says that he thinks a classmate’s answer is “wrong,” and then modifies the framing of his statement to say that he “disagrees” with the answer after being prompted by a member of his group.) As students resolve a disagreement, other students in the group frequently join the discussion, attempting to clarify claims and arguments submitted by their peers. The students are engaged in meaningful and important mathematics, linking visual and symbolic representations of rational numbers and articulating their reasoning about the relative sizes of these numbers. As the group of students in the video reaches consensus regarding the correct matching and ordering of the rational numbers and their representations, the group places the cards in an array, creating a record of the students’ thinking that is readily visible and can be assessed quickly by the teacher. Thus the video serves as an illustration of how a carefully designed activity can support teachers in addressing all five dimensions of the TRU Math framework, though Schoenfeld emphasizes that certain dimensions, such as Access and Agency, cannot be fully addressed by activities or curricular materials without thoughtful implementation by the teacher who uses them.

In his presentation, Schoenfeld pointed out that TRU Math should not be viewed as a tool, but rather as a perspective on mathematics teaching and learning. Although TRU Math was originally conceived as an organizing framework for research on what makes mathematics teaching effective, the Algebra Teaching Study (ATS) at the University of California, Berkeley. The University of California, Berkeley, has developed tools to support teachers in improving these dimensions of teaching and to help teacher educators and instructional coaches have effective conversations with teachers about the five dimensions. These tools include:

- **Formative Assessment Lessons**, or “Classroom Challenges,” published by the Mathematics Assessment Resource Service. Each lesson is centered around a classroom activity based on an important mathematical idea, accompanied by a brief pre-lesson

assessment that students complete prior to the activity, in class or as homework. Lessons provide examples of anticipated student thinking, both correct and incorrect, and suggest verbal prompts and questions that teachers can use when they observe such thinking as the lesson unfolds. The materials also suggest follow-up lessons that teachers can use to review the mathematical content of the activity. The MARS website has approximately 100 such lessons addressing major ideas in grades 6 through 12.

- ***The TRU Math Conversation Guide***. This guide provides a structure for conversations between teachers, coaches, and administrators about the five dimensions of powerful mathematics teaching (Baldinger & Louie, 2014). The guide recommends that users schedule a classroom observation and then select questions from the extensive lists provided to help guide discussion of all or a subset of the five dimensions.

Schoenfeld's research team has also developed a rubric that can be used to measure the alignment of instruction with the five dimensions in the TRU Math framework (Schoenfeld, et al., 2014). However, during his presentation, Schoenfeld noted his team's strong recommendation that administrators and coaches use the Conversation Guide as the primary tool for coaching in conjunction with the TRU Math framework, and use the scoring rubric primarily as a tool to measure growth rather than to incentivize or punish teachers. We discuss the danger of conflating tools for research, tools for professional learning, and tools for evaluation in more detail in the final section of this essay, *Next Steps: Directions for Future Study, Action, and Advocacy*.

Because TRU Math packages many of the important questions people ask about a mathematics classroom under five overarching categories, using TRU Math as a framework for observation may afford observers the opportunity to "adjust" the complexity of the observation and the feedback generated. An observer may choose to focus on one dimension (possibly based on the recommendation of the teacher being observed) and give brief reports on the others, or may elect to report on all five dimensions in some detail. Because the five categories describe elements we hope to see in any effective classroom, regardless of the form or specific content of the teaching we observe, the framework adapts to different styles of instruction; it can be used in a classroom where students receive direct instruction or one where students are engaged in inquiry-oriented activity. At the same time, the framework sets forth several "non-negotiables" that might impel an observer to notice what is not present in a lesson as well as what appears before his or her eyes. For example, in a classroom dominated by direct instruction, an observer using TRU Math might question whether a lesson contains adequate cognitive demand, or whether that lesson offers opportunities for students to build a sense of mathematical agency with the content under discussion.

The Teaching for Robust Understanding of Mathematics (TRU Math)

A. H. Schoenfeld's rubric shown below endeavors to describe and enumerate the characteristics of teaching that consistently produce powerful mathematical thinkers.

The Mathematics	Cognitive Demand	Access to Mathematical Content	Agency, Authority and Identity	Uses of Assessment
How accurate, coherent, and well justified is the mathematical content?	To what extent are students supported in grappling with and making sense of mathematical concepts?	To what extent does the teacher support access to the content of the lesson for all students?	To what extent are students the source of ideas and discussion of them? How are student contributions framed?	To what extent is students' mathematical thinking surfaced; to what extent does instruction build on student ideas when potentially valuable or address misunderstandings when they arise?
Classroom activities are unfocused or skills oriented, lacking opportunities for engagement in key practices such as reasoning and problem solving.	Classroom activities are structured so that students mostly apply memorized procedures and/or work routine exercises.	There is differential access to or participation in the mathematical content, and no apparent efforts to address this issue.	The teacher starts conversations. Students' speech turns are short, one sentence or less, and constrained by what the teacher says or does.	Student reasoning is not actively surfaced or pursued. Teacher actions are limited to corrective feedback or encouragement.
Activities are primarily skills-oriented, with cursory connections between procedures, concepts and contexts (where appropriate) and minimal attention to key practices.	Classroom activities offer possibilities of conceptual richness or problem solving challenge, but teaching interactions tend to "scaffold away" the challenges, removing opportunities for productive struggle.	There is uneven access or participation but the teacher makes some efforts to provide mathematical access to a wide range of students.	Students have a chance to explain some of their thinking, but "the student proposes, the teacher disposes": in class discussions, student ideas are not explored or built upon.	The teacher refers to student thinking, perhaps even to common mistakes, but specific students' ideas are not built on (when potentially valuable) or used to address challenges (when problematic).
Classroom activities support meaningful connections between procedures, concepts and contexts (where appropriate) and provide opportunities for engagement in key practices.	The teacher's hints or scaffolds support students in productive struggle in building understandings and engaging in mathematical practices.	The teacher actively supports and to some degree achieves broad and meaningful mathematical participation; OR what appear to be established participation structures result in such engagement.	Students explain their ideas and reasoning. The teacher may ascribe ownership for students' ideas in exposition, AND/OR students respond to and build on each other's ideas.	The teacher solicits student thinking and subsequent instruction responds to those ideas, by building on productive beginnings or addressing emerging misunderstandings.

The visual above is adapted from the rubric appearing on page 5 of the following pdf:
<https://pdfs.semanticscholar.org/abf5/3fbede1d69cba70ed2f931952c43b2bfdeb3.pdf>

SECTION 3

OBSERVING TO LEARN: OBSERVATION AND VIDEO AS SOURCES OF INSIGHT AND PROFESSIONAL GROWTH

In the previous section, we saw how frameworks for observation can focus our attention on specific aspects of mathematics teaching and learning, such as the mathematical content of a lesson, mathematical quality of instruction, students' and teachers' thinking about mathematics, and how students and teachers convey expectations for one another. During the 2016 CIME workshop, researchers and practitioners shared some examples of insights about mathematics teaching and learning that they have gained from careful, systematic observation. The aim of this section is to summarize some of these findings and provide an overview of several different types of learning — including scientific discovery about the mechanics of teaching and learning, as well as professional learning for practicing teachers — that in-person and video observation have to offer.

Insights from observation: some highlights of prior research

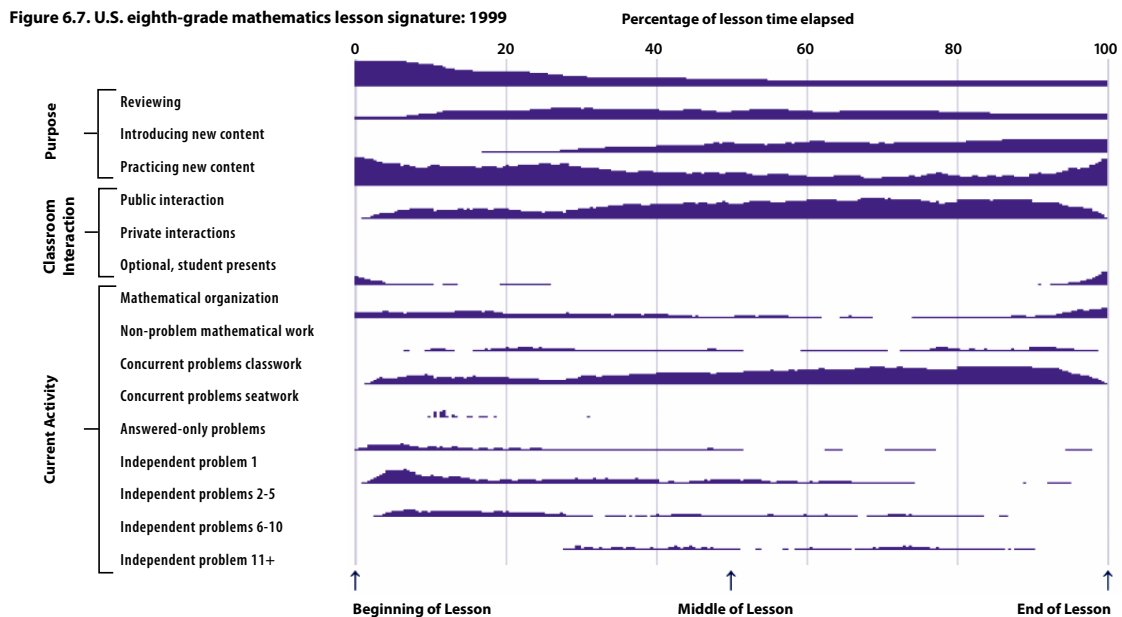
Researchers in mathematics education frequently use classroom observation as a way of learning about the work of teaching. Depending on the focus of a research study, mathematics educators may observe how teachers choose to introduce ideas, launch tasks, or highlight student thinking; they may observe how students work individually or collaboratively to assimilate new ideas and solve problems; they may observe teacher or student discourse about mathematics; they may observe how the physical space of a classroom facilitates or inhibits students' mathematical activity. However, the observables themselves are ephemeral; the opportunity to observe and record them is often brief, and researchers not in the classroom when they occur may have to rely on observers' accounts for information about what transpired during a lesson. The advent of widely available video recording technology removed one of the key practical obstacles to collaborative research that requires observation of mathematics teaching: the inability to observe the same event multiple times and through multiple sets of eyes.

I present here some examples of prior research studies that have extensively used video observations, either as a way to better understand the practice of teaching or as a lens through which to view the professional knowledge needed for teaching.

Classroom video observation as a window into the practice of teaching

The 1999 TIMSS video study (Hiebert, et al., 2003) examined mathematics teaching in eighth grade classrooms in seven countries. By viewing and coding classroom videos from the countries selected, the research team performed comparative analyses of teaching practices in the seven countries, taking into account the structure and purpose of lessons, the mathematical content of lessons, and the ways in which teachers and students engaged with the mathematical content. Prior to this study, the 1995 TIMSS video study had suggested sharp distinctions between mathematics instruction in Japan and that in the United States and Germany (Stigler & Hiebert, 2009). The findings of the 1995 study were so pronounced that they gave many stakeholders the mistaken impression that the only way to achieve national success on assessments of students’ mathematics achievement was to emulate Japanese teaching practices. The 1999 TIMSS video study painted a more nuanced picture, examining practices in seven countries whose assessment scores ranged from near the international average to well above the

TIMSS Lesson Signature



NOTE: The graph represents both the frequency of occurrence of a feature and the elapsing of time throughout a lesson. For each feature listed along the left side of the graph, the histogram (or bar) represents the percentage of eighth-grade mathematics lessons that exhibited the feature — the thicker the histogram, the larger the percentage of lessons that exhibited the feature. From left to right, the percentage of elapsed time in a lesson is marked along the bottom of the graph. The histogram increases by one pixel for printable dot for every 5 percent of lessons marked for a feature at any given moment during the lesson time, and disappears when fewer than 5 percent of lessons were marked (due to psychological limitations). By following each histogram from left to right, one can get an idea of the percentage of lessons that include the feature as lesson time elapsed.

SOURCE: <https://nces.ed.gov/pubs2003/timssvideo/> A pdf is available at: <https://nces.ed.gov/pubs2003/2003013.pdf#page=140&zoom=auto,-47,792>

average. The authors of the study report found that while eighth grade mathematics teaching in Japan is markedly distinct from teaching in other countries in several ways, featuring fewer problems worked per lesson, greater problem complexity, and more frequent opportunities for students to make connections among concepts, the other six countries in the study have relatively few differences. Nonetheless, the authors were able to develop a more textured composite picture of the eighth grade mathematics classroom in each country by creating a *lesson signature*, a diagram that maps the relative frequencies of various features of instruction — the number of problems worked by the class, the mode of interaction (whole-class instruction, private student work), and the purpose of instruction (reviewing prior material, developing new concepts, practicing new material) — as functions of time, from the start of class to the end of class. By doing so, the authors were able to examine the interplay between the purpose and the structure of classroom interaction and identify some subtle differences among the composite “classrooms” in different countries. For example, eighth grade classes in both the Netherlands and the U.S. placed heavy emphasis on review of prior material at the beginning of class; however, teachers in the Netherlands tended to conduct review by having students work problems semi-publicly in front of the class, while teachers in the U.S. tended to conduct review in a whole-class format led by the instructor.

The approach taken by the 1999 TIMSS video study illustrates several of the affordances of video technology for research in mathematics education, as well as some methodological considerations that may help researchers gain additional insight from video. The use of video permitted the research team to view each class multiple times through different lenses, coding each class for the structure and purpose of instruction, the focus and cognitive demand of the lesson content, and the ways in which teachers and students interacted with the content, including the resources they used, opportunities for teachers and students to talk, and the level of cognitive demand maintained by the class in its orchestration of mathematical tasks and discussions. Researchers set a minimum level of inter-rater reliability for each coding scheme and rewatched each video until the desired level of reliability was achieved. The use of video permits this iterative re-watching and re-coding in a way that in-person observation cannot.

The research team also designed its analysis of the large video collection it had accumulated in a way that allowed for a rich interplay between quantitative and qualitative aspects of the study. Keeping in mind that video has the power to create vivid anecdotes that can override a level-headed analysis of the data, the team conducted a quantitative analysis of each

feature of instruction it studied, taking care to note whether perceived differences rose to a level of statistical significance in view of the size of the data collection. Yet the team also used general impressions from the videos to inform its analytic approach and highlight significant findings while urging caution on others that may not be significant. For example, in their analysis the authors found that Japanese teachers presented problems using diagrams significantly more often than their counterparts in other countries. However, the video library used to analyze Japanese classrooms, collected during the 1995 TIMSS video study, had been captured entirely during the part of the year in which classes focused on geometry; therefore, this finding was likely an artifact of this feature of the data collection. On the other hand, the use of video allowed the team to notice and highlight the practice, particular to Dutch classrooms, of calling students to the front of the room to solve problems and be publicly “graded” on their work. This observation of classroom traditions specific to various countries, the authors point out, allows educators in each country to view various teaching practices not as givens but as choices that can be examined, critiqued, and modified.

Classroom video analysis as a window into mathematical knowledge for teaching

Kersting et al.’s 2010 study on how classroom video analysis (CVA) may provide a window into the kind of mathematical content knowledge that can be operationalized and deployed in actual teaching situations (2010). Prior research had shown that while mathematical knowledge for teaching (MKT) is necessary for effective teaching of mathematics, it is not sufficient; teachers who have robust content and pedagogical knowledge of mathematics may have difficulty accessing and utilizing this knowledge in the practice of teaching. Kersting et al. developed a thirteen-item video analysis instrument to determine whether teachers’ ability to access MKT when analyzing authentic classroom episodes dealing with fractions correlated with their MKT as measured by a more traditional instrument or with student learning of fractions content. In the CVA instrument, teachers view thirteen brief video clips, taken from videos of actual fifth- and sixth-grade lessons on fractions, and analyze the mathematical content and student thinking in each clip. Responses are scored on four dimensions: analysis of mathematical content, analysis of student thinking, suggestions for improvement of teaching in the episode, and depth of interpretation. The teachers in the study also completed an assessment consisting of fraction-related items from an established MKT instrument (Hill, Rowan, & Ball, 2005), and a subset of the teachers collected student responses, both before and after units on fractions, to a multiple-choice quiz on fractions assembled from several state standardized achievement tests.

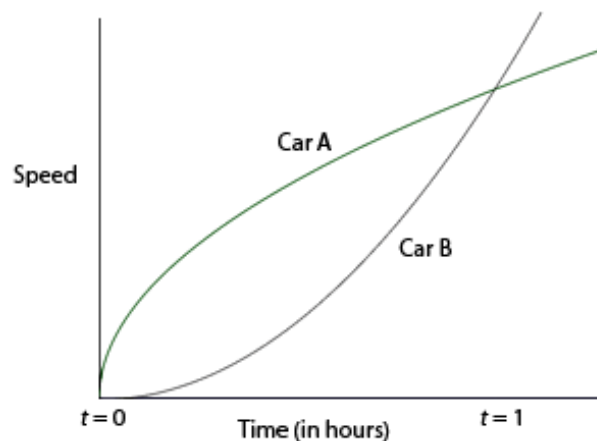
The researchers found a significant correlation between scores on the mathematical content (MC) dimension of the CVA instrument and scores on the MKT instrument, and a strong correlation between scores on the suggestions for improvement (SI) dimension of the CVA instrument and student learning as measured by the fraction pre- and post-tests, with one standard deviation change in the SI score corresponding to about half of a standard deviation change in student learning gains. No correlation was found between MC scores on the CVA and student learning gains. The study suggested that while the mathematical content scores on the CVA seemed to measure teachers' knowledge of content related to fractions, the other dimensions of the CVA, such as suggestions for improvement and analysis of student thinking, may better simulate the type of operationalized knowledge that teachers must deploy in the day-to-day work of teaching.

Understanding the work of teaching: observing mathematical knowledge for teaching in practice

While most speakers at CIME 2016 shared examples of insights gained from observations of mathematics classrooms, Natasha Speer, University of Maine, demonstrated that we can learn about the work of teaching and the knowledge required for this work by observing teaching practices that take place outside of the classroom. Having found that research on mathematics teachers' practices at the post-secondary level is relatively scarce (Speer, Smith, & Horvath, 2010), Speer conducted a study on how experienced mathematicians and graduate students read student work and analyze mathematical thinking. She collected some responses to the following task (Monk, 1992; Carlson, 1998) from college calculus students, and conducted interviews with college mathematics instructors in which they were asked to analyze the student responses:

The given graph represents speed vs. time for two cars. (Assume the cars start from the same position and are traveling in the same direction.)

Question: State the relationship between the position of car A and car B at $t = 1$ hour. Provide an explanation for your answer.



Speer presented this task to workshop participants and asked them to consider ways of thinking, productive or unproductive, that students might have when presented with this task. Participants suggested a variety of possible ways of thinking about the problem. For example, a student might reason that since car A is faster than car B at all times in the interval $(0, 1)$, car A will be ahead of car B at time $t = 1$. Alternatively, a student might confuse position with speed, and interpret the graph as saying that car A travels at a high speed initially and then slows down, and car B travels slowly at first and then speeds up and catches up with car B at time $t = 1$. A student who has difficulty interpreting a graph as a representation of covariation between two quantities may incorrectly infer from the two graphs shown in the diagram that there are two different roads, one taken by car A and one taken by car B, and that the two cars reach the same point at $t = 1$. Speer stated that the students who responded to the task demonstrated all of these ways of thinking.

Speer then played audio clips of her interview of “Brent,” a mathematician who was asked to respond to several student responses to the task. Brent was presented with the following student answer:

Car A and Car B probably cross each other at 1 hour because there might be 2 roads to take.

Brent seemed taken aback by this response, and was initially impressed that the student may have noticed that the problem statement did not specify that the two cars are moving in the same direction. However, after confirming that the problem statement does specify this, he had difficulty interpreting the student’s response as evidence of the student viewing the graph as a visual representation of two different trajectories. Brent was then presented with the following response:

They are in the same place even though at first A was going much faster, but B later increased and caught up.

This response is consistent with a student interpreting the graph as a position-time graph, and treating each graph’s slope as an indicator of the corresponding car’s speed. Brent understood that the student’s claim that car A is moving much faster at the start of the hour is incorrect, but beyond this was unable to make sense of the student’s thinking. He then read the following response:

They are at the same position. They took different ways of getting there, but at $t = 1$ the cars are the same amount of distance.

This time, Brent interpreted the student’s response as indicating that the student interpreted the graph as a picture of two roads. He was then able to relate this way of thinking to the

first student response, though he still did not believe that the first student had exactly this interpretation of the picture (based on the student's claim that there "might be" two roads).

The interview of Brent illustrates that viewing a collection of student responses that follow a specific pattern may help an instructor uncover a common way of thinking students may have about a mathematical idea or task. This provides a window into the process of building pedagogical content knowledge for teaching college mathematics, the mechanics of which remain largely undiscovered by researchers (Seymour & Lehrer, 2006). The study also has implications for the professional development of various groups involved in college mathematics teaching, including faculty, graduate teaching assistants, and undergraduate graders who must regularly make sense of student work that may be idiosyncratic and provide only narrow windows into the underlying thinking. In her study, Speer found that many interview participants were able to describe only one way of thinking about the task given, and conjectured that student responses would be variants (correct or incorrect) on this one way of thinking. However, she also found that some interview participants, when asked to describe possible ways of thinking before viewing actual student responses, were able to envision a variety of actual student approaches, even though they had not seen the task prior to the interview. This suggests that some instructors have a knowledge of content and students (KCS) that allows them to anticipate ways in which students might think about the underlying mathematical content, regardless of their lack of direct experience with student work on this particular task (Hill, Ball, & Schilling, 2008).

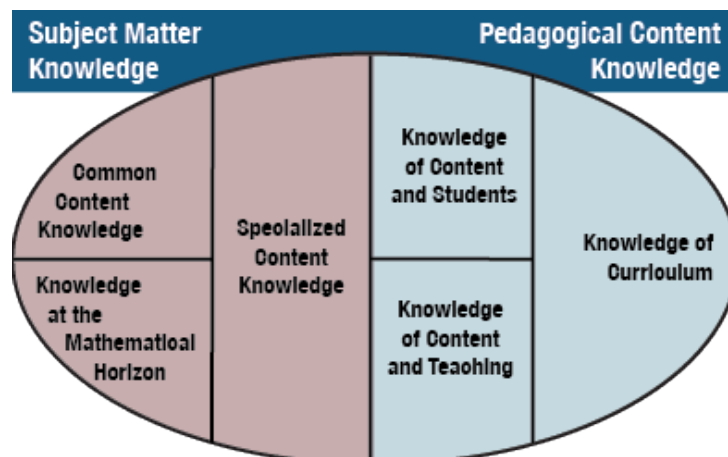


Figure 1. Mathematical Knowledge for Teaching (MTK)
(Ball, Thames, & Phelps, 2008)

Observations as tools for professional learning: the case of video clubs

In addition to providing insight for mathematics education researchers that may not be obtainable from other kinds of data, video observations can provide useful stimulus material for discussion and professional learning among practicing mathematics teachers. Videos can allow teachers to view their own classrooms through different eyes than their own, and create opportunities to examine certain aspects of teaching and learning more closely than one can when steeped in the complex and taxing work of teaching and managing a classroom.

Video clubs have recently emerged as forums in which teachers can discuss issues of teaching and learning while maintaining a shared focus and using a shared body of evidence. A *video club* is a group of teachers who meet and participate in structured discussions of videos of students working on mathematics. Videos are often contributed by teachers in the group; discussions are typically focused on the mathematical thinking of students in the classroom under observation, with facilitators prompting participants to justify their claims with evidence that can be found in the video.

Miriam Sherin and Elizabeth van Es presented their research on video clubs for mathematics teachers. In their research, Sherin and van Es explore three questions:

1. *What kinds of video are useful for teachers to discuss?*
2. *How should we facilitate discussion of videos with teachers?*
3. *How might we help teachers collect video from their their own classrooms?*

To illustrate features of classroom videos that are conducive to discussion in video clubs that focus on students' mathematical thinking, the speakers presented two clips from the same classroom in a girls' school in urban Chicago. Both clips contain discussion of the same mathematical task, taken from a preparation manual for the ACT college readiness assessment (ACT, 1997):

<i>Umberto's mother expects</i>	<i>A. \$36,005</i>
<i>an increase of 5% in her</i>	<i>B. \$36,180</i>
<i>current annual salary of</i>	<i>C. \$37,800</i>
<i>\$36,000. What would her</i>	<i>D. \$41,000</i>
<i>new annual salary be?</i>	<i>E. \$54,000</i>

Most of the first clip is dominated by public, teacher-directed discussion, with students ruling out (A) as the correct answer, reasoning that five percent is not the same as five dollars. The conceptual depth of the mathematical discussion in this clip is relatively shallow. In the second clip, two students present solutions leading to two different answers, (C) and (E). The student who selects (C) reasons that \$360 is one percent of \$36,000, and finds five percent by multiplying this amount by 5. The student who selects (E) multiplies \$36,000 by 0.5 to get the amount of the raise, and adds this to the original salary. After reading her classmate's solution and discussing her own approach with the class, she changes her answer to (C). The second clip arguably affords greater opportunity for in-depth discussion of mathematical thinking, since the disagreement between the two students prompts both to clarify their reasoning and resolve the conflict between their answers. However, one participant pointed out that the first clip has value in that it reflects a complexity inherent in actual classroom teaching: students often utter answers or thoughts in which the mathematical thinking may be difficult to unpack, and teachers must be able to make sense of these as well.

This comment foreshadowed one of the findings of Sherin's and van Es' research. The researchers identified three dimensions of videos that may influence their usefulness as stimuli for discussion in video clubs that focus on student thinking:

- **Windows:** *Is there evidence of student thinking in the video clip?*
- **Depth:** *Are students exploring substantive mathematical ideas?*
- **Clarity:** *How easy is it to understand the student thinking in the video?*

The researchers classified a collection of classroom videos along each of these three dimensions, rating each clip as "Low," "Medium," or "High" on each dimension. They initially hypothesized that clips that were high on windows and depth but low on clarity would lead to the most productive discussions: such videos contain ample evidence of student thinking on mathematically substantive issues, but do not reveal student thinking with so much clarity that video club participants lose the opportunity to make sense of the students' reasoning for themselves. To test their hypothesis, the researchers studied a video club of seven elementary school teachers who viewed a total of 26 clips, each coded along the three dimensions above. For each clip, the researchers determined whether the ensuing discussion sustained focus on students' mathematical thinking, had a substantive mathematical focus, and allowed for collective sense-making on the part of the teacher participants.

The researchers found that clips with high mathematical depth tended to lead to productive video club discussions, but only if the depth was sustained; discussion participants often did not take up instances of student thinking that were revealed only in brief comments or interludes. Sometimes, low-depth videos led to high-quality discussions, particularly if teachers discussed the students' mathematics at a level deeper than that enacted in the video. Additionally, Sherin and van Es found that high-clarity videos as well as low-clarity ones could lead to productive discussions; on occasion a video revealed a way of thinking on a student's part that was novel for teachers and led to a substantive mathematical discussion.

The presenters then turned to the question of what facilitation practices support productive discussions in video clubs; in particular, what skills or dispositions are needed to help maintain a club's focus on evidence of students' mathematical thinking and prevent discussion from veering into issues tangential to the video. They identified several categories of practices that help maintain the quality and productivity of a discussion of a video clip:

- ***Facilitation moves that maintain the focus of discussion on the video.***
Facilitators may ask club participants to return to discussing what is in the video, or may highlight a specific aspect of the video for discussion.
- ***Facilitation moves that encourage participants to focus on specifics.***
Facilitators may point to specific stimuli in a video, such as a student's mathematical actions in a clip, or prompt other participants to explain and clarify their thinking using evidence in the video.
- ***Facilitation moves that open up discussion when participants are not sure how to analyze what they see.***
A facilitator may choose to model how to analyze student thinking in a video by providing a substantive explanation for something that happens in a clip.

Sherin's and van Es' presentation highlighted some of the considerations involved in organizing and facilitating a successful video club, from the selection of videos that are conducive to rich discussion of mathematical thinking to the practices necessary to keep discussion on a trajectory that benefits participants' professional learning. A subsequent presentation allowed CIME participants to see some of these skills in action in video clubs organized by Math for America.

Using observations to improve professional learning: learning from video club facilitation

Michael Driskill and Kristen Smith gave a presentation on Math for America's efforts to organize and improve video clubs for K-12 mathematics teachers. Math for America (MfA) works in several cities across the United States to develop master teachers of mathematics and science by providing advanced professional development and leadership opportunities for highly qualified teachers. In New York, MfA organizes video clubs as part of a slate of professional development activities in which fellows participate. Like the video clubs discussed in Sherin's and van Es' presentation, MfA video clubs focus on interpreting and analyzing student thinking in video clips, and adhere to strong norms that encourage participants to make claims based only on evidence in the videos presented.

Focusing on student thinking is the explicit goal

Driskill and Smith presented two clips of videos from MfA video clubs: one in which a group of teachers analyzes a video of students working on a geometry task that introduces the geometric mean, and one in which Smith leads a group of teachers in discussing students' work on a task involving successive discounts. Both video club clips illustrated the natural tension that arises when club participants want to discuss mathematical ideas and possible pedagogical decisions that "branch off" from what is happening in the classroom video under discussion. In the first clip, teachers discuss different possible interpretations of the idea of "middle" or "average," such as the median, the arithmetic mean, and the geometric mean, in response to a student's description of "splitting the numbers equally." One of the co-facilitators of the group quickly steps in to urge participants to stick to the scripted discussion questions and base their claims on what the students in the video are saying. This redirects the discussion toward analysis of what the student understands about the average as a representation of the "middle" of two numbers.

In the second clip, teachers discuss student work on the following task:

During a sale, a store offered a 40% discount on a particular camera that was originally priced at \$450. After the sale, the discounted price of the camera was increased by 40%. What was the price of the camera after this increase?

In the classroom video, students base the 40% increase on the amount of the discount, \$180, rather than on the discounted price, \$270; the clip of the video club features a discussion of this point by several middle school teachers. In the clip, one participant suggests that the key issue for students is a misunderstanding of the phrase “discounted price,” and conjectures that if the phrase “discounted price” had been replaced with “decreased price,” more students would have completed the task correctly. Several other hypothetical discussions spin off from this one: for example, one teacher asks what students would have done if the discount had been 10%, and therefore less prone to be confused with the discounted price, rather than 40%.

The facilitator asks what might have happened if the task had been simply to find the discounted price rather than to consider a subsequent increase. CIME participants noticed that in this clip, because the facilitator allowed more discussion of hypothetical scenarios not represented in the classroom video clip, the teachers’ discussion focused less on the particulars of the student thinking actually shown in the video. While discussion of possible alternatives to a task may tap into useful parts of teachers’ professional knowledge, such discussion in this case deviated from MfA’s explicit goal of focusing on student thinking as evidenced in the video clips under discussion. On the other hand, Smith noted that, as the facilitator of the group, she wanted to allow her groupmates to pursue an intellectual thread that was interesting to them and allowed them to make better sense of the mathematics in the video.

This contrast between the two videos suggested a new line of inquiry, which Driskill explored during the second part of the presentation: how can organizers of video clubs support facilitators in conducting productive discussions and in honoring the diverse perspectives and interests of club participants? To this end, Driskill and his colleagues began conducting video clubs for facilitators, in which facilitators view and analyze video clips of teacher video clubs. In doing so, they use a protocol analogous to the one used with teacher video clubs, select a video and do the mathematical task that students work on in the classroom video:

- Watch a clip of teachers discussing the classroom video, analyze teachers’ thinking
- Suggest possible alternative facilitation moves
- Make connections

This activity taps into facilitators’ mathematical knowledge for professional development (MKPD; see Borko, Koellner, & Jacobs, 2014), as it requires making sense of teachers’ thinking about an episode of mathematical work, envisioning possible directions for discussion, and identifying connections across content areas and grade levels that may be relevant to teachers.

Using observation to improve teaching: the continuous improvement framework

James Hiebert and Dawn Berk gave a presentation on the continuous improvement process they have applied to their mathematics content courses for elementary preservice teachers (ePSTs) and their intermediate algebra and precalculus courses at the University of Delaware (Berk & Hiebert, 2009). Hiebert, Berk, and colleagues developed a process for the continuous improvement of courses centered around the following five principles:

1. *Adopt a shared, stable, and precise set of learning goals.*

Hiebert emphasized that in order for ePSTs to develop content knowledge that is robust enough to be deployed in classroom practice, they need to spend a large amount of time on a focused set of mathematical topics. This implies a responsibility on the part of mathematics teacher educators to select a relatively narrow set of learning goals and commit to spending a significant amount of time on those goals, even when this means removing other topics that may seem important from the course. He also noted the importance of keeping these goals stable over a long period of time; he pointed out that it is virtually impossible for any community of teachers to become proficient at helping students meet learning goals when these goals change every few years, as they have in most of the United States.

2. *Use a shared, detailed, lesson-level curriculum.*

In order to be able to make consistent improvements in the course and in students' attainment of desired learning outcomes, it is necessary to agree upon a lesson-level curriculum with prescribed activities and assessments. It is not sufficient to agree simply to cover specified parts of the course text at given times; different instructors' visions for how to cover a given part of the course text may vary so widely that it is impossible to have a focused discussion about how students develop mathematical understanding and how they struggle with a lesson.

3. *Treat lesson plans as empirical hypotheses to test and refine over time.*

Each lesson is treated as an experiment. When modifying one of the shared lesson plans, the team forms hypotheses about student learning, making conjectures about how students will interact with a lesson, what misconceptions they are likely to have, and how they might make progress toward the specified learning objectives for the lesson. The team then uses these hypotheses to define objectives for upcoming observations.

4. *Use common assessments across all sections of a course to compare learning across sections and across time.*

This creates a lasting record of how improvements to the course affected student learning globally and helps guide priorities for the improvement process over subsequent semesters.

5. *Repeat the cycle of improvement every semester.*

Expect the process to take time. Lesson plans are passed from one semester to the next, and instructors use data from previous semesters to inform possible future improvements. Lesson plans are annotated to indicate changes that have been made and the rationales for those changes so that the team does not make the same mistakes repeatedly.

In the case of the elementary mathematics content courses at the University of Delaware, the team decided to focus the first two of three courses on the concept of place value, as it plays out with whole numbers and decimals, and the concept of fraction. This choice generated some pushback among the instructional team because it required jettisoning other topics that instructors may have found important. However, the choice to narrow the topics of these courses also allowed the team to define more focused learning goals for each course and then subject these goals to systematic study.

Another source of pushback was the use of highly specified lessons with prescribed activities and assessments. Many teachers associate the word “prescription” with scripted instruction and have had negative experiences with the latter. Furthermore, many instructors express concern that using scripted lessons will transform teaching into an intellectually vacuous activity. Hiebert suggested that many past negative experiences with scripted instruction have been with lessons that were poorly designed, in large part because these lessons were developed by authors with minimal classroom experience and no grounding in what would be likely to work in a classroom setting. Furthermore, far from turning instruction into a mindless activity, the collaborative development of shared lessons elevates the intellectual work of teaching to the critical attention of a discerning group of educators, and offers recognition and prestige to those who can propose changes in lessons that enhance student learning. Hiebert stated that instructors on the team are welcome to make adjustments to a lesson or propose an alternative, provided that they record what those adjustments or alternatives are, and collect data so that the team can determine whether the adjustments have a positive overall effect and should be incorporated into the shared lesson. He gave the example of the Stern School of Business at New York University, which designed lessons

for every session of its introductory course for MBA students. Faculty at the school would regularly examine possible alternatives to these lessons proposed by instructors; if they found that an instructor's proposed lesson outperformed the one in the standard curriculum, they would replace the lesson in subsequent editions of the curriculum and give credit to the proposer. Thus the kind of collaboration over lesson design and implementation that continuous improvement entails can, in fact, elevate the intellectual and professional status of teaching.

When the instructional team began the process of developing and refining lessons, classroom observation became an indispensable tool for testing possible improvements to the activities used in the course. Hiebert articulated two principles that guided the team's use of classroom observations:

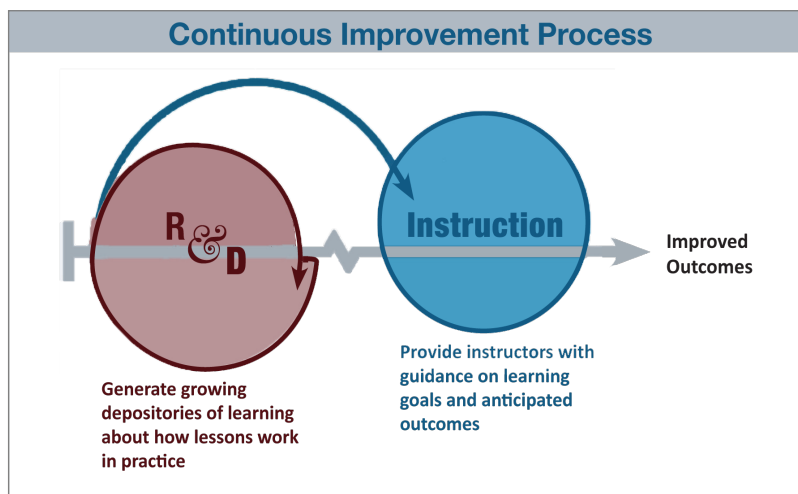
1. *Clearly identify the changes that are being made to the lesson being observed, and form hypotheses about these changes that can be confirmed or refined through observation.*

If one wants to determine through observation whether improvements to a lesson have the intended effect, one cannot simply walk into a class and freely observe what the teacher and students are doing. Instead, Hiebert and his colleagues specified which parts of a lesson they wanted to focus their attention on, and formed hypotheses about how students would respond to specified changes. Hiebert related the story that at one point, the team observing a lesson was so eager to see whether students would respond as expected to a certain change that they watched the lesson unfold with great anticipation. When the teacher presented the part of the lesson that the team had changed, the observers were greatly gratified to see students respond exactly as they had anticipated.

2. *Focus on observing teaching, not the teacher.*

Hiebert emphatically pointed out that when the continuous improvement team at the University of Delaware observes a lesson, they are focused on the teaching that takes place using the prescribed curriculum, rather than on the teacher presenting the lesson. Many observations of teaching tend to focus on the actions and decisions of the teacher, and records of these observations tend to be evaluative rather than descriptive. In the continuous improvement process, all lesson materials are the shared intellectual property of the team; accordingly, the team focuses on observing how the aspects of the lessons that are shared by the entire team appear to influence student learning.

In the continuous improvement process, lesson plans play a dual role. On the one hand, they provide guidance for instructors, laying out the team's shared learning goals, prescribed activities, and anticipated student responses. However, the lesson plans also serve as a repository for the team's learning about how lessons work in practice.



Continuous Improvement Process: image is based on Cody Patterson's summary of the March, 2016 CIME workshop presentation by James Hiebert and Dawn Berk at the Mathematical Sciences Research Institute, Berkeley CA.

When the team observes a lesson, they record summaries of what they have observed and make suggestions for possible future improvement. These notes become part of the lesson plans, which gradually evolve into richly detailed documents. Because the lesson plans serve as a record of changes to the lessons and activities and the reasons for those changes, the team can avoid repeating interventions that have proven not to be successful.

Hiebert shared some assessment data from the elementary content courses that illustrates some of the successes of the process. In the first few years of the continuous improvement process, ePSTs demonstrated significant growth on two fraction tasks.

1. On a task that asked ePSTs to write a story problem that can be represented by the division $1\frac{3}{4} \div \frac{1}{2}$, the success rate increased from 32% prior to the start of the continuous improvement process to 70% four years later.
2. On a task that asked how much milk is left over if John has $2\frac{1}{3}$ cups of milk and uses this milk to make cakes requiring $\frac{3}{4}$ cups of milk each, success rates increased from 31% to 67% over the same period.

Furthermore, in a longitudinal study that followed these students through the teacher preparation program and the first three to four years of teaching, the team found that on a wide range of teaching-like mathematics tasks, students from the University of Delaware program performed significantly better on topics that were part of the focus of the elementary content courses than on those that were not. They also found that spending more time on these topics in the elementary content courses increased the likelihood that graduates of the program would use the content knowledge they developed in their teaching.

Dawn Berk talked about her team's recent efforts to apply the continuous improvement framework to intermediate algebra and precalculus courses at the University of Delaware. Berk and her colleagues were motivated to undertake a coordinated improvement of these courses by DFW rates in these courses that consistently hovered between 20% and 40%. Berk's discussion highlighted some of the challenges that occur during the early stages of the continuous improvement process: getting faculty buy-in, ensuring that participating faculty have sufficient time and resources to devote to the process, developing a shared lesson-level curriculum from the ground up, and sustaining faculty interest in the process when initial improvement attempts fail.

- ***Faculty buy-in***

The improvement process entailed several substantial changes to the courses that generated resistance among faculty. College algebra and precalculus courses tend to be very broad and revisit certain topics (such as quadratic functions) repeatedly; the continuous improvement team had to cut some peripheral topics and commit to developing each topic in the curriculum once in adequate depth. One of the continuous improvement team's pedagogical decisions was to flip instruction for the entire course; therefore, instructors teaching the course needed to be persuaded to adhere to the flipped model.

- ***Faculty time and resources***

During the early stages of the continuous improvement process for the elementary content course, the team had access to a number of graduate students who could devote 20 hours per week to the project. Faculty teaching the intermediate algebra and precalculus courses tended not to have as much time to devote to the continuous improvement process. As a result, starting the process of observing lessons has proven difficult; however, the team plans to begin doing so in the immediate future.

- ***Developing a curriculum***

Changing to a flipped model of instruction necessitates a curriculum that can support this pedagogical approach. The continuous improvement team consists of four faculty members, each with different areas of expertise; thus each team member can focus on developing different parts of the curriculum. However, building a curriculum takes time, and coordinating efforts across multiple faculty members is challenging. At times, the team had to make deliberate decisions to keep improvement efforts focused on a limited area at a given time.

- ***Sustaining faculty interest***

In any substantial effort to improve instruction, observable improvements are usually slow to appear. It is therefore necessary to manage faculty expectations for the improvement process and remind team members and other stakeholders that the process, by design, is an incremental one and takes time to generate noticeable results.

Although Berk and her team struggled with various aspects of the project's launch, they have already noticed some improvements in the outcomes they hoped to affect. In the intermediate algebra course, the DFW rate dropped by over 10% in the first semester of the continuous improvement process; in the precalculus course, the DFW rate dropped by over 5%. These effect sizes increase if one counts as "passing" students who performed at a passing level but failed a course due to attendance penalties the team instituted as part of the continuous improvement project.

<i>DFW Rates of University of Delaware Mathematics Courses Under Continuous Improvement</i>			
Course	Previous five years	2015	2015 (no attendance penalty)
M010, Intermediate Algebra (Spring)	36% (N = 134)	24% (N = 63)	21% (N = 63)
M115, Precalculus (Spring)	32% (N = 919)	25% (N = 245)	23% (N = 245)
M010, Intermediate Algebra (Fall)	26% (N = 1148)	33% (N = 235)	29% (N = 235)
M115, Precalculus (Fall)	22% (N = 2329)	18% (N = 560)	16% (N = 560)

At the end of the presentation, Berk and Hiebert were asked to talk about concerns, frequently raised by faculty in mathematics departments, that continuous improvement processes in which faculty are required to adopt shared, scripted lessons infringe upon faculty members' academic freedom. Hiebert alluded to a quote by Al Shanker, former president of the American Federation of Teachers, who said that his dream was for the best that the teaching profession knows to become standard practice. Hiebert added that within other professions, such as law and medicine, when practitioners do not take into account shared professional knowledge and agreed-upon best practices, we regard it as malpractice. He emphasized that if faculty and students on the instructional team believe they can do better than what standard practice prescribes, they are free to try something different, provided that they document the change and collect data to determine whether the change worked as intended, so that the team could adopt the change if the data so warranted. Berk added that "academic freedom" invoked in this way raises some equity concerns; if faculty operate within individual silos, it becomes much more difficult to improve instruction in ways that will benefit students. Furthermore, students across different sections may make different progress toward learning outcomes as a result of variations in teaching and be assessed and graded according to different standards. These disparities lead to valid ethical concerns about the professional practice of instructors who claim the right to neglect research-based best practices under the guise of academic freedom.

**"Within other professions, such as law and medicine,
when practitioners do not take into account shared
professional knowledge and agreed-upon best practices,
we regard it as malpractice."**

— James Hiebert

SECTION 4

NEXT STEPS: DIRECTIONS FOR FUTURE STUDY, ACTION, AND ADVOCACY

At the end of the CIME workshop, James Hiebert and Anna Sfard offered some closing remarks to synthesize key points of the workshop and challenge participants to continue moving the work of observing and improving teaching in productive directions. In this section, I summarize Hiebert's and Sfard's remarks, and offer my own commentary about possible directions for future work.

How do we get better when resources are limited?

In his closing remarks, Hiebert asked, "How do we get better at observing classrooms, and how do we get better at connecting those observations to improving learning opportunities for students?" He remarked that in the closing of Atul Gawande's book *Better* (2007), the author comments that doctors often ask him what they can do to improve their practice; the author's response is that one can start by measuring something. Hiebert asked, given that one purpose of classroom observation is to understand more deeply what happens in the classroom, how can we measure whether our observations are actually advancing our understanding, and how can we measure our ability to use observation data to influence the kind of learning that happens in schools?

**"It occurs to me that measuring stuff
is a good start to getting better
at what we do ..."**

— *Jim Hiebert*

Hiebert's questions are particularly timely in view of the increasing volume of skepticism about the value of classroom observations by school administrators. In a blog entry hosted by the Brookings Institution, Mark Dynarski posits the opinion that classroom observations are a waste of limited resources, noting a lack of correlation between teacher observation scores issued by administrators and student scores on achievement tests (2016). He notes that most teachers, based on observations by administrators, are rated "effective" or "highly effective," and thus the potential of observations to spur improvements in practice is lost in most cases.

Some states, frustrated with the inability of most observation systems to distinguish more effective teachers from less effective ones, have recently moved to limit the extent to which a teacher's effectiveness rating may be determined by administrator observations. This trend may further lessen the impetus for administrator and peer observation in a professional environment that is already characterized by isolation.

Speakers at the 2016 CIME workshop illustrated the benefits of classroom and video observation not only as a tool for research on mathematics teaching and learning, but as a way to build community among teachers, investigate the efficacy of lessons, and provide timely and useful coaching for apprentice teachers.

"... So the question is: what do we measure? Observing classroom interactions, I think, helps us understand the details of interactions in a way we couldn't otherwise. So given that's the purpose of observing, how do we measure whether we're understanding more during our observations, and create observations, videos, artifacts, that allow us to understand more deeply what's happening in the classroom? Once we understand classrooms more deeply, how do we measure our ability to use that information and influence the kind of learning opportunities students have in schools? ..."

— Jim Hiebert

However, observation is costly in terms of time and effort. For researchers who use data from classroom observations, it is often insufficient to attend a class and take field notes; a thorough analysis of the data often requires transcription and coding. Master teachers and administrators who wish to use observation as a coaching tool often find it useful to have pre-observation and post-observation meetings with instructors, and typically generate written reports of their observations. Most of these tasks require hours of effort above and beyond the time needed for the observations themselves, and many can be performed only

“... I don’t know the answers to these questions. But I think that, in my own work, when I ask myself that question, it usually sort of keeps me on track in terms of doing what I like to do. As I said, I don’t think this distinguishes between teachers and administrators and researchers and whoever else is in this room. Because I think we all can get better.”

— Jim Hiebert

by professionals with some experience and subject-area expertise. Efforts to automate or simplify these tasks (for example, by creating checklists for post-observation feedback and coaching) often sacrifice some of the nuance and complexity of insight that observation affords.

Developing a way to measure what we gain from observation — the insight we obtain about teaching and learning, and the opportunity to improve mathematics teaching — may help us focus our efforts in directions that are likely to be productive. It may also help us justify the expense associated with using classroom observations, whether we are applying for research funding or making the case to deans and department heads that classroom observation, for community improvement of teaching and for coaching of apprentice teachers, should be recognized as a part of the faculty workload” to “K-12 and college teachers’ workload.

How do we keep words (and numbers) from getting in our eyes?

Yet any effort to measure our progress entails a responsibility to be mindful of what measurement can and cannot accomplish, and safeguard against uses of measurement that may derail efforts to improve teaching. In her closing remarks, Anna Sfard returned to the question of how our words — as observers of mathematics teachers, and as observers of those who

observe — influence what we see. She noted that 2016 CIME workshop participants had various ways (“frame,” “lens,” “rubric”) of focusing their observations in order to avoid the trap of walking into a classroom, hastily taking notes, and only seeing what is most readily apparent. She also observed that while most participants described ways of focusing observations, the specific foci were varied: “mathematics,” “cognitive demand,” “collaborating,” “equity.”

Sfard also took note of words that were not prevalent at the workshop. In particular, she observed that verbs, which describe characteristics of teaching that may be ephemeral, prevailed over nouns, which signal impermanence. She observed that words indicating form-content dichotomy were largely absent. She cheered the relative absence of words that signal evaluation, such as “good” or “better,” “poor” or “worse.” Words pertaining to assessment and evaluation of teachers were replaced, in large measure, with words conducive to formative assessment and ongoing development of practice.

Sfard pointed out that it is neither possible nor advisable to abandon evaluative language entirely; to do so would be contrary to the field’s mission of improving learning opportunities for students. However, we must be careful in our efforts to evaluate, and particularly guarded in our efforts to develop measurement tools, whether these tools are used to analyze teaching or to analyze our efforts to improve teaching through observation. When measurement tools fall into the hands of people in search of easy solutions to problems of performance evaluation, they can be reductive rather than informative. Given a way to measure some aspect of teacher performance using an observation tool, administrators may be tempted to flatten the entirety of a teacher’s professional practice into a single metric. It is up to our field to ensure that the metrics we develop are used in ways that elevate the mathematics teaching profession rather than demeaning it.

How do we sustain a commitment to measuring teaching and not teachers?

In order to preserve classroom observation as both a stimulus for the improvement of teaching and a tool for the professional development of teachers, I submit that we must do the following:

1. ***Mind the teacher-teaching duality.*** Bear in mind that an observation only generates data on a specific episode of teaching; because a teacher's actions and decisions are always situated in various practical, cultural, and policy contexts, any inferences we make about a teacher from observing a classroom episode should be made with great caution.
2. ***Use observations for different purposes, and be clear about the purpose of each observation.*** The purpose of an observation may be to provide coaching for a teacher, to analyze the efficacy of a particular lesson, to generate data for the evaluation of a teacher's effectiveness, or to research a problem of teaching and learning that cuts across different contexts. Schools should consider using observations for both coaching and evaluation purposes, and separate these purposes to the greatest extent possible, so that teachers have the opportunity to respond to coaching feedback before being evaluated.
3. ***Identify observables that are predictive of student learning gains, and focus observations of mathematics teachers on these if allowed by local regulations.*** Teaching mathematics draws upon a unique set of professional skills and technical expertise. Observations should focus on the characteristics of classrooms that are most predictive of student learning, such as those outlined in Schoenfeld's TRU Math framework, and provide feedback that might allow teachers to strengthen facets of their practice that may not yet be optimal for student learning. Although union rules in some parts of the United States require teachers to be evaluated using uniform rubrics that do not vary by subject area, feedback specific to the mathematics classroom may still help teachers improve in areas such as maintaining cognitive demand, using formative assessment, and ensuring equitable access to mathematical thinking.
4. ***Measure our own success.*** In keeping with Hiebert's advice, we must identify clear goals for the improvement of practice, both in teaching and in the observation of teaching. By identifying observable characteristics of teaching that produces powerful mathematical thinkers, we can begin the arduous effort of measuring our progress toward the kind of practice we consider desirable. By making a commitment to measure our progress at regular intervals, we can more readily discard observation strategies that do not produce actionable feedback or that do not provide sufficient focus to stimulate improvement in professional practice.

**Where we go
from here:
observation as a
tool for research
on teaching and
learning**

The use of classroom observation in research on mathematics teaching and learning, and the use of video observation in particular, have rich histories. To set a single agenda for the entire enterprise of observation-based mathematics education research would fail to do those histories justice. However, the 2016 CIME workshop called attention to one particular vein of inquiry that has the potential to inform a variety of studies in mathematics education.

When we observe a mathematics classroom, we see many different aspects of teaching and learning in play simultaneously. We see how the physical space of a classroom influences students' interactions with mathematical ideas and with each other; we see how the teacher uses language and gestures to convey mathematical thinking; we see the rich interplay among students, teachers, and mathematical tasks. One way to manage the sheer volume of information we receive during an observation is to filter a large part of it out. This is often appropriate if we have identified a specific facet of mathematics teaching or learning on which we would like to focus. However, the work shared by Rasmussen and Sfard during the workshop suggests an alternative approach. We can gain insight about how the discourse of a mathematics classroom develops by bridging the form-content divide that is often present in our analysis of teaching, and considering the interplay between mathematical ideas in a classroom and pedagogical factors such as the use of words to describe objects or signifiers, the types of questions asked by the teacher, and the sources of authority cited when making mathematical claims. We can gain insight about how students' mathematical thinking develops in a cooperative learning setting by coordinating theoretical perspectives on peer collaboration, mathematical argumentation, and disciplinary practices.

If we wished to measure the success of such a line of inquiry, we might ask ourselves whether we can build models for classroom activity that can help us predict, with some accuracy, whether students will develop specified ways of thinking about mathematical content, based on various observable factors, such as the discourse patterns of teachers and students, the tasks presented to students and how interactions with these tasks are managed, and the attention given to ensuring that students enjoy equitable access to mathematical activity. This is an ambitious agenda; it is widely understood that many factors influence the quality and efficacy of teaching, and coordinating these factors in a way that is both manageable and meaningful presents a staggering analytical challenge for the mathematics education research enterprise. However, this work of synthesis has already begun, and we should be encouraged by its progress.

The way forward: coordinating perspectives for the improvement of mathematics teaching and learning

Schoenfeld's TRU Math framework offers a way to manage the diverse array of factors that influence teaching effectiveness by organizing these factors into five clusters: factors that influence the mathematical content of a lesson, factors that influence the cognitive demand that students experience, factors that mediate students' access to the mathematical thinking of a lesson, factors that influence the development of students' mathematical identity, and factors that provide opportunities for students and teachers to monitor learning. It is insufficient to consider these five clusters of variables as if they are independent: a lesson characterized by high cognitive demand in the service of a worthwhile mathematical goal might prove stimulating and challenging for students,

while a lesson characterized by high cognitive demand in the service of an unmotivated mathematical goal might lead to frustration and disengagement. In order to provide a meaningful analysis of a mathematics lesson, we must consider the mathematical content that is taught and the manner in which it is taught simultaneously.

Mathematics educators who wish to use the TRU Math framework as a starting point for professional development may benefit from discussing the five clusters of variables in concert with one another. For example, a lesson may involve appropriate cognitive demand, but does it ask for substantial mathematical involvement from all students, or only from a few? A lesson may attend nicely to the development of a specific mathematical meaning, but does it do so in such a way that students will be inclined to see this meaning as intellectually worthwhile, and to develop their own sense of ownership of this meaning? These inter-domain considerations may provide useful explanations of why lessons succeed or fail in achieving their intended outcomes, and may help to focus discussions on features of teaching rather than on teachers.

The continuous improvement work of Berk and Hiebert reminds us that if we wish to make progress in the enhancement of mathematics teaching, we must first adopt a shared definition of what "progress" should mean, and set hypotheses accordingly. Once we have adopted goals and articulated some hypotheses about how we might make progress toward those goals, we can use observations to test and refine those hypotheses. The great promise of classroom observation as a component of this process is that it allows us access to a

multitude of factors that influence the effectiveness of teaching, and allows us the opportunity to consider these factors in concert rather than in isolation. In order to fulfill this promise, we must avoid the trap of allowing our language to set unwarranted restrictions on what we notice in teaching. While it is tempting to regard frameworks as rigid and restrictive, a well-organized and inclusive framework can expand our attention and help us to coordinate multiple perspectives in ways that enhance the explanatory power of each.

We can gain insight about how the discourse of a mathematics classroom develops by bridging the form-content divide that is often present in our analysis of teaching, and considering the interplay between mathematical ideas in a classroom and pedagogical factors such as the use of words to describe objects or signifiers, the types of questions asked by the teacher, and the sources of authority cited when making mathematical claims.

We can gain insight about how students' mathematical thinking develops in a cooperative-learning setting by coordinating theoretical perspectives on peer collaboration, mathematical argumentation, and disciplinary practices.

— *Cody Patterson*

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I also wish to thank the 2016 CIME workshop organizing committee for putting together a stimulating and exquisitely organized program; their framing and sequencing of the talks during the workshop served as an excellent starting point for the development of this e-booklet. Workshop presentation videos are at <https://www.msri.org/workshops/793>.

Finally, I wish to express my gratitude to all of the speakers at the workshop. Many of the presentations at CIME reflected months (or years) of work undertaken without the purpose of this particular workshop in mind; it is therefore remarkable that so many of the speakers were able to set up their ideas so that they could build upon one another and contribute to a greater whole. Much of the synthesis of the ideas in this e-booklet occurred due to the active (if unintentional) facilitation of the educators who presented them. This e-booklet would not have materialized without their help and inspiration.

— *Cody L. Patterson*



A number of workshop attendees gathered for a photo at the entrance to MSRI *Photo: David Eisenbud*

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TESSELLATION TANGO



“Tessellation Tango” is a tile mural composed of handmade porcelain tiles and crushed mosaic whiteware tiles, plus aluminum line elements. The mural was designed in a collaborative effort by Linda Vanderkolk and Scott Frankenberger, both of West Lafayette, Indiana.

The porcelain tiles consist of approximately 950 whole or partial tiles of 9 different colors, about two-thirds of which have a variety of numbers, expressions, concepts, or names inscribed. The composition features two different tessellations made from the same two-base tile shapes (a pair of complementary shaped rhombi), and a transitional middle section. The patterns may be loosely characterized as a “tumbling block” design and a Penrose tiling.

We explore possible patterns within the tessellations, and then patterns within the patterns, augmented by color repetitions or shape development. The frame of the mural is not the limit of the visual field. We can imagine more patterns and new forms implied, unfolding in the imaginary zone beyond the edges. This is not unlike finding intriguing relationships within progressions of numbers (Fibonacci series, multiples of 9, for example) and imagining what happens well beyond what we can comfortably quantify. The kind of thinking that artists often use – exploration, discovery, creativity, synthesis — is the very same at its root as that used by mathematicians. The different ways of looking at the same forms and patterns, both far away and close up, leads to enlightened new thinking, and inspiration.

The interaction between the two vigorous patterns, made more active by the colored subgroups within each tessellation, suggests a dance or dialogue. The transitional central section suggests a communication, an effort to find elements in common, or that can be morphed into each other.



Further, the insistent geometries of the porcelain tiles are held in suspension by, and in subtle contrast to, the chaotic background mosaic tiles.

We hope that viewers can enjoy the mural at a distance, contemplating the tension and flow between the two tessellations, and enjoying the many shapes and patterns created by color placements. And we hope that viewers can further enjoy the mural through up-close inspection, discovering the many references to mathematicians, expressions, fun formulas, or arcane concepts impressed into many of the tiles.

The mural was installed in early 2006 by the artists Linda Vanderkolk and Scott Frankenberger with assistants Brishen Vanderkolk and Barry Rubin.



The Mathematical Sciences Research Institute (MSRI), located in Berkeley, California, fosters mathematical research by bringing together the foremost mathematical scientists from around the world in an environment that promotes creative and effective collaboration. MSRI's research extends through pure mathematics into computer science, statistics, and applications to other disciplines, including engineering, physics, biology, chemistry, medicine, and finance. Primarily supported by the U.S. National Science Foundation, the Institute is an independent nonprofit corporation that enjoys academic affiliation with more than 100 leading universities as well as support from individuals, corporations, foundations, and other government and private organizations.

MSRI's major programs, its postdoctoral training program, and its workshops draw together the strongest mathematical scientists, with approximately 1,700 visits over the course of a year. At any time, about eighty mathematicians are in residence for extended stays. Public outreach programs and VMath, the largest mathematical streaming video archive in the world, ensure that many others interact with MSRI throughout the year.

MSRI created the Critical Issues in Mathematics Education Workshop Series in 2004. This series of workshops addresses key problems in education today and is designed to engage mathematicians, mathematics education researchers, and K–12 teachers. The workshops provide participants a unique opportunity to learn about research and development efforts in this area. In addition participants develop ideas about methods for working on these problems and get to analyze and evaluate current or proposed programs. These workshops offer a space to make connections and exchange ideas with others concerned with the same issues in their fields.

Most workshops are held at MSRI and last for a few intensely secluded days. Each workshop attracts approximately 150 participants. Workshop organizers make sure to ensure diversity and relevant expertise by reaching out to mathematicians from a broad cross-section of colleges and universities.

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