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Mathematical Modeling in K–16: Community & Cultural Contexts

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Mathematical Modeling in K–16: Community and Cultural Contexts

A companion booklet to CIME Workshop 16, March 6–8, 2019

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Author: Evelyn Lamb. Editor: Tracy Hicks. Booklet coordinator: MSRI Deputy Director Hélène Barcelo.

The Critical Issues in Mathematics Education 2019 Workshop took place from March 6-8, 2019. It was the sixteenth workshop in this yearly series.

This booklet was written by Evelyn Lamb, summarizing and reporting on the individual presentations and sessions of the workshop, and edited by Tracy Hicks. Julie Rehmeyer contributed to the summaries of some presentations. Workshop presenters were generous with their time and energy in responding to numerous questions and drafts, and lead organizers Cynthia Anhalt and Rachel Levy read multiple drafts of the completed booklet with a careful eye for both detail and tone. The chapters attempt to capture the spirit and energy of the conference, and any errors are solely the responsibility of the author and editor, not the presenters or organizers.

We express our deep gratitude to all contributors whose work is included in this volume.


MATHEMATICAL SCIENCES RESEARCH INST

— MARCH 6–8, 2019

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INTRODUCTION

INCLUDING MATHEMATICAL MODELING as part of the curriculum is one of the most powerful ways students can learn mathematics. When students are confronted by a real-world question of genuine significance and asked to bring mathematical tools to bear in trying to make sense of it, learning mathematics becomes about more than just getting a good grade on a test: It becomes a powerful method of understanding the world around us.

Mathematical modeling has gained increased visibility in the education system and in the public domain, to the point where it currently appears as a content standard for high school mathematics and as a mathematical practice standard across the K–12 curriculum (Common Core Standards; and other states' standards in mathematics education). For those trained in the mathematical sciences, job opportunities are increasing in business, industry, and government. Quantitative reasoning is foundational for civic engagement and decision-making for addressing complex social, economic, and technological issues. As such, effective and equitable teaching and learning of mathematical modeling from kindergarten through graduate school should be a priority for teachers and schools.

For more details, including videos, notes, and handouts from the sessions, visit the workshop website: www.msri.org/workshops/919

Benefits and Barriers

For decades, a group of educators that has been increasingly enthusiastic about the educational potential of mathematical modeling has worked both individually and in small communities to create effective mathematical modeling tasks for students while meeting the pedagogical challenges mathematical modeling presents and supporting one another in learning more. They have discovered enormous benefits: students become more sophisticated and engaged mathematical thinkers; mathematics becomes connected to the cultures and communities of the students themselves, rather than being dry and abstract; students are more motivated to learn mathematics content when it is a tool to get answers they care about; and equity and social justice move naturally into the mathematics curriculum.

These benefits have convinced these educators that mathematical modeling should be at the center of much of mathematics teaching. The barriers that exist to adopting mathematical modeling as a larger part of K–16 mathematics education range from teachers' preparation to assessing mathematical modeling tasks on standardized tests.

Workshop Planning and Priorities

Organizers of the 2019 CIME workshop aimed to address those barriers and help stakeholders understand how to make mathematical modeling a far more widespread way of teaching mathematics. Lead organizers Cynthia Anhalt and Rachel Levy designed the workshop to be a natural successor to the 2018 workshop, "Access to Mathematics by Opening Doors for Students Currently Excluded from Mathematics," and invited several of the previous workshop's organizers and presenters to join the organizing committee and addressing questions related to equity and justice specifically in the context of mathematical modeling education.

Participation. Attendees came from a variety of professions. They were K–12 teachers, mathematicians, teacher educators, leaders of mathematical organizations, representatives from testing companies, and professionals who use mathematical modeling in their work. Together, they learned and shared strategies for using mathematical modeling in the classroom in effective and accessible ways.

Goals. The three primary goals driving the workshop were to

1. Critically examine current mathematical modeling educational policies and practices
2. Develop a vision for mathematical modeling education
3. Encourage participants to take action beyond CIME

Tension & balance. One planning theme of the workshop was tension: educators and leaders are often holding multiple concerns in tension. Open-ended mathematical modeling tasks are in tension with the kinds of assessments that are constructed to be easy to grade by constraining the acceptable set of answers. The desire for widely-available professional development materials to prepare teachers to include mathematical modeling in their classrooms is in tension with the desire for mathematical modeling tasks that address the culture of the local communities in which teachers and students live. Part of the hard work of all education, including that in mathematical modeling, is to address multiple concerns in a balanced way.

Organization of this Report

Panels and discussions at the workshop fell into four major themes: understanding the current landscape and recent progress in mathematical modeling education; equity and justice in mathematical modeling education; how mathematical modeling education can help students prepare for careers in STEM fields; and broadening the impact of mathematical modeling education outside of participants' own classrooms and communities. This report is divided into parts based on these themes.

Part 1. The Current State of Mathematical Modeling Education

Three CIME panels addressed the current state of affairs, challenges, and opportunities in mathematical modeling education, including

- Actions that have moved the needle with respect to adoption of mathematical modeling practices in educational settings
- Where mathematical modeling education is taking hold in K–16 education
- Where teachers are currently getting their education in mathematical modeling and the challenges and opportunities in this arena
- Balancing meeting requirements of pacing/assessment with the creation/implementation of new teaching modalities
- Including community and cultural contexts

Part 2. Recent Progress in K-12 Mathematical Modeling Education

Part 2 continues the investigation of these themes, specifically in K–12 education.

Part 3. Equity and Justice in Mathematical Modeling Education

Participants in the CIME workshop are interested in improving mathematical modeling education at large and small levels, from their own classrooms to national curricula. It is no secret that the field of mathematics is male- and white-dominated. Can including mathematical modeling in curricula work to address the biases and inequalities that exist in mathematics? This section discusses

- How equity and social justice frameworks support and/or challenge mathematical modeling education
- Connecting equity and social justice to research initiatives
- Outcomes and crucial, replicable elements of programs that have effectively addressed social justice in mathematical modeling contexts
- Impacts on student learning, experiences, identity, and engagement when teaching from an equity/social justice stance
- Lessons for mathematical modeling education from student experiences
- Ways to approach the sociopolitical aspects of mathematical modeling: Supporting teachers as they navigate the mathematical, community, and political contexts

Part 4. Beyond K-12: Non-STEM and Professional Pathways

How do educators and administrators build a mathematical modeling pipeline that gives all students access to the benefits of mathematical modeling activities and prepares those who wish for jobs that use mathematical modeling? This section discusses

- Mathematical modeling learning pathways at the undergraduate and graduate levels
- Aspects of these pathways that are working or need improvement
- Modeling skills that employers look for at the undergraduate level and postgraduate level
- Collaboration between industry and academics to provide industry-specific modeling training and experience

Part 5. Systemic Change in Mathematical Modeling Education: Structures and Assessment

CIME participants wanted to reach beyond their own classrooms. Including effective mathematical modeling instruction in the classroom and in curricula provides one key contribution to the improvement of mathematics education in the United States. Two panels addressed current research in mathematics modeling education (addressed by individual chapters in each of the previous sections), the role of assessments in mathematics modeling curricula, and how to take action to make systemic change in mathematics modeling education. Part 5 discusses structures and assessments that contribute to systemic change, including

- Overview of the implementation of U.S. mathematics modeling education in K–12
- Influence of state and national standards on curricula, implementation, and assessment
- Lessons for the U.S. from the international research and education communities
- Current assessment frameworks for mathematics modeling education
- Structures and mechanisms that help build partnerships

Part 6. Emerging Critical Issues and Closing Thoughts

In discussions and breakout work that followed each panel session, the CIME panelists and participants identified critical issues and questions that should motivate continuing work and improvements in the field. These emerging critical issues are listed in Part 6 along with the organizers' closing thoughts. The issues range from very local, personal issues in the classroom all the way to large, structural issues.

Connections, Support, and Growth

CIME 2019 participants are dedicated to change in their own classrooms and projects, as well as to providing support, resources, and policy direction for stronger mathematics modeling education and research across the K–16 spectrum. They want to build on the strengths where modeling, community, and cultural connections already exist and to create new structures where they are needed in order to make mathematics modeling more prolific in the curriculum.

While this report aims to summarize the issues addressed through the workshop's panels and discussions, it cannot completely capture the atmosphere of the workshop, which fostered rich reflections and conversations and new connections between educators, mathematicians, and industry representatives who might not have been in the same room without this kind of workshop. The educators and leaders who attended are bringing those reflections and conversations to their communities, large and small, in many ways.

PART 1

THE CURRENT STATE OF MATHEMATICAL MODELING EDUCATION

Perspectives from
the classroom and the literature,
from national and international policy,
and from the role of student

CHALLENGES AND OPPORTUNITIES IN INCREASING MATHEMATICAL MODELING AMONG K–12 TEACHERS

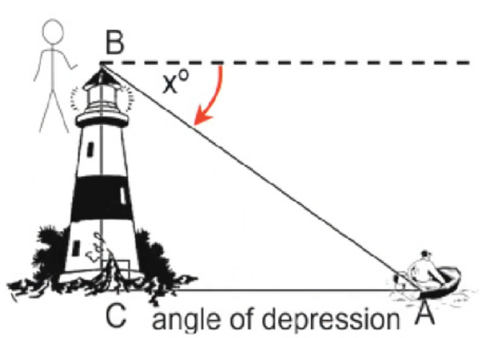
Based on the panel presentation by Patrick Honner of Brooklyn Technical High School

HOW DO WE MAKE THE USE of mathematical modeling more widespread among K–12 teachers? To begin with, we have to understand the barriers that currently stand in the way.

Three Challenges to Using Mathematical Modeling in K–12

Misunderstanding about what mathematical modeling actually is. Among teachers, there may not be a common understanding of the meaning of mathematical modeling or how to engage students in mathematical modeling. For example, some teachers may see the left figure below as an example of mathematical modeling with trigonometry. Another might think of the right figure as an example of modeling with geometry:

Are these examples of modeling?



Teachers often understand that "mathematical modeling" means that there must be some kind of application, but the application they choose might turn out to be a superficial add-on that ends up being ludicrous, like the pseudo-context offered in the problem below:

What is the volume of the solid generated when the region bounded by the graph of $x = \sqrt{y-2}$ and the lines $x = 0$ and $y = 5$ is revolved about the y-axis?

Pseudo-context does not make it modeling…

* Btw, this is the volume of a priceless Victorian era teacup

Instead of different ways of presenting modeling to different teachers, we need adequate preservice and in-service teacher preparation in mathematical modeling for all teachers so that they are aware of the options and can make appropriate choices depending on the prior experiences with modeling that their students bring to the classroom and the current learning objectives.

Open-ended activities are challenging for instructors. An additional challenge is that when teaching mathematical modeling, one never knows the type of mathematics that will be required, and some of it may not be mathematics with which a particular teacher is familiar. In addition, the act of mathematical modeling itself is new content to most teachers.

Testing often ignores process. Standardized testing poses one more challenge, because what is tested often dictates what is taught, and mathematical modeling questions on exams rarely capture students' understanding of the mathematical modeling process. Consider, for instance, the test question of how many ways five books can be arranged on a bookshelf. The intended answer is 120, or five factorial. But that answer is correct only if you restrict arrangements to the type pictured below on the left. You must not consider other arrangements, as in the illustration on the right.

This question, and so many others like it, teaches students that when they come to a mathematics problem, they have to shut off any part of their brain that thinks creatively or outside the box, and focus only on the answer we want them to give.

What the test expects…

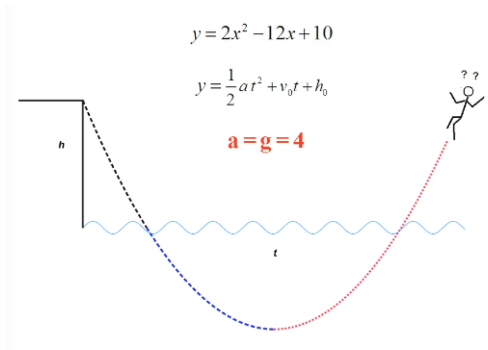
But what about … ?

Turn off your creativity for the test…

Honner shared an all-time favorite terrible test question, which is misleading in both the mathematics and the reality of the scenario:

The height of a swimmer's dive off a 10-foot platform into a diving pool is modeled by the equation $y = 2x^2 - 12x + 10$*, where* x *represents the number of seconds since the swimmer left the diving board and* y *represents the number of feet above or below the water's surface. What is the farthest depth below the water's surface that the swimmer will reach?*

An all-time terrible test question



Everything about the real world in this problem is window-dressing. The quadratic equation suggests that the acceleration due to gravity is 4. The given formula completely breaks down after the swimmer leaves the water. The problem simply wants students to find the vertex of this parabola, while ignoring everything we should want them to attend to in modeling. Such problems send the same message to teachers.

Two Suggestions for Dealing with These Challenges

At the school level, use learning teams. Honner offered some suggestions for dealing with these challenges. First, teachers should work together on professional learning teams. Honner has found that there is a better chance of success and sustainability if he trains a group of teachers rather than working with them individually.

For policy change, use the magic word ("STEM!"). Additionally, teachers who wish to advocate for more and better modeling education should know that "STEM" is a magic word. STEM (science, technology, engineering and mathematics) is one of the most powerful policy-moving words in education. People who make policy and spend money on it love the word STEM and hear almost nothing else. Mathematical modeling integrates all of these, so there is a very natural place for modeling in STEM. Similarly, computer science is a natural place to integrate mathematical modeling, and computer science has enormous policy energy right now. For the past few years, Honner has been focused on integrating mathematics and computer science, and he has found that when mathematics is taught computationally, every programming problem is explicitly a modeling problem. As students are working in mathematical modeling, they are exploring a problem space, identifying important variables, making parameter decisions and organically iterating the process to improve their models. At the intersection of mathematics and computer science lies modeling.

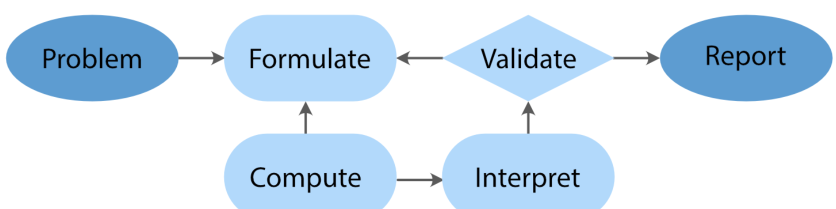
"THE LITERATURE SAYS…"

Based on the panel presentation by Jennifer A. Czocher of Texas State University

JENNIFER CZOCHER AIMED TO GIVE TEACHERS and other stakeholders an idea of what research has been done about mathematical modeling education, what it says about best practices in teaching mathematical modeling, and to walk through some ways of understanding students' thinking about modeling tasks. She also wanted to show that there are a lot of open questions remaining and therefore a need for more research on this topic.

Mathematical Modeling Education: Benefits & Structure

Mathematical modeling is underrepresented in mathematics education research journals; nonetheless, research on modeling shows that including mathematical modeling in mathematics curricula has several benefits. While it is positively associated with three predictors of persistence in mathematics, such as interest, self-efficacy, and proficiency, innovations that target interest only succeed when they also develop both proficiency and self-efficacy. For women in particular, self-efficacy in mathematics, one's belief in their ability to complete a problem or task, may play a larger role than proficiency in their persistence.



Mathematical modeling cycle

From Common Core State Standards (2010), page 72.

Mathematical modeling is a creative and complex project. On one hand, part of the power of mathematical modeling is the fact that essentially the same mathematics can be used to represent different real-world situations. On the other hand, a small change in real-world conditions can suddenly make the mathematics required to model the situation more complicated. Mathematical modeling is frequently represented as a cycle, as shown above. This representation helps students conceptualize and predict student work.

The cycle is descriptive, not prescriptive. Domain knowledge, from both the situation being modeled and the mathematics itself, is important, but the most difficult skills developed in modeling are recognizing and imposing mathematical structure, mathematizing the problems, and validating proposed solutions.

Student Reasoning Makes Problems More Realistic

Students are themselves also creative and complex. Czocher shared two examples from the research literature of student reasoning on modeling tasks, which show that authenticity is a powerful way to engage students in mathematical modeling.

(1) *Suppose you have two job options: One pays you $7.50/hour and the other one gives a fixed amount of $300/week. Which option would you take?*

Some students said the problem did not provide enough information because they did not know how many hours to expect to work. Some chose $300/week because then you would earn $1200 per month; others chose $7.50/hour because you could earn, for example, $750 per week by working 100 hours. The rest said there is no difference, reasoning that 40 hours, a typical work week, gives you $300/week at the $7.50/hour rate. Students considered whether a 40-hour work week was reasonable, the distance between work and home, whether they could set their own work hours, and how they would get to and from work. By considering those factors, they made the problem more realistic.

(2) *You just won Gasoline for Life prize. Should you take the option of a lump sum of $250,000 instead?*

In this case, teachers advised students to consider questions such as: *Will I be living in a city where I can take public transportation? Will my job have a long commute that I have to make each day by car? What if I want to take a long trip this summer?* Students, on the other hand, often opted to take $250,000 up front because they could use the money immediately either to buy things they want (including college tuition) or to invest. "Who knows where I'll be in 10 years! I'll take the cash!" said one. Another said, "$250,000 NOW. I won't get to drive until I am 16 and even then I probably won't have money to buy a car—so even assuming that I will get one, say, in seven years, I am almost sure that by then we will have solar cars, and no need for fuel." Once again, students transformed the problem into one that was more authentic and complex than the one originally posed.

Scaffolding to Reach Larger Problems

The two examples also highlight an important question teachers have: If students have such varied reasoning and answers on these fairly simple problems with real applications, what can a teacher do to implement larger, more authentic problems? Czocher shared some ideas for how teachers can scaffold these activities in their classrooms. They need to know the problem thoroughly, understand students' current thinking, understand students' possible cognitive operations, and give them strategic help.

Two recent studies characterized effective scaffolding as having three dynamic aspects. Effective scaffolding

- Responds to students' current ways of thinking, which is based in their relevant mathematical and non-mathematical knowledge,
- Is flexible enough to reduce the level of support within a problem because the student is capable of performing the next steps independently
- Anticipates long term reduction of support across many problems in order to transfer responsibility to the student.

One effective strategy for uncovering students' current ways of thinking is to ask them explicitly to explain the state of their work.

Research Gaps

The most under-researched area of modeling is how to educate teachers to best implement it. As with other areas of math teaching, there are parallel problems: educating teachers in the content they need and educating them in pedagogical practices. Two issues that arise are trying to communicate what modeling is and understanding what it is for. Is mathematical modeling a vehicle for teaching other mathematics content or the content itself? Is it a mechanism for policy change? Efforts to train teachers to teach mathematical modeling must give them ample time to reflect on modeling problems and pay extra attention to how to handle time constraints.

One of Czocher's primary goals was to share research articles about mathematical modeling that will allow teachers to advocate for including modeling in curricula and understand how to teach modeling effectively. A brief list of references shared by Czocher is included at the end of the report.

AN INTERNATIONAL OVERVIEW OF MATHEMATICAL MODELING EDUCATION

Based on the panel presentation by Sol Garfunkel, of the Consortium for Mathematics and Its Applications (COMAP)

"WE ARE NOT ALONE," says Sol Garfunkel of the current challenges in the U.S. to increase the prominence of mathematical modeling in K–16 education. Back in the 1960s in England, an effort was made to introduce mathematical modeling into the curriculum. In the 1970s and 1980s, a Dutch curriculum based on applications came out. Denmark also changed its curriculum to emphasize modeling. In 2003, Germany did the same. China is currently embracing modeling wholeheartedly.

Two Key Factors for Success

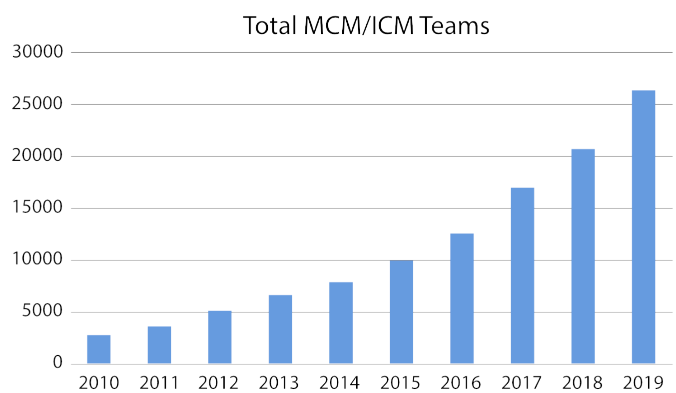
Time. One key factor to make these efforts successful is time. The 1989 NCTM standards and the Common Core ran into problems because they were introduced too abruptly. By contrast, the Dutch took five years to write their modeling-based high school curriculum and another five to develop the tests and to do teacher training. Only after ten years did a single word get taught.

Another example is Germany after the "PISA shock" in 2003. PISA, the Program for International Student Assessment, is a test for 15-year-olds of practical problems like, *Can you read a bus schedule and go from here to there sensibly?* It is not curriculum-based. The assessment is offered every three years; mathematics is the major focus every nine years. In 2003 (a mathematics year), Germany scored only one country above the United States. This caused incredible angst, because German educators and citizens thought of themselves as being one of the top countries in the world for mathematics education. As a result, they took the next nine years to revolutionize the national standards, curriculum, and exams — and for the next mathematics PISA in 2012, they were among the top four or five countries in the world.

Coordinated research and education communities. A second key factor is that the mathematics and mathematics education communities have to be on the same page. In 1989 (at the introduction of the Common Core), the "math wars" hindered progress. Later on, the Common Core had its political problems as well because testing and curriculum got mixed up in people's minds. If the U.S. is really going to have change, it needs to have a national commitment with as much national consensus at the governmental level as possible. Further, the research and education communities have to speak together with one voice. Otherwise, efforts are viewed merely as nice experimentation by individual teachers.

COMAP and Other Competitions as a Catalyst in China

China has committed themselves to educational involvement in mathematical modeling, as is evident in entries (and success) in modeling competitions like MCM, COMAP's Mathematical Contest in Modeling (described by Jo Boaler in the presentation reported on page 41). COMAP started MCM with a grant from the Fund for the Improvement of Postsecondary Education (FIPSE) in 1984. The first contest was run in 1985 with 90 teams of three members each from 70 colleges in the United States. Around 2010, Chinese teams started joining, and the total number of teams had grown to approximately 2500. By 2018, there there were approximately 25,000 teams with approximately 500 American teams and another 100 from 20 other countries. The graph below shows the rapid rise in involvement of teams in MCM and Interdisciplinary Contest in Modeling (ICM) competitions.



Rapid growth in contest participation

The Shanghai Experimental School (SES) is a high school funded by the city of Shanghai. Their students have won almost as many modeling contests as the North Carolina School of Science and Math. At SES, not only does each student have their own computer, but each team has their own server. Classrooms have been designed specifically to support mathematical modeling.

A Call for Commitment

Garfunkel is not a great believer in contests for their own sake, but he believes that the effects on the educational system in China have been impactful. Modeling is now a very important subject, and they have changed the high school curriculum to emphasize modeling and are working on changing the national tests to include modeling. There are now even colleges and universities in China with the word "modeling" in their names. Garfunkel explains the U.S. is not alone in doing mathematical modeling, but he fears that we may very well wind up alone if we do not take the call to add it to the curriculum seriously and make the kinds of commitments that we need to make in order to make mathematical modeling education effective.

CAN ALL LEARNING BE VIEWED AS MODELING?

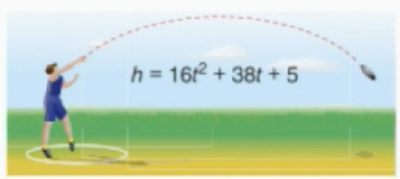
Based on the panel presentation by Dan Meyer of Desmos

DAN MEYER DESCRIBES his primary work at Desmos (a company that develops software tools for math education and recently launched a middle school math curriculum) as using digital technology to design modeling tasks. He is hopeful for the future of math modeling, even though he finds it lacking in some ways at present.

Two Extremes of "Mathematical Modeling"

On the one side, you have what Meyer describes as "the charlatans, the grifters," who will take any problem set outside the classroom and apply the "modeling" label to it. As an example, Meyer shows a standard ballistics problem set outdoors at a track meet.

9. CCSS | MODELING Ken throws the discus at a school meet. (Example 4)



a. What is the initial height of the discus?

b. After how many seconds does the discus hit the ground?

Should this be considered modeling?

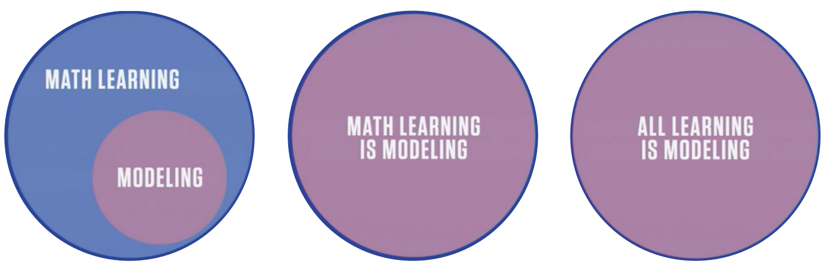
On the other side, Meyer cites the GAIMME (Guidelines for Assessment and Instruction in Mathematical Modeling Education) report. Meyer believes the report is well-intentioned, but nevertheless too long and dense for a classroom teacher to digest and incorporate into their teaching. In his view, the report relies on too many ambiguous adjectives such as "open," "genuine," and "real-world." As a result, he believes GAIMME puts mathematical modeling out of reach for most teachers.

Thinking of Modeling as a Continual Revision of Knowledge

Meyer challenged the audience to expand what they think of as mathematical modeling. To introduce the idea, he used an example of a mystery sequence of numbers: The first number was 1. "What might the next number be?" he asked. He revealed the next number — 2 — then 3, then 5, and talked through what possible next numbers could be after each reveal. Eventually, some participants became fairly confident about what the sequence was.

Meyer argued that tasks like this should count as mathematical modeling because participants started with early ideas about what was happening, revised their models as more information was revealed, and eventually felt confident that they understood the pattern.

Meyer broadens this idea to see modeling as present in abstract sequences of numbers and even, for example, concepts as abstract as love. He explained that a person's idea of what love is starts in childhood, perhaps influenced by the relationships they see around them, and is refined over time. In young adulthood, they may take that idea out into the world and discover it is lacking. The model may be tested—perhaps painfully through a bad breakup—but continues to sharpen over time. That, too, is modeling.



How should we regard modeling within math learning?

Drawbacks of Treating Modeling as a Subset of Math Instruction

Meyer notes that in conversations about modeling, people often treat modeling as a subset of mathematics learning. He says that to draw the distinction is to undermine the goal of modeling — because this implies for teachers that on some days (the days they are working in the modeling "subset"), they should respect and draw upon student knowledge; but on other days (non-modeling days), that respect is not necessary. Or, likewise, that in one place, they should respect students' initial ideas about what is going on mathematically, whereas in another, they need to issue them predetermined models and try to transfer these models from the teachers' brains to the students'. Or, finally, that in one situation, the teachers give interpretive feedback that helps students draw and redraw and sharpen their models, but in another, they tell them that they are either right or wrong.

To avoid these contradictions, Meyer believes the perspective should be that all of mathematics learning is modeling, and in fact, he proposes that all learning is modeling. He drew a parallel between students' learning and the experience of the participants in the workshop: "We are right now modeling our understanding of mathematical modeling. We all came in here with different ideas about what this thing is, and we're all having experiences and hearing things and having conversations, and we will all walk out of here with that feedback, with a different model of mathematical modeling."

Meyer's presentation was somewhat controversial. In later discussions, panelists and others at the workshop discussed the value of having a shared definition of mathematical modeling that is more specific than "all learning is modeling."

A SHARED MATHEMATICAL MODELING TASK

Presented by Julia Aguirre of the University of Washington Tacoma

IN ORDER TO GIVE CIME PARTICIPANTS an opportunity to experience and reflect on the student perspective, Julia Aguirre facilitated a mathematical modeling task that she has used in many contexts, including in mathematics methods classes for preservice K–12 teachers, practicing teachers, and middle schoolers.

Modeling Tasks

The exercise was based on the water crisis in Flint, Michigan and was framed by two tasks: First, to evaluate a plan of water donation by large corporations to the school children of Flint, and then to determine the amount of waste generated by the plastic bottles.

To begin the activity, Aguirre asked the workshop participants to discuss what they knew about the water crisis in Flint, including what they knew about the crisis, why lead in a water supply is dangerous, and how the problem was or could be addressed.

Aguirre then showed a clip from CNN Student News that gave a background on the crisis, including the dangers, the difficulty in "uncovering" the crisis, the mechanism by which the water became contaminated, and the the social and personal impacts on the community.

After viewing the short video, participants engaged in discussion about a real-world modeling task based on true events:

> *On January 26, 2016, Walmart, Coca Cola, Nestlé and PepsiCo said they would "collectively donate bottled water to meet the needs of 10,000 schoolchildren for the balance of the calendar year." They planned to send 176 truckloads of bottled water, with up to 6.5 million bottles.*

A. *Is that enough water?*

B. *Is the companies' plan a good one? Nestlé alone sent out 33,000 bottles worth of water in Flint three times a week, providing about 100,000 bottles worth of water each week until Labor Day. So how large is the environmental problem created by the plastic waste from Nestle's water bottles? What about that of all four companies? What size bottle minimizes waste while meeting the need?*

Participants at the workshop worked on these problems in small groups for 45 minutes, producing a poster for each question that described their model and the corresponding assumptions.

Reflections about the Task

Afterward, Aguirre asked participants these questions:

- How does the open-ended nature of the modeling tasks provide access for everyone in your group to contribute to solutions?
- What role did having to make assumptions and decisions play in moving your group forward when you found that you did not have all the information required to solve the problem?
- What other social issues were discussed in your group related to the problem?
- How do these tasks address issues of equity, inclusion, and social justice?

Connecting through emotions. One participant commented that a problem like this can elicit an intense emotional response; as educators, it is important to create space for people to process those emotions. This partly depends on the individuals in the group. Some might get narrowly focused on the math, while others might more fully take in the social justice issues. In this problem, and how it is framed, the social justice issues feel authentic rather than contrived. She said it was a great experience for her group.

One participant from another group said that the task reminded her of the recent focus on "rehumanizing mathematics." "Tasks like this really tug at students heartstrings," she said. "When they do a community-based activity like this, there is a sense of empathy the students start to feel, and then they really feel this ownership because it is about community and people… When students take on tasks like this they start to feel like they can actually make a small change, a change that matters."

Mathematical questions and insights. Another participant said that her group got absorbed in the mathematics. They had to think about what "enough" water meant. Was it one bottle of water per student per day? Was it enough for them at home too? For their families? For drinking, for cooking, for washing dishes, for showering? Her group came up with an estimate that the students would need two million bottles of water every day. "What made this problem so pertinent," she said, "was that it was something we really care about."

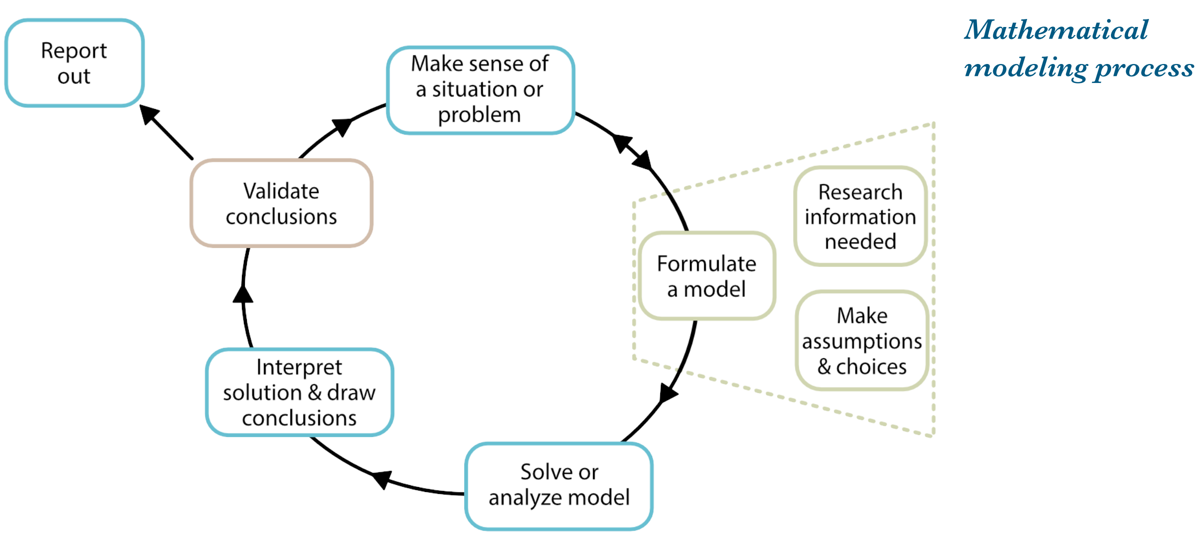
A participant from another group said they appreciated how mathematical modeling provides a way to think about a giant, overwhelming problem — like that of water in Flint — and break it into small pieces. The idea that we can start by making some assumptions and deal with just one part of the overall issue is very empowering.

Another participant said that they appreciated the way this task would help students visualize the difference between a million and a billion. It would also give them an opportunity to think about the environmental impact of bottled water and whether there are more efficient ways to distribute those resources.

Diversity of thought. Another participant commented on how important it is to have a diverse group. One of her group members, who is Native American, asked, "Did anyone ask these people what they wanted or needed, or is this just put on them?" The participant, a white woman, thought, "Oh my goodness, I didn't even go there!" Her group member's comment made her think about the importance of diversity in her student population as well. Students are often trying not to stick out, because diversity tends not to be valued in the classroom; this task showed her the importance of valuing that diversity, both educationally and in order to generate good mathematical models.

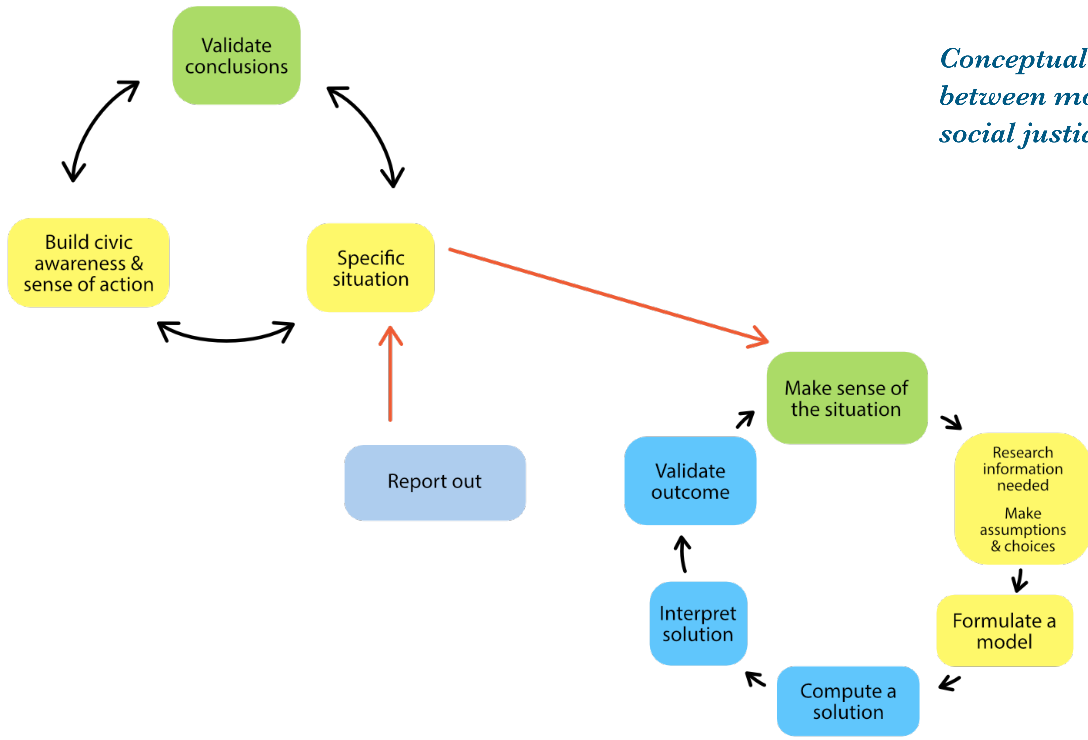
Frameworks for the Modeling Process

Aguirre offered this diagram, developed by Anhalt, Cortez, and Been Bennett (2018), as a way to think about the mathematical modeling process:



Aguirre et al. (2019).

Aguirre also offered the following diagram to visualize how mathematical modeling interacts with social justice issues, which she published in a recent paper with others:



Conceptual interaction between modeling and social justice

Expanded from Anhalt et al. (2018); Aguirre et al. (2019).

References

Cynthia Oropesa Anhalt, Ricardo Cortez & Amy Been Bennett (2018) The Emergence of Mathematical Modeling Competencies: An Investigation of Prospective Secondary Mathematics Teachers, Mathematical Thinking and Learning, 20:3, 202-221, DOI: 10.1080/10986065.2018.1474532

Aguirre, J., Anhalt, C., Cortez, R., Turner, E., & Simic-Muller, K. (2019). Engaging Teachers in the Powerful Combination of Mathematical Modeling and Social Justice: The Flint Water Task. Mathematics Teacher Educator, 7(2), 7-26. doi:10.5951/mathteaceduc.7.2.0007

PART 2

RECENT PROGRESS IN K-12 MATHEMATICAL MODELING EDUCATION

Introducing mathematical modeling in the classroom, supporting current teachers, and training future teachers

MORE CHALLENGES AND OPPORTUNITIES IN INCREASING MATHEMATICAL MODELING AMONG K–12 TEACHERS

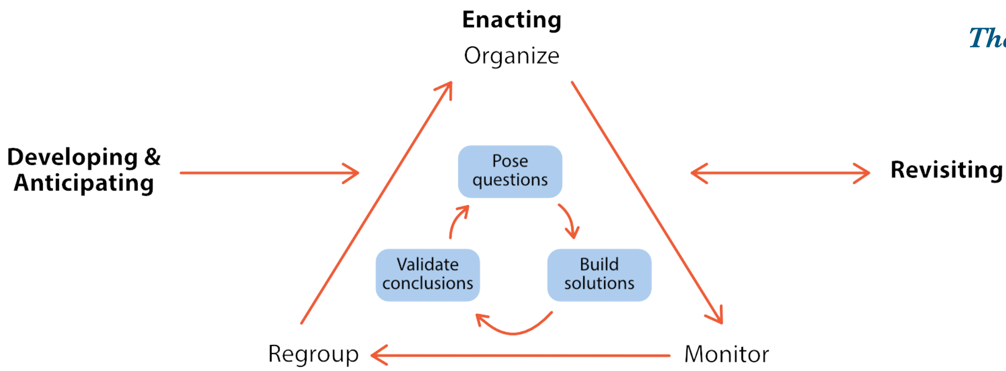
Based on the panel presentation by Maria Hernández of the North Carolina School of Science and Mathematics

THE NORTH CAROLINA SCHOOL OF SCIENCE AND MATHEMATICS, where Maria Hernández teaches, is a public boarding school that serves the entire state. The school has a strong focus on mathematical modeling and has developed its own precalculus and calculus textbooks that are infused with modeling. In addition, they have standalone courses that cover mathematical modeling, mathematical modeling with differential equations, complex systems, and complex systems and modern networks.

Hernández explained that by opening the door to mathematical modeling, teachers can invite students into the mathematical conversation, offering them opportunities to work collaboratively with their peers on real world problems.

Modeling Puts Teachers in a New Role

Hernández also noted that there are challenges to opening that door. Like Patrick Honner (see pages 13 and following), Hernández has found that teachers feel a lot of tension around standardized testing focused on curriculum standards. Teachers say at the beginning of a training, *Where will this fit into our curriculum standards?* They are often worried about when they will have time to squeeze in modeling when they have a long list of standards to cover. Parents and students also have to be convinced that mathematical modeling will be good for the students and that focusing more on modeling will not hurt their standardized testing scores. Teachers need support because their role changes when they teach mathematical modeling, and there can be more uncertainty about the shape of a class period. Sometimes the real-world contexts of the modeling tasks are new for the teachers. They often ask, *What happens if the students ask a question and I don't know the answer?*



The teacher's role

Carlson et al. (2016).

The figure above is a schematic of the teacher's new role. The interior of the figure shows what the students are doing: posing questions, building solutions, validating their conclusions, and then posing more questions. The teacher, in the meantime, is not lecturing at the front of the room. Instead, they are helping the students organize their thoughts, anticipating their questions, figuring out the kind of task that would be interesting to students, enacting the task, monitoring the class, and bringing everybody back together to point things out like, *Where is the mathematical content that we've studied this year coming up?* The situation can be even more complicated when the mathematics is being developed by the students for a task.

Given all these challenges and tensions, how do we help teachers when they first engage with mathematical modeling?

Steps to Open the Door to Mathematical Modeling Instruction

To ease the transition to the new types of activities and instruction mathematical modeling requires, Hernández tries to find places for teachers to incorporate mathematical modeling into their classrooms in small ways. She looks for tasks that are tied to the curriculum, encouraging teachers to commit to incorporating one open-ended mathematical modeling problem per term or unit.

She shares resources like the COMAP (Consortium for Mathematics and Its Applications) curricula, the GAIMME (Guidelines for Assessment and Instruction in Mathematical Modeling Education) report, modeling handbooks, and MMHub. MMHub is a website of the NCTM/SIAM/COMAP Committee on Modeling Across the Curriculum that provides mathematical modeling resources for both students and educators and allows people to submit rich modeling tasks that others can use. Further support can be found in conferences such as the NCTM and SIAM education conferences, which provide access to resources for creating modeling tasks as well.

Hernández encourages teachers to tackle just one little thing, rather than feeling like they have to revolutionize their teaching entirely. And she provides opportunities for teachers themselves to engage in the modeling process, so they can experience modeling from the students' perspective.

Accomplishment and Perspective

One teacher from one of Hernández's workshops said, "I had experiences in math modeling that helped me understand what it would be like for my own students." And a student from one of Hernández's classes said, "Math for me was mostly focused on doing what the teacher wanted. Doing the first problem here opened my eyes to real math problems. While trying to solve the problem, I became thoroughly confused and frustrated. Solving it with my group felt great since it felt like we accomplished something." Such reflections can be powerful for getting parents on board as well.

Accomplishment and perspective!

Hernández' Modeling Workshop Participants, Anja S. Greer Math and Technology Summer Workshop, Philips Exeter Academy

Reference

Carlson, Mary Alice, Megan H. Wickstrom, Elizabeth A. Burroughs, and Elizabeth W. Fulton. 2016. "A Case for Mathematical Modeling in the Elementary School Classroom." In Mathematical Modeling and Modeling Mathematics, edited by Christian R. Hirsch, pp. 121–29. Annual Perspectives in Mathematics Education 2016. Reston, VA: National Council of Teachers of Mathematics

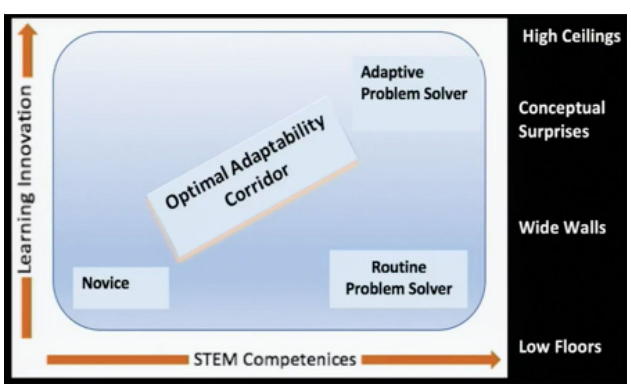
INTRODUCING MODELING TASKS TO YOUNG CHILDREN

Based on the panel presentation by Lyn English of Queensland University of Technology

MATHEMATICAL MODELING is increasingly important in a STEM-based world, and early foundations lay the groundwork for secondary school and beyond. Mathematical modeling develops students' skills in future-oriented thinking (systems thinking, design processes, creative and innovative thinking, critical thinking, teamwork, flexibility, and adaptability) and facilitates interdisciplinary learning. Most of Lyn English's work has been based on developments in the U.S. She has worked on model-eliciting activities, data modeling, STEM-based modeling, and modeling with design, an iterative activity that has overlaps with both mathematical modeling and engineering design applied to complex problems. She focused her presentation on mathematical modeling education research in younger grades.

Framework for Creating Modeling Experiences for Elementary Grades

Creating modeling problems for any grades, particularly for elementary school, is not easy for curriculum developers or teachers. English has developed modeling experiences for the elementary grades that fit in with the teachers' curriculum. She has worked on problems that intersect with STEM subjects, cultural and global issues, history, geography, literacy, and philosophy. Mathematical modeling helps students develop skills that are relevant to all of those other subjects, such as looking at assumptions, recognizing the points of view of others, and questioning one's own thinking and understanding.



Learning Innovation framework

English uses a framework she calls “Learning Innovation” to illustrate important features of modeling problems. Good modeling problems have low floors, so all students can enter. They have high ceilings, so students can go as far as they wish. Even students who might have been labeled as mediocre or struggling in mathematics can become adaptive problem solvers, adapting what they have learned to related situations. Modeling tasks have “wide walls” in the sense that you can tell your peers about your learning during the activity.

Two Examples of Tasks

Fancy feet. English shared two examples from her experience in mathematics education. The first was “fancy feet,” an activity she did with fourth graders. The activity started with human feet: measuring feet and shoes, gathering data about shoe sizes, looking at the materials shoes are made of, and so on. Students also designed and created shoes. She shared a video from a company called “Shoes of Prey” that helped students understand the design and manufacturing process. Students had to consider things like materials, functions, and budgets in their designs. They tested their shoes and redesigned and reconstructed them as necessary.

Swimming progress. Another modeling problem she has used (with both fourth graders and seventh graders) is related to sports. She showed a graph of Australian women’s 100m freestyle swimming results from 2012–2014. Students gathered data about swimmers’ times and ages and were given the task of selecting the Australian team for the next Olympics. Some students tried to compare the effect of less versus more competitive competitions (national finals are less competitive than Olympics or Commonwealth Games), and some looked at how swimmers’ times might change based on age. Some just looked at their personal best times, and others criticized that approach as too simplistic. English has found that students generate their own mathematizations on ranking activities. By fourth grade, students have not been formally introduced to mode, median, frequencies, ranks, sorting, quantifying, weighting, or optimizing, but they develop those notions on their own.

Women’s 100m Freestyle Results (inc relay events) recorded by Australian Competitors (in sec)

Event No.	Competition	Alicia Coutts (Current age 27)	Ami Matsuo (Current age 19)	Brittany Elmslie (current age 20)	Bronte Barratt (Current age 25)	Bronte Campbell (Current age 20)	Cate Campbell (Current age 22)	Emily Seebohm (Current age 22)	Emma McKeon (current age 20)	Melanie Schlanger (Current age 28)	Sophie Taylor (Current age 17)	Yolane Kukla (Current age 19)
1	July 2012 Olympic Games	53.90		53.41			53.19	54.24		52.65		54.61
2	April 2013 Australia Swimming Championships	53.42	55.30	54.20	54.98	53.72	52.83	54.81	54.17	54.29	56.64	54.98
3	June 2013 Swimming Australia Time Trials	55.10	55.75		56.53	54.35	52.89		55.60			
4	July 2013 FINA World Championships					54.57	52.33					
5	December 2013 McDonald’s Qld Championships			54.71	56.41	53.85	52.69		54.69	55.22		56.36
6	Jan/Feb 2014 NSW/Vic Open Championships	54.77			55.33			56.02	53.80		56.36	55.59
7	April 2014 Australian Swimming Championships		54.72	54.06	54.75	53.02	52.68	55.76	55.43	53.78	56.08	54.95
8	June 2014 Swimming Australia Grand Prix 3		55.28	55.29						53.97	57.65	55.16
9	July 2014 Commonwealth Games					52.86	52.68		53.61			
10	August 2014 Pan Pacs			53.72	55.28	52.88	52.62		54.96	52.97		
	Personal Best Times www.swimming.org.au	**53.42**	**54.72**	**53.41**	**54.75**	**52.86**	**52.33**	**53.96**	**53.43**	**52.65**	**56.01**	**54.08**

Blank: Did not compete

MATHEMATICAL MODELING IN ELEMENTARY SCHOOL

Based on panel presentations by Erin Turner of the University of Arizona and Nina Miller of the Tucson Unified School District

THE MATHEMATICAL MODELING WITH CULTURAL AND COMMUNITY CONTEXTS (M2C3) project focuses on third, fourth, and fifth grade teachers because the project leaders believe that young children have experiences in homes and communities outside of school that produce knowledge and practices that can be resources for their work in mathematics. For the project — funded by the National Science Foundation and based in the Tucson, AZ and Renton, WA areas — Erin Turner and her colleagues design modeling tasks that allow children to leverage the experiences that they bring from outside the mathematics classroom, and outside of school altogether, in their mathematics classes.

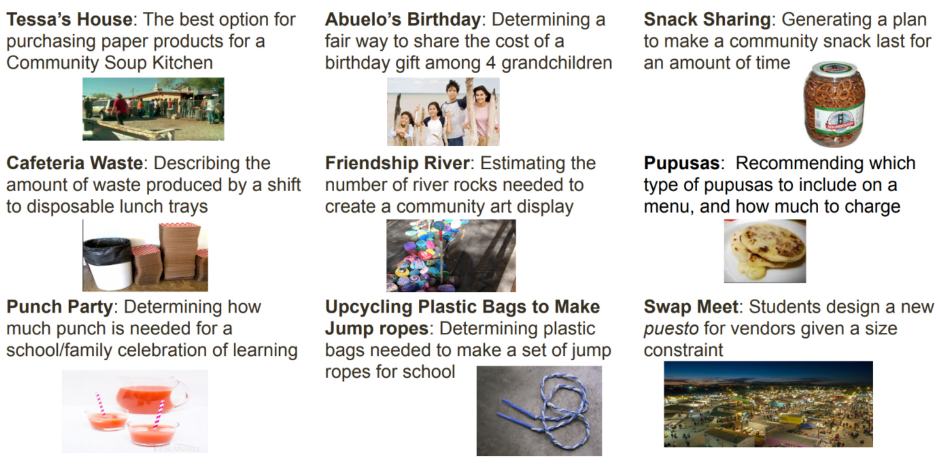
The project focuses on two driving questions:

- How can professional development support teachers in developing pedagogical knowledge and skills related to mathematical modeling?
- How can mathematical modeling support teachers in connecting to students' communities, cultures, and experiences?

Support for M2C3 Cohorts

The researchers have been working for two years with teachers who have a wide range of previous experience, as well as with instructional coaches and mathematics leaders from the districts. In the summer, they meet with the teachers to provide a week-long professional development institute. During the school year, they convene every month in a study group to discuss the tasks they are implementing in their classrooms, to analyze student work, and to brainstorm additional tasks that connect to things that are happening in their schools or communities.

Turner shared sample tasks from the study groups: Some connect to cultural family practices such as birthdays or buying gifts for relatives, others to community settings such as a soup kitchen or swap meet, and others to values such as environmentalism, like up-cycling plastic bags. “The thing that stands out to me,” she says, “is the sense of energy and the sense of connection that teachers have found in using mathematical modeling tasks with young children.”



Sample modeling tasks

Support Beyond the Cohorts

After Turner presented, Nina Miller spoke about her role as mathematics interventionist in the Tucson Unified School District. She taught fifth grade for ten years before taking on a role as a K–8 mathematics curriculum specialist and coach for both teachers and students. She coaches teachers with their instruction and planning and also helps finding tasks for students.

She also coaches student mathematics teams, which are run on an interest basis. Students are invited to come practice with the team and check out what they do if they are interested in mathematics. The district tries to encourage students with a wide range of backgrounds to participate. Students compete in Mathleague competitions, which are closed tasks, not mathematical modeling. Miller has noticed that as their teachers have taken on more mathematical modeling tasks in the classroom, students have stayed more engaged in other mathematics problems and in competitions. They have gained the ability to attack paragraph-long problems with content that could be three to five grade levels above their current learning.

Miller is in each third, fourth, and fifth grade classroom at least once a week to teach or to support the teacher. She often brings mathematical modeling into classrooms that might otherwise not have experiences with it; this allows teachers to see what mathematical modeling would look like with their students. Her collaboration with Turner and the M2C3 cohorts at the University of Arizona has made a noticeable difference in both the quality and quantity of mathematical modeling in the district. In the past, teachers had a different conception of mathematical modeling because they had never experienced mathematical modeling in their teacher preparation — for example, they considered using manipulatives, such as base-ten blocks, as modeling. More recently, they have received instruction and support on seeing modeling as a process of open-ended, authentic problems.

Miller's district has taken this collaboration with the University of Arizona and extended it to professional development sessions in the summers. Interested teachers have participated in one-day workshops, and from there, enthusiasm has spread by word of mouth between teachers. There is a hunger for more real-world mathematics in elementary school classrooms: Teachers want to move beyond the narrow, scripted curriculum to something more authentic.

Benefits of Norms and Habits

In discussion following the panel, Turner and Miller commented on how teachers can help students learn to mathematize the world, for example by asking open-ended questions about what they notice and wonder about the world and how to quantify some of their questions. They emphasized the importance of routines; that is, of getting students in the habit of asking mathematical questions by establishing routines and classroom norms that encourage this type of thinking.

At the end of her presentation, Miller showed a video of a fourth grade classroom with a teacher who was working with mathematical modeling tasks in her class for the first time. The work changed the students' motivation in mathematics and also showed how these norms and habits can spill into other subjects. The teacher reflected:

> *I found that students at first were waiting for me to tell them what to do and how to do it, a 'just give me the mathematics problems and tell me how to solve it' kind of attitude. So at the beginning of the year, it was very frustrating for them and for me. Eventually, through our mathematizing routines, they began to think differently about mathematics and how it could help them solve real-world problems. I remember when a group of visiting principals were in my class observing how the limited-English students were interacting with their native-English-speaking peers. We happened to be in the middle of a mathematics routine where my primary goal was to have students focus on some assumptions they might have, and why they assumed these things. As the visitors walked in, I was showing a picture of a pumpkin patch. The students all busily wrote in their notebooks about what they noticed and what assumptions they could make about what they saw. My very limited English speakers were very engaged, too, in their sharing. I noticed that the students had gone beyond just noticings and wonderings, but were now naturally coming up with mathematical questions, such as how many pumpkins were in the picture, how many pumpkins could be grown in the field behind, and how much money the owners made. Throughout the course of the year, they began to use the language of math modeling and other areas. Without prompting, they were saying, 'I noticed that…' or 'I wonder if…' They even made assumptions about what they viewed. And not only that, but they held each other accountable for what they assumed by asking each other 'why' and 'how' questions. They also became so creative at solving problems, even when the mathematics involved was just out of their reach. All this from a group that was unable to ask any questions or start any problem at the beginning of the year.*

STEPS TOWARD ADOPTING MATHEMATICAL MODELING IN A HIGH SCHOOL CLASSROOM

Based on the panel presentation by Annalee Salcedo of Cate School in Carpinteria, CA

MATHEMATICAL MODELING CAN SEEM OVERWHELMING. Annalee Salcedo explained how, despite that feeling, she was able to gradually incorporate it into her teaching with small steps. She sought out mentoring in several places, including an annual conference at the North Carolina School of Science and Mathematics, a program sponsored by the MAA that took place at the University of Nebraska, the SIAM conference in applied mathematics, and the Mathematical Modeling Hub Faculty Mentoring Network. Taking these small steps every year allowed her to incorporate their lessons into her classroom effectively.

Salcedo has been able to share the lessons from her professional development with the rest of her department, adding more mathematical modeling tasks to the students' experience every year. After one recent task, one student wrote:

> *Hi Ms. Salcedo,*
> *I couldn't wait to show you because the pattern suddenly became so clear, and it's quite beautiful. I took what J had said at the very end of class into consideration and inputted the* 'x' *values into the scatter plot instead of the* 's' *values. Sure enough, linear regression showed a slope of 0.56, and when I went back to the* 'p' *and* 'x' *values, each* 's' *value was precisely 56% of the* 'p' *value. So the general pattern seems to be* $x = 0.56p$. *To find the sum of the areas, the* 'x' *value can simply be inputted into the equation A and K derived. Super interesting!*
> *See you Monday,*
> *J*

Actions and enthusiasm like this show that rich modeling tasks generate excitement with students.

Four Key Factors

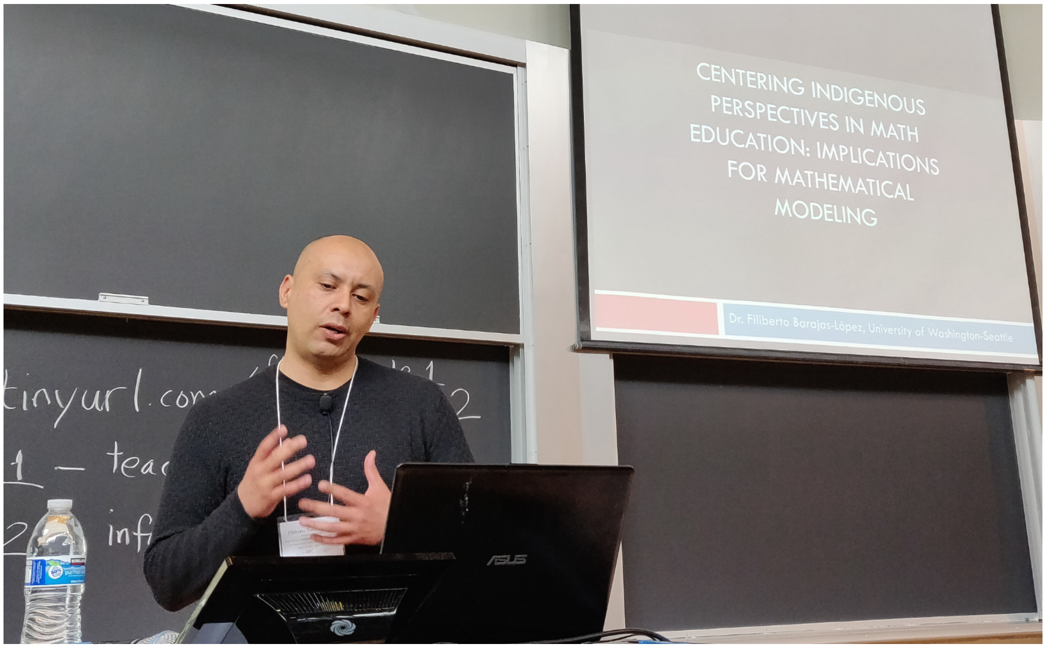
Salcedo notes four key factors that have helped her department adopt more mathematical modeling.

- The department embraced small changes because small changes each year add up to large changes over time.
- Exposing teachers to mathematical modeling tasks multiple times has helped them get more comfortable using such tasks in their classrooms.
- Mentoring can help teachers develop new skills quickly.
- Feedback from students about what they learned and how they felt working on modeling problems can help teachers adjust tasks as necessary.

Tensions: Finding Time and *Will this Be on the Test?*

Salcedo continues to wrestle with a few issues, including how to balance open and closed modeling tasks and how to balance mathematical content and the modeling process. In the end, many of the questions she has are special cases of a larger question: *What is the best use of time in a high school mathematics curriculum?* Teachers have a limited amount of time with their students, and sometimes when students are reflecting on tasks, they will say something such as, "It's not going to be on the test, is it?"

Students feel a tension between the value of the lessons in mathematical modeling classes and their desire to get good grades and do well on standardized tests. Teachers must wrestle with these tensions for themselves.



PART 3

EQUITY AND JUSTICE IN MATHEMATICAL MODELING EDUCATION

Creative ways of addressing equity and justice by focusing on relevance

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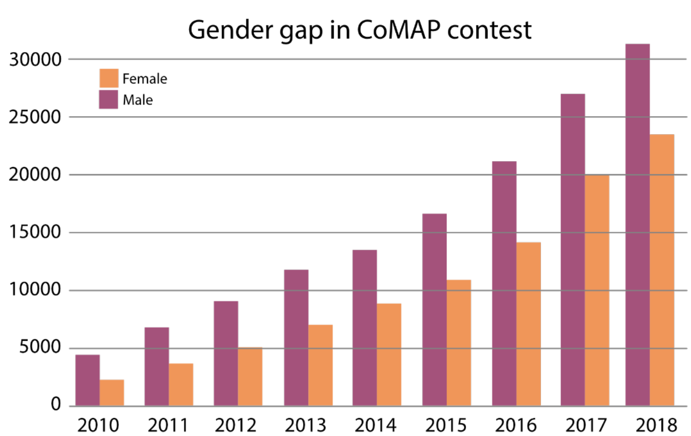
GENDER EQUITY IN MATHEMATICAL MODELING COMPETITIONS

Based on the panel presentation by Jo Boaler of Stanford University

THE GENDER RATES OF PARTICIPATION and outcomes in mathematics competitions are bleak, according to Jo Boaler. Every year the U.S. sends a team of six high school students to the International Mathematical Olympiad. In the last 11 years, they have not sent a single girl. So we know something has to change.

By contrast, 43 percent of both participants and winners of the 2018 COMAP Mathematical Contest in Modeling were women. (COMAP is the Consortium for Mathematics and Its Applications; the contest website is www.comap.com/undergraduate/contests/.) So Boaler was very intrigued when Sol Garfunkel, who runs the COMAP competition, asked her if she was interested in studying why the gender participation rates and outcomes were marked and consistently better than other mathematics competitions. She and her team spent a year on the study (reported on page 18 of the February/March 2019 issue of the Mathematical Association of America's *MAA Focus* magazine).



COMAP gender gap

COMAP Offers Collaboration and Ability to Contribute Individual Strengths

COMAP is a competition both in universities and in high schools. It is collaborative; each team of three people chooses its own question from a list of possibilities, and they work on that question for four days, with access to any non-human resources they would like. Each question is a scenario that involves an application of mathematics in real-world context, with variables and constraints to consider in order to address the problem and provide a recommendation based on the group's analysis. Each team submits a mathematical paper describing their methods as well as a short paper for a wider audience. These written products call for deep mathematical and statistical analyses of models, understanding of ideas from other disciplines, and, in some cases, a policy recommendation for decision-makers. In 2018 there were 20,000 participants from 23 countries.

When discussing the gender statistics in other mathematics competitions, people often say things like, "Well, girls don't want to enter mathematics competitions," or, "Maybe they can't be successful in these very high-level mathematics competitions." COMAP provides a clear counterexample that shows that is not true.

The research study drew from a rich array of primary sources: Boaler's research group gave surveys to 59 STEM faculty members across 20 universities and interviewed eight of them. They studied the teams working on the competition in two different locations over 42 hours, and they conducted an online survey of over 1300 students from 10 countries. They first asked the students why they entered the competition. This is a word cloud of the results:

Why did you enter the competition?

Participants often said things like, "I can write about math," "I'm a good communicator," "I'm a leader," or "I can use Python." They wanted to contribute their different strengths to the team. It engaged them as whole people.

Key Motivating Factors for Participation

The research team found three different factors that were especially important in accounting for the very high participation rate of women in this competition.

Collaboration. First, collaboration was an important factor for the people in the competition, particularly the women. One of them said, "It involves a lot of collaborative work, allowing multiple people to have input in the solution to a problem."

Multi-faceted mathematics. The second key factor for students was the opportunity to model and engage in more multi-dimensional mathematics than they had previously experienced. One student commented, "The problem is more creative and closer to life. The competition motivates us to think more and deeply about the world we live in, which can't be achieved in other competitions." Another said, "We were using knowledge from a bunch of different classes, and math classes overlap a little bit but you don't really see the actual overlapping until you have to apply something like that."

Freedom (to take risks … and from grades!). The third factor was that students found that, even though they were competing, they felt free in ways they had not felt free in mathematics before. For example, one student said, "I've been a student my whole life, and so it's kind of nice to test our skills, but not being in such a harsh grading criteria. That's definitely different. It makes you think about the effort more and you're more willing to take risks, I feel." Another student said, "My favorite thing was definitely the feeling of using all of our resources that we've learned in school for something that's not a test and not a graded project."

Motivation Beyond the Competition

Over two-thirds of the students reported in surveys that engaging in this competition would change their future career pathways. One student said, "I got an unforgettable memory during those four days. As a student who majored in math, it's the first time I've applied the knowledge to solving problems that are so close to real life, which makes me very excited!"

Another said, "That was awesome. … I'm on the right track now and I know that. It's no longer a questioning: What am I gonna do after college? I have an idea, I know what I wanna do, and I feel good about it."

One mathematics faculty member commented, "MCM/ICM [Mathematical Contest in Modeling and Interdisciplinary Contest in Modeling] is a different kind of experience compared to mathematics competitions like the Putnam. In my opinion, it is a more accurate reflection of what professional and academic mathematicians do (reading, writing, working on a team, exchanging mathematical ideas, attacking problems that are not initially well-defined, spending time on a problem instead of having a shorter time period, etc.). Among other reasons, I recommend this competition for students who want to get a taste of what mathematics 'research' is like, and I recommend it to students who want to go to industry jobs directly after graduation."

"Performance Mathematics" vs Mathematical Freedom

Boaler argues that most mathematics competitions exclude women and girls not because of the problems in the competitions, but because the competitions are part of a culture that emphasizes speed and performance pressure, which women often find repellent. She contrasts "performance mathematics" — short, closed questions, which they have to answer quickly, often using a small set of methods they have memorized — with "mathematical freedom," which uses open-ended questions to get students to think deeply about the ideas, communicate them with others, and find new connections. Many mathematics competitions are "performance mathematics," while COMAP offers the opportunity for "mathematical freedom." She also argues that although mathematical modeling is applied mathematics, this same approach could be taken with pure mathematics competitions. Such competitions are one way of giving students a taste of mathematical freedom.

FRAMEWORKS FOR SOCIAL JUSTICE IN MATHEMATICAL MODELING EDUCATION

Based on the panel presentation by Ksenija Simic-Muller of Pacific Lutheran University

IN 2017, FRANCIS SU GAVE A TALK, now fleshed out into a book called *Mathematics for Human Flourishing*, that identified five reasons to do mathematics: play, beauty, truth, justice, and love. Ksenija Simic-Muller believes Su's talk has helped open up conversations in mathematics departments about what mathematics is, beyond just a search for truth and beauty. In the section devoted to justice in his 2017 talk, Su focused on access to mathematical education — Simic-Muller agrees that access is important, but explained that there are other ways to think about justice in mathematics.

Justice in Mathematics

Simic-Muller shared a framework, based on work of Jacqueline Leonard, Joy Barnes-Johnson, and Robert Q. Berry III, that identifies four different approaches to social justice and mathematics:

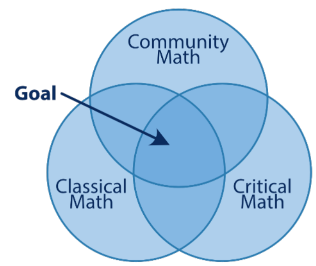
- Access to high-quality mathematics instruction for all students
- Curriculum focused on the experiences of marginalized students
- Use of mathematics as a critical tool to understand social life, one's position in society, and issues of power, agency, and oppression
- Use of mathematics to transform society into a more just system

Most of Simic-Muller's work focuses on the third item, the use of mathematics as a critical tool to understand social life, one's position in society, and issues of power, agency and oppression. The mathematics community has been increasingly interested in this issue. She quoted from the book *Mathematics for Social Justice: Resources for the College Classroom*, edited by Gizen Karaali and Lily Khadjavi: "Scholars from all echelons of the academy have been working to ensure that students from all backgrounds receive the best mathematics education they can. One way to achieve this goal is to offer all students a range of opportunities for authentic engagement with mathematics addressing issues that are relevant and timely."

Equity in Math Education

What is a good definition of equity in mathematics education? Simic-Muller likes Rochelle Gutiérrez's description: you are achieving equity if you are unable to predict student patterns (for example, achievement, participation, ability to critically analyze data/society) based solely upon characteristics such as race, class, ethnicity, gender, beliefs, and proficiency in the dominant language. "We are nowhere near that," she says.

One way to get closer is to explicitly address the problems in the classroom. She shared the "three C's" of mathematics classrooms, an idea of Eric (Rico) Gutstein: classical mathematics, community mathematics, and critical mathematics. These areas form an overlapping Venn diagram, but much of mathematics education focuses only on classical mathematics—often the mathematics students need to pass tests. Community mathematics is mathematics based in students' communities; critical mathematics, which Simic-Muller focuses on, asks how mathematics can address pressing societal issues.



Addressing the Overlap

Teaching mathematical modeling can address the overlap of the "three C's" in the diagram, and Simic-Muller shares her understanding of mathematical modeling in the work that she does with people who have not seen themselves as "successful" in mathematics in a traditional way. For her, mathematical modeling requires

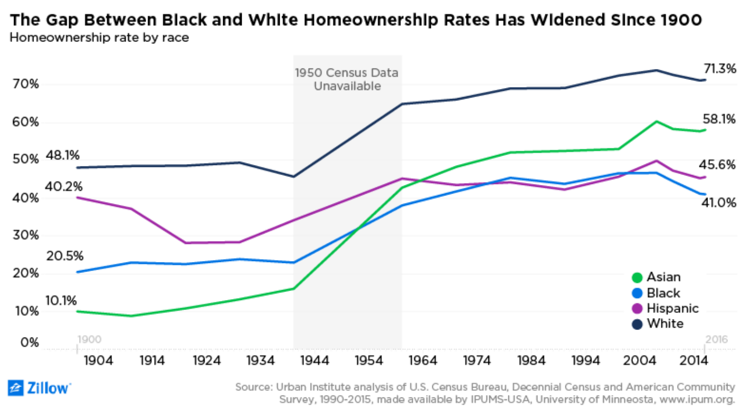
- Authentic data (for example, she spends a lot of time on the U.S. census website and seeking other sources of real-world data)
- Compelling problems
- Assumptions that must be made
- Solutions that must be interpreted
- Multiple entry points, solution methods, and answers
- A relationship to problem posing; students get better at mathematical modeling when they get better at asking the right questions and posing problems well

The connection between mathematical modeling and social justice can be very fruitful. Simic-Muller finds that the Common Core standards have been helpful because there are both modeling and content standards, and social justice problems fit naturally into mathematical modeling. She also thinks modeling can be an equitable practice because, as some research shows, students who might not traditionally have been successful in mathematics classes can still do well in modeling because the focus is not on procedures as much it is in the process.

Motivation and Empowerment

What are the impacts on student learning, experiences, identity, and engagement when teaching from an equity/social justice stance? Simic-Muller shared the impact of a quantitative literacy course she taught in January 2019. The class was diverse, and the students in it were on the whole not drawn to math. The course focused on race and housing over time, from the foreclosure crisis all the way back to redlining.

They looked at the home ownership gap between Black and white families and tried to decide whether it has widened since 1900 — the answer depends on how you define the gap.



Is it accurate to say the gap has widened?

Skylar Olsen on Zillow.com, "Black and White Homeownership Rate Gap Has Widened Since 1900," Apr. 10, 2018.

The gap is stark, and trying to understand the gap generated rich conversations. Since the course centered around topics that one could say were depressing, she was concerned about whether it was too emotionally difficult for students. When she asked the class how all this information made them feel, one of the Black students said, "This makes me feel empowered because I now can tell people I am not crazy. I have data to back up my experiences." Other students related to that. She says it is important to bring in student voices and validate their experiences.

When introducing a topic, one practice that Simic-Muller uses is "notice and wonder." For example, in a quantitative literacy course, she shared two pictures from Tacoma that relate to the city's gentrification. She uses photos like this in order to spark conversations about both student experiences and mathematical questions that they can ask about the photos. It is a relatively easy practice to implement, and it gives everyone a voice and allows students to generate lessons.

INDIGENOUS MAKING AND MATHEMATICAL MODELING

Based on the panel presentation by Filiberto Barajas-López of the University of Washington, Seattle

FILIBERTO BARAJAS-LÓPEZ' PERSPECTIVE CENTERS on acknowledging the historical erasure of Indigenous peoples across the world, and he believes that if we care about equity in mathematics education and society more broadly, then the conversations we have about equity must be situated in the proper historical, political, and social context, which includes acknowledgment of the continued presence of settler colonialism in modern society. He spoke about his work with Megan Bang at Northwestern University on making and tinkering.

A Problem of Today (Not the Past)

Settler colonialism is not a single event that happens and then is over. Rather, it is a continual process that educators, among others in society, still engage in even after the original "settlement" has passed. Barajas-López believes that educators can either contribute to the erasure of Indigenous people or contribute to their sovereignty.

Why start with Indigenous erasure and settler colonialism? Barajas-López believes that this starting point requires educators to confront the limitations of their existing frameworks, particularly around learning and equity, and to consider more appropriate approaches that center Indigenous worldviews and cosmologies and decenter the existing settler paradigms that currently take up most of that space.

Barajas-López moves beyond critique of existing paradigms and tries to understand what viable solutions exist that are grounded in Indigenous ways of knowing and doing. In the context of the CIME workshop, he understands that these questions can cause tension, but there are ways that the different perspectives present can inform and transform each other.

Mathematics is not "Universal and Culture-Free"

Barajas-López notes that mathematics education is Eurocentric; one aspect of that Eurocentrism is that the mathematical practices and contributions of Indigenous people have largely been erased or described as deficient or unsophisticated. The problem goes deeper: many people think of Indigenous people as living in the past, but in truth they are a part of society today, and they are engaged in the practice of education in today's classrooms.

Many people see mathematics history as European even though what we think of as mathematics is the accumulation and development of knowledge from many cultures from several different continents. The field of ethnomathematics, defined in 1985 by Ubiratan D'Ambrosio as "the mathematics which is practiced among identifiable cultural groups such as national-tribe societies, labor groups, children of certain age brackets and professional classes," makes explicit the relationship between the development of mathematics and the culture that develops, countering the false narrative that mathematics is universal and culture-free. Students of mathematics should know that different cultures develop different ways of understanding and doing mathematics.

Indigenous STEAM Collaborative Camp

As one example of addressing this challenge, Barajas-López has been involved with a science, technology, engineering, arts and mathematics camp run by the Indigenous STEAM Collaborative. The camp began in the Pacific Northwest and welcomes over 80 Indigenous youth from across North America every summer. Camp activities center the land and water, using open outdoor spaces as a classroom, and show how millennia-old Indigenous technologies such as weaving and clay work are connected to STEAM disciplines. The directors of the camp incorporate insights from community elders, families, and youth storytellers artists, rather than just academics.

Storying changing lands and waters

Relational reciprocity — an Indigenous worldview that does not center humans as the most important beings — is important to the work of the camp. Humans are in relationship to "more-than-humans," the animals and landscapes of the natural world. The directors of the camp try to help young people build relationships with both humans and these "more-than-humans."

Indigenous clay work is one activity at the camp that allows students to engage with many disciplines, including science, mathematics, and arts. They mix organic materials, they learn to understand the different states of clay from dry to wet, and they learn to work with non-standard measures as they are making pieces. That part in particular involves quite a bit of mathematical modeling. These are all embodied forms of mathematics. When students begin doing this work, they try to tell stories using just the flat clay. But as they use the clay more, their work shifts and they start to think more three-dimensionally. They engage in an iterative process of constructing, which is similar to what engineers do when they use modeling in their work.

Indigenous clay work

Perspectives Change Priorities

Barajas-López says one important thing he and Bang have learned is that making in Indigenous cultures has different priorities than making in western cultures. The western perspective sees making and tinkering as entrepreneurial and product-based, while Indigenous perspectives emphasize the relationships between people and land. Barajas-Lopez sees his work as an act of cultural resurgence rather than participation in the U.S. economy.

He shared three principles to consider as mathematical modelers and educators:

- A critical analysis of educational injustice should be a priority of mathematical modeling.
- Historicized approaches to mathematical modeling as a cross-cultural activity are needed to decenter western views.
- Explicit attention to pedagogical philosophies and practices of mathematical modeling.

In discussion following the panel, Barajas-López noted that when mathematics education breaks out of typical classroom dynamics, it can take students by surprise. Sometimes students at his camps, for example, expect to have typical school experiences and feel uncomfortable with a curriculum that centers different perspectives and has different goals. Teachers must grapple, then, with how to get student buy-in to new models and, if necessary, how to present the value of different types of pedagogies and learning environments.

MATHEMATICS FOR SPATIAL JUSTICE

Based on the panel presentation by Laurie Rubel of the University of Haifa

WHEN WE REFER TO SPATIAL REASONING in mathematics education, we usually refer to space void of humans. A growing body of work, however, considers space as it is socially constructed: that is, how space is shaped by social forces, and how space in turn shapes our sociality. As humans, we make space and space makes us. Laurie Rubel shared a quote from late UCLA geographer Edward Soja: "Everything that is social is simultaneously and inherently spatial, just as everything spatial, with regard to the human world, is simultaneously and inherently socialized." She pointed out that there is an "if and only if" relationship in that quote.

Discovering Social Relationships Through Maps

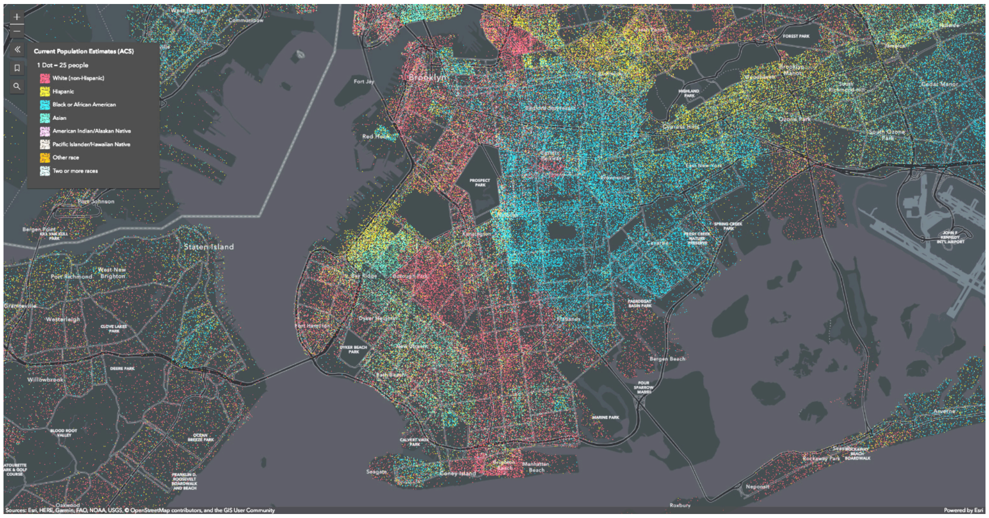
Social relationships inevitably take spatial form. As an example, Rubel looked at race. Race takes spatial form in many ways, and she highlighted one way that is visible in a map of Brooklyn (top of next page). The map shows the population density of white residents in red and Black residents in blue — and shows a stark spatialization of race in Brooklyn. Other places in the U.S. would yield similar maps.

One story that maps tell is about the systematically uneven distribution of resources across the space. Legal scholar David Delaney said, "Race in all of its complexity and ambiguity, as ideology and identity — is what it is and does what it does precisely because of how it is given spatial expression."

Discovering Social Relationships Using Photographs

What about gender? What kinds of places are gendered? Rubel looked up images to go with the term "locker room." Metaphorically and literally, locker rooms are places that are typically marked as being for men only, and also as spaces where misogynist discourse is acceptable. Beyond locker rooms, women are limited, sometimes even restricted in mobility, as to where they can go and at what time they can be there, what they can wear, and so on, because of harassment or violence.

Spatialization of race in Brooklyn



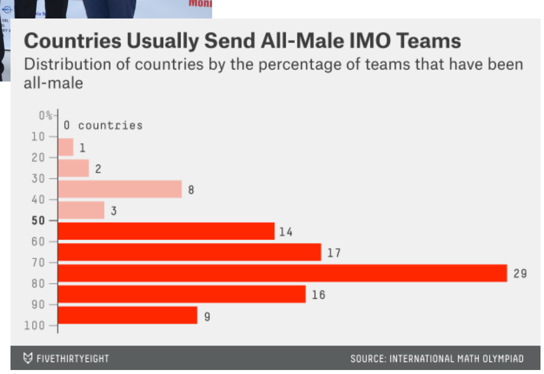
Sources: Esri, HERE, Garmin, FAO, NOAA,USGS, ©OpenStreetMap contributors, and the GIS user community [1]

The social system of the patriarchy in the United States maintains exclusion for women in all kinds of places, which means that the social relations that occur in those places are restricted too. This includes corporate boardrooms, Wall Street, athletic fields, military battalions, the White House, and mathematics. Rubel showed a picture of the 2018 U.S. Mathematics Olympiad team. (The young woman in the photo below is not on the team.)

She then showed a graph of the distribution of countries who participate in the International Mathematical Olympiad arranged according to the percentage of their teams over time that have had zero women. The U.S. falls in the second to last bar, where close to 100 percent of teams have had zero women.

Who is not on the Math Olympiad team?

Mathematical Association of America [2]



FiveThirtyEight [3]

Mathematical Modeling Education in Support of Spatial Justice

Mathematics educators can change this. As Frederick Douglass said, "Power concedes nothing without a demand." Mathematics educators can make this demand. She shared another quote from Soja: "Spatial [justice] is not a substitute or an alternative to social, economic, or other forms of justice but rather a way of looking at justice from a critical spatial perspective. From this viewpoint, there is always a relevant spatial dimension to justice, while at the same time all geographies have expressions of justice and injustice built into them."

Justice has a geography, and this occurs at all levels of spatial scale, from bodies to neighborhoods, cities, countries and beyond. This framing is full of hope and possibility because places are always in flux, meaning that changes are possible, or even already happening. Rubel shared four examples of how mathematical modeling might support spatial justice.

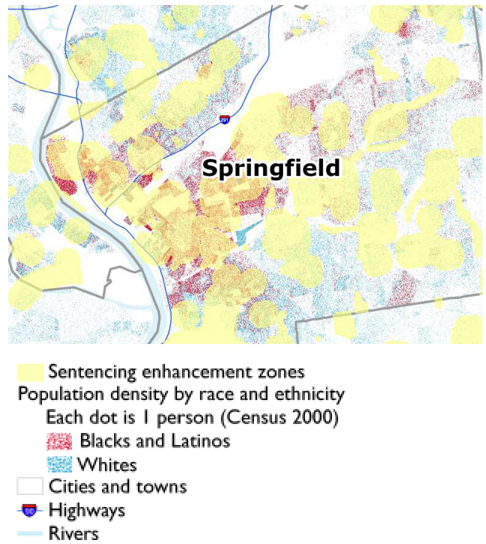
Mobility. The first example was mobility justice, the freedom to move. People have different access to mobility, including with respect to gender, but more generally, Doreen Massey said, "some people are more in charge of it than others, some initiate flows and movement, others don't; some are more on the receiving-end of it than others; some are effectively imprisoned by it." As an example, consider the planning of American highways. The places that have been chosen to locate highways have historically been guided by models that have specific values embedded in them. The kinds of things that have not been valued have been who is going to be impacted environmentally by the location of the highway, who is going to be displaced when the highway gets put in, who is going to lose access to mobility by the neighborhood getting divided, and so on.

Lehman College Library (CUNY). All Rights Reserved.

"Highways… help move people very easily across these elaborately segregated landscapes." (Connelly)

Rubel shared a photograph from New York City in the 1950s, showing the Cross Bronx Expressway being built. The expressway went through a neighborhood in the South Bronx that was (and still is) a low-income African American and Puerto Rican neighborhood. The highway was built there because it enables the commuters from the suburbs to get to the center city faster in cars. Highways are in one way the center of the civil rights movement now because of their history. Highways impacted mostly African American communities across many American cities, and they continue to reinforce segregation and limit mobility. They are the places where a lot of police stops and then shootings happen, and they are also places that are very effective sites of social protest. Mathematical modeling might be a way to reimagine how highway systems could be modeled, foregrounding the considerations that up until now have mostly been ignored.

Punishment. The second example comes from the geography of punishment. Most states changed their penal codes as part of the war on drugs, giving lengthier penalties for drug-related offenses that happen within 1000 feet of a school. Under an assumption that people and schools are equally spread across space, that would seem to be a neutral law. But in, for example, Hampden County, Massachusetts, the law ended up producing longer sentences for Black and Latinx people, shown in the map by red dots, because they tend to live in areas with greater population and school density.



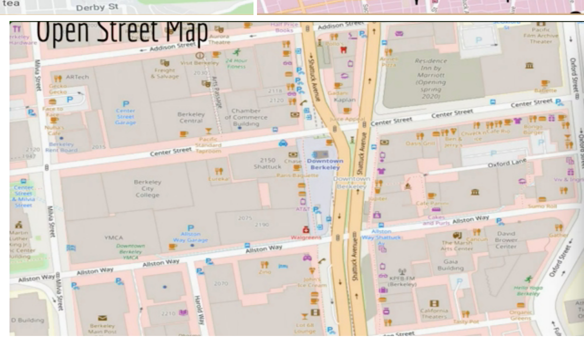
Prison Policy Initiative [4]

The map is a mathematically-guided visualization of sentencing enhancement zones relative to population density and race. The visualization makes the invisible injustice visible. Digital mapping can be a great way to generate representations of spatial dimensions of a data set.

Who Determines the Map? For her third example, Rubel illustrated how the sources of information used in a map matter by comparing several maps of Berkeley, California. One is Google's map, in this case highlighting coffee shops. Another was from *Queering the Map,* a community generated counter-mapping platform for digitally archiving LGBTQ2IA+ experience in relation to physical space. Many of the entries were from Berkeley High

Google Maps [5]

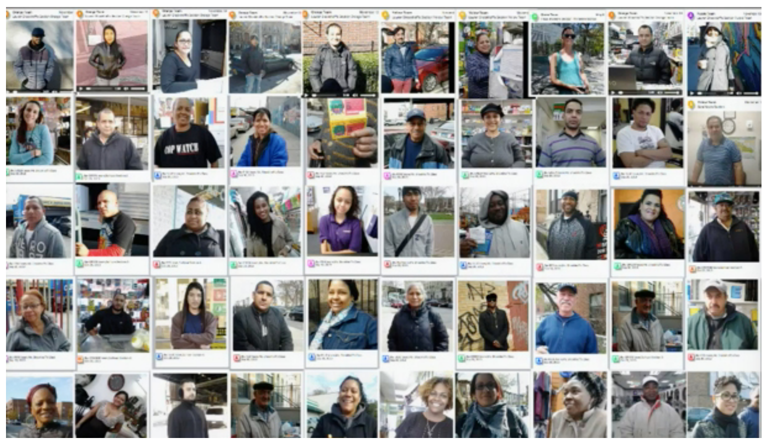
Queering the Map [7]



Open Street Map [6]

Schoolers and showed a queer history of the area as remembered by them. Yet another was an OpenStreetMap, which users contribute to. Most of the participants in OpenStreetMaps are men, so they tend to include things like sports arenas, strip clubs, and bars. If you are interested in fertility clinics, abortion clinics, domestic violence shelters, and so on, it is harder to find those on the maps—not because those locations do not exist, but because they are not in the dataset. Rubel encourages people to create representations of place that include voices and stories from all people in those places.

Location of institutions. Rubel and collaborators did a project with New York City High school students and their teachers about pawn shops and banks. Part of what they did was mapping their neighborhoods to see where these different institutions are located and to get stories from the people they encountered.



Through the mathematical modeling exercise, they aimed to illustrate the predatory nature of the pawn shops and other kinds of alternative financial institutions. Interviews brought out other stories, sometimes complicating the narrative. For instance, one person talked to the students about how seeing more banks in his neighborhood actually meant that the neighborhood was gentrifying, and it was not welcome. Another thing that the students encountered was that the banks were not always welcoming to the students, whereas people at the pawn shops were happier to talk with the kids and often had employees who could speak Spanish with them.

Further resources. Rubel shared a site called Missing Maps (missingmaps.org), which hosts map-a-thons that teachers could find useful in mathematical modeling classes. See also [8].

Image Credits and Resources

[1] Esri. "Current Population Estimates (ACS)". Scale Not Given. "Current Population Estimates (ACS)". <ekenes.github.io/esri-js-samples/4/visualization/dot-density/population-race.html>. July 21, 2021.

[2] Mathematical Association of America. From press release, "Team USA Returns to First Place in Olympics of High School Math". July 12, 2018.

[3] FiveThirtyEight. "Girls Are Rare At The International Math Olympiad". July 2, 2015.

[4] Prison Policy Initiative. Aleks Kajstura, Peter Wagner, and William Goldberg. "The Geography of Punishment: How Huge Sentencing Enhancement Zones Harm Communities, Fail to Protect Children". July 2008. <www.prisonpolicy.org/zones/vermont.html>. Image from <www.prisonpolicy.org/zones/fullpagemap.htm>

[5] Google Maps, 2019. Berkeley area around University of California campus. <maps.google.com>.

[6] OpenStreetMap, 2019. Berkeley area just west of University of California campus. <openstreetmap.org>.

[7] Queering the Map, 2019. Berkeley area around University of California campus. <queeringthemap.com>.

[8] Laurie H. Rubel & Cynthia Nicol (2020) The power of place: spatializing critical mathematics education, Mathematical Thinking and Learning, 22:3, 173-194, DOI: 10.1080/10986065.2020.1709938.

PART 4

BEYOND K–12: NON-STEM AND PROFESSIONAL PATHWAYS

College-level courses for both STEM and non-STEM students, graduate-level training, and requirements for a mathematical career in modeling

INTRODUCING MATHEMATICAL MODELING TO NON-STEM MAJORS

Based on the panel presentation by Ricardo Cortez of Tulane University

INDUSTRY EMPLOYERS AND GRADUATE SCHOOL admission committees, Ricardo Cortez has observed, have different but related desires. Industry employers are generally looking for candidates who can code algorithms efficiently, use statistical methods to analyze data, think independently, work collaboratively, and communicate their conclusions to both expert and non-expert colleagues and clients. Graduate schools, on the other hand, tend to accept students who understand theoretical aspects of modeling (particularly proofs), have familiarity with "classical" models (such as the heat equation and the predator-prey model), think independently, and communicate mathematics to other mathematicians.

Mathematical modeling courses for undergraduate and graduate students need to balance these goals. Prerequisites for these courses at the university level mean that they are usually reserved for undergraduate junior and senior STEM majors. At Tulane University, Cortez wanted to increase access to mathematical modeling among all undergraduates, in accordance with recommendations like the Mathematical Association of America's Common Vision (2015), which recommends mathematical modeling courses for students in their first two years.

A Mathematical Modeling Course Open to All Students

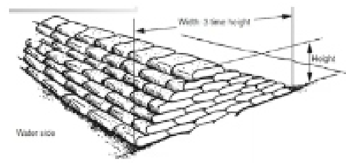
Cortez created a course called "Mathematical Modeling to Understand the World," which is open to all undergraduates of any major and year. The only prerequisite is Calculus I — though Cortez says that in reality, he simply asks, "Have you heard the word calculus?" If they have, they are in. The course is problem-driven and student-centered, with collaborative group work, written reports and presentations, and a focus on the modeling process instead of on specific models. In class, there is almost no lecturing by the instructor. The time is spent listening to students' ideas and encouraging them to pursue them and reflect on the outcome.

One example of a modeling situation the class might tackle addresses fighting floods by stacking sandbags, which act as a barrier to prevent streams from overtopping levees. Floods and sandbags are a part of life in New Orleans, so while Tulane students would not typically have stacked sandbags themselves, they would certainly be familiar with them. To prepare for teaching with the sandbags modeling problem, Cortez read two manuals about how to stack sandbags, one from the Army Corps of Engineers, another from the Missouri Department of Natural Resources. While these two groups seemed to agree on how to build the wall, they disagreed on the number of bags needed; the class activity challenges the students to engage with the discrepancy and to develop their own method for estimating the number of bags needed.

A problem-driven activity

Problem-driven activities: Example

Fighting floods: how to stack sandbags



Height of sandbag wall	Army Corps of Engineers estimate	Missouri Dept. of Natural Resources estimate	Your estimate
1 foot	600 bags	500 bags	
2 feet	2,100 bags	1,000 bags	
3 feet	4,500 bags	2,100 bags	
4 feet	7,800 bags	3,600 bags	

The table shows the estimates for a 100 foot long wall of various heights. Why do you think the estimates are so different? Develop your own procedure to estimate the number of sandbags and compare.

Benefits to Students

The class has many benefits. It allows a larger number of students access to mathematical modeling than traditional classes with more advanced prerequisites. These students develop experience choosing appropriate mathematics to describe, explain, understand, or predict daily life situations, bridging the gap between everyday situations and classroom math. Students learn to use the mathematics they already know in addition to learning new topics in math, and they gain experience identifying instances in their daily lives when mathematics is useful. The everyday contexts of problems are often relevant to their other interests, and they can address issues of social justice, science, politics, community, health. The material can be tailored to the interests of the students.

Challenges for Faculty

The course also comes with some challenges for faculty. Many modeling scenarios in textbooks require more advanced mathematics than these students know, so instructors have to make sure students have the tools they need to work effectively on those problems.

Additionally, the varied backgrounds of the students can cause disparities in learning outcomes. Faculty who are used to teaching more traditional modeling courses may need to rethink their pedagogy. Traditional courses tend to teach models rather than modeling, de-emphasizing student creativity. In this class, the models are driven by students' ideas, and students must learn to provide structure to the problems. For this reason, the pace and topics are not always predictable. The syllabus falls into place at the end of the semester.

DESIGNING AN UNDERGRADUATE MATHEMATICAL MODELING CAPSTONE COURSE

Based on the panel presentation by Joceline Lega, University of Arizona

THE UNIVERSITY OF ARIZONA'S upper-division mathematical modeling course since 2019, "Introduction to Mathematical Modeling" is required for students who are pursuing applied mathematics and life sciences emphases in their mathematics major. The course is usually taken in the spring semester of the senior year and serves as a capstone and writing emphasis course. The course also covers competencies in discovery and professionalism to address a university structure called "100% engagement," so the course has several goals to satisfy at once, as shown by the course objectives:

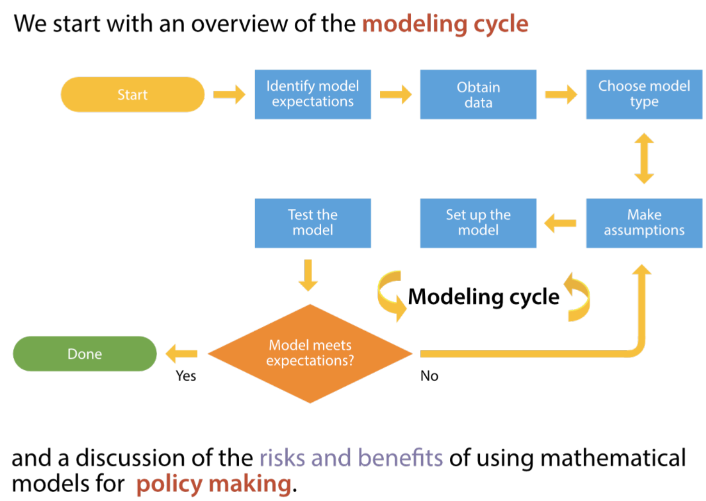
- Students understand what models are and how they are being used
- Students critically evaluate and extend selected mathematical models in the current scientific literature
- Students apply computational methods and concepts from prerequisite mathematical content to analyze scientific problems
- Students work in small groups to read, understand, and reproduce articles in the current scientific literature
- Students effectively communicate their work to non-specialized audiences in written and verbal form

The course has several mathematical prerequisites: linear algebra, differential equations, a programming course, and completion of at least one senior-level applied mathematics course. In principle, students come in with a good background to build and analyze models.

Course Structure

The course objectives are addressed through the lectures and in-class activities as well as semester-long projects.

Foundations of MATH 485



In class. The in-class component of the course includes lectures, activities, videos, and numerical simulations. The goal is to provide a global context to the work students will be doing in their projects. They discuss typical methods and approaches that people use when they develop models. Another goal is to show students that the mathematics they learn in the prerequisites is not just there because they want to get a degree but because it can be applied to solve real-life problems.

The course begins with an overview of the modeling cycle (that you make assumptions and test them and repeat if necessary) and addresses the risks and benefits of using mathematical modeling for policy decisions.

After an introduction to the idea of mathematical modeling, the course covers various types of models. The simplest are agent-based models, starting with people doing “the wave” in a stadium. Then they move on to models from classical mechanics, including the nonlinear pendulum; population dynamics and epidemiology; and spatial effects, including diffusion and pattern formation.

Outside class — Homework. As part of homework assignments, students read articles from science journals that show the real-world implications of mathematical models and discuss decisions made using the models. Two such examples are a 2001 article by M. Enserink called “Rapid Response Could Have Curbed Foot-and-Mouth Epidemic” and a 2017 article by Erik Stokstad called “U.K. expands kill zone for badgers in fight against bovine TB.” Students write reflections on these articles as a partial fulfillment of the writing emphasis of the class.

Outside class — Projects. The projects are done by small groups. Over the course of the semester, each group reproduces and extends the results of a research article with supervision by a postdoc or graduate student mentor in applied math. They present their work in written and oral form and at a judged poster session.

The most recent time she taught the course, Lega's students developed projects on disease transmission in social networks, fire propagation, the immune system, Ebola, melt ponds in the Arctic, and "super-models," which combine several models to describe complex systems. Students are receptive to the modeling process and the idea of combining analysis and computations to do projects with real-world relevance.

Continuing Improvements

Lega has run into some problems with the course that she hopes to improve. These include students with lower-than-expected mathematical proficiency who often are not aware of what they do not know. Few students use the course to reinforce their knowledge of previous courses, and it can be difficult to get students to go beyond rote, procedural learning. Students are not used to more open-ended, creative tasks in mathematics, so some of them are unsure where to start. That problem is what led Lega to include graduate student and postdoctoral mentors in the course when she redesigned it. It is difficult to design assignments in class, other than the semester-long project, that build those skills.

Course Effectiveness

Benefits of projects. Lega says the projects are effective in several ways. Students enjoy the real-world applications discussed in their projects and feel empowered by their ability to read and reproduce results from the research literature. Many of them feel ownership of their contributions to their project, and they are able to give professional presentations about their work by the end of the semester.

Students who take the course in their junior years and want to keep working on mathematical modeling can continue working on their project with their mentor or take a guided research course with another faculty member.

Challenges with projects. Some students have difficulty working in a group on the project, and students sometimes take too many courses and do not put enough time into their projects. Mentors can help by noticing when these problems start to happen. Lega also requires progress reports that include a summary of everyone's contributions.

Benefits for both undergrads and grad student mentors. Many students report that the course has helped change the way they see applied mathematics. They understand better its utility in the real world. The graduate students and postdoctoral mentors appreciate the opportunity to mentor undergraduate research, often for the first time.

WORKSHOPS AND CAMPS IN MATHEMATICAL MODELING FOR GRADUATE STUDENTS

Based on the panel presentation by Linda Cummings of New Jersey Institute of Technology

LINDA CUMMINGS EXTENDED THE CONVERSATION from K–16 to K–21 education by describing two workshops that train graduate students in using modeling to solve real-world problems directly relevant to industrial partners: the annual Graduate Student Mathematical Modeling Camps (GSMMC) and the Mathematical Problems in Industry (MPI). Cummings co-organizes these workshops with colleagues at a consortium of universities located primarily in the Northeast.

The MPI workshops, which inspire much of her current research, stem from similar industrial workshops first organized in the UK in the 1960s, a concept that has since spread

In the capstone lab

to countries around the world. MPI are five-day workshops, currently sponsored by the National Science Foundation, the Institute for Mathematics and its Applications, and the participating companies. Academic participants at MPI span the whole range from senior faculty members, who guide modeling efforts, through to graduate students, who do much of the actual model solving.

MPI agenda. On Day 1 of MPI, industry participants present real problems facing their industry. Following a Q&A session, each problem is assigned a breakout room. Participants choose which problems they want to work on, and brainstorming begins. On Day 3, there is an interim feedback session with all groups to share developments and exchange ideas. On Day 5, graduate students make final presentations to make recommendations and summarize progress, which is usually very impressive.

Examples of Projects

Cummings shared examples of past problems that such workshops have addressed.

Pregnancy test design. One working group tested the feasibility of a new design for a pregnancy test. Pregnancy tests often work by detecting a hormone called hCG in urine. The proposed test would have used a small microfluidic device with two channels and a reagent that reacts with hCG in one of the channels, increasing the viscosity of the fluid in that channel if hCG is present. If hCG is present in a urine sample, the fluid in the channel without the reagent will flow more quickly than the fluid in the channel with the reagent. If hCG is not present, the fluid in the two channels will flow at the same speed. The group carried out mathematical modeling and simple experiments with microfluidics, ultimately concluding that the new device would not work because the reaction occurred too slowly to affect the time of the flow in the devices.

In further examples:

- Vodafone brought a problem seeking to use cell phone signals from moving vehicles to identify speeding drivers by monitoring when a car's phone signal switched from one cell tower to the next.
- The Bristol Zoo needed to incubate penguin eggs successfully. In the wild, penguins hold their eggs on their feet and rotate them regularly, but the function of the rotation, and how best to replicate it in an incubator, was unknown. The workshop group studied internal flow within an egg to determine how the rotation affects convection and transport processes within the eggs.
- Jaguar brought an engineering problem: how to minimize the jerk in automatic car braking, which the workshop group was able to formulate as a calculus of variations problem.
- Novartis asked the workshop about ways to model the growth of tumor populations in cancer patients; in other words, when cancer is metastatic and tumors form in various parts of the body, how do they grow? The group formulated a model for how these tumor populations evolve and how drugs can inhibit their growth. They also explored issues of tumor dormancy.

Benefits to Partners, Students, and Faculty

Cummings and her colleagues have cultivated a network of industry contacts to find these and other industrial problems. Participating companies pay a modest fee to present a problem at a workshop, in return for which they get a final presentation, an executive report, lots of new ideas and, in many cases, smart new recruits out of graduate school. At this point, several current industry attendees are alumni of the program as students.

Academics and students benefit from the program in many ways. They learn about new problems, which can lead to new publications, grants, and even case studies to look at in modeling classes. Students get training in modeling and working on a team, as well as access to networking opportunities and exposure to non-academic career paths. Some of them get recruited by companies that participate.

Preparing Students for MPI — the GSMMC Workshop

The MPI organizers take care to prepare participating students for the workshops. The industrial problems are rarely neatly formulated in the language of mathematical modeling, which can be daunting for students. For this reason, MPI organizers run the GSMMC beforehand. It runs a week prior to the MPI workshop with about 28 graduate students and four faculty mentors, who bring real-world modeling problems, usually from their own research.

GSMMC is a smaller version of the MPI workshop, preparing students for what they might face at the following MPI workshop. Just as with MPI, problems are presented at the beginning of the week and students decide what to work on. Under the guidance of the faculty mentors, the teams formulate mathematical models for the problems and present their work at the end of the week. Cummings and her colleagues have found that the camp prepares students well for the next week at MPI.

Overwhelmingly Positive Assessments

Cummings and the other organizers have collected feedback from assessment surveys from both GSMMC and MPI. Feedback is overwhelmingly positive, with almost all participants wishing to return again.

VARIOUS TYPES OF MATHEMATICAL MODELING CAREERS

Based on the panel presentation by Genetha Gray, Salesforce

WHERE MIGHT YOUR MATHEMATICAL MODELING STUDENTS *end up after they graduate?* was the question Genetha Gray asked the educators in the room to consider.

People Analytics

Gray works in people analytics, also known as workforce analytics or talent intelligence. The term was coined at Wharton Business School to describe the "data-driven approach to managing people at work." Gray was initially interested in this area because she saw that her work in the field could make a difference in how people work. Companies started investing in the field because they had found that their workforces were not as efficient as they could be and they needed to compete more for skilled workers. One of Gray's tasks is to figure out how to attract skilled mathematical modelers to work at her company instead of, say, Google or Facebook.

People analytics grew from industrial and organizational psychology. Workers in the field have degrees in everything from data science and mathematics to business and the law. Gray works on problems related to skills taxonomy (that is, understanding what people know and how it relates to other areas of knowledge), team dynamics, employee benefits usage, diversity and inclusion, and workforce planning.

Skills for "Dealing with Data"

When schools train students to be ready for mathematical modeling jobs, one requirement is that they need to know how to "deal" with data. This encompasses all tasks related to collecting, storing, protecting, retrieving, analyzing, and reporting data. Some of these pieces are sometimes overlooked.

- *Collecting* does not refer only to collecting data in a lab setting: it could entail anything from finding the applicable publicly-available reports to implementing surveying schemes that will attract useful samples. Once data is collected, it needs to be stored to both protect privacy as necessary and allow authorized people to retrieve it. For example, training programs need to consider if students are learning how to retrieve data efficiently without using too many resources.
- *Storage* and *protection* require thinking about both security and cost. Companies often have a global employee base or work with international partners or clients, and the web of laws about what information can be retrieved by whom is important to understand.
- *Analysis* is probably the most obviously mathematical part of the process and what students and their instructors think of first when they think about data.
- *Reporting* is important, but it is often overlooked in training. It is important for students to learn both how to visualize information in ways that are useful for the intended audience and how to effectively communicate results and findings to non-experts.

Preparation Can Start by Using Existing Programs

Gray emphasized the importance of student participation in programs such as the Mathematical Association of America's Preparation for Industrial Careers in Math (MAA PIC Math). PIC Math provides professional development for mathematics faculty so that they can develop courses that give mathematics students opportunities to work on applied problems posed by industry or community partners. These experiences prepare students for internships and careers in mathematical modeling.

Advice for Prospective Mathematical Modelers

Gray ended her presentation with four final thoughts for prospective mathematical modelers:

1. *You do not need to do everything well*, but you do need to cultivate expertise in at least one area and recognize when you are not an expert so you can ask for help.
2. *You need to become a data storyteller.* When you are applying for a new position, you have to describe the mathematical modeling projects you have done in a way that interests everyone regardless of their areas of expertise as well as explain and visualize the supporting data in a way that everyone can understand.
3. *The more practice you can do* in many different application areas, the better.

4. *Do not be afraid to try new things.* Educators can help by making their students a little bit uncomfortable by encouraging them to investigate the unknown. This will help students build the skills they need to have a successful career in mathematical modeling.

PART 5

SYSTEMIC CHANGE IN MATHEMATICAL MODELING EDUCATION: STRUCTURES AND ASSESSMENT

CIME participants wanted to do more than just change their classrooms: In approaching systemic change, the workshop considered frameworks for student learning, teacher support, and assessments

MAKING SYSTEMIC CHANGE IN MATHEMATICAL MODELING

Based on the panel presentation by Diane Briars of the Conference Board of the Mathematical Sciences (CBMS)

DIANE BRIARS FRAMED HER PRESENTATION with two motivating questions:

- In what aspects of our organizational initiatives does the question of mathematical modeling, community, and culture show up? How does that already exist in our organizations?
- How can organizations and CIME participants work together to provide support, resources, and policy direction for stronger mathematical modeling work across the K–16 spectrum?

Providing Support for Mathematical Modeling Education

The Conference Board of the Mathematical Sciences is an umbrella organization of 18 different mathematics and mathematics education societies, all of which have as one of their primary objectives the increase or diffusion of knowledge in one or more of the mathematical sciences. The board meets twice a year to discuss issues and collaborate to promote research, improve education, and expand the uses of mathematics, with other projects happening in between.

The different CBMS societies support mathematical modeling in various ways, including

- Providing materials and professional learning experiences that promote understanding of mathematical modeling—what it is and what it is not;
- Providing resources and professional learning experiences that support teachers and administrators in infusing mathematical modeling into K–16 education;

- Making the case that mathematics—and mathematical modeling—is essential for K–16 education, both because of the potential for employment and because it helps support students' mathematical development more generally, including the development of positive beliefs about the usefulness of mathematics and about their ability to do mathematics.

Many of the CBMS societies have resources that can help educators advocate for and implement more and better mathematical modeling in curricula. Most notably, the Society for Industrial and Applied Mathematics (SIAM) and the Consortium for Mathematics and Its Applications (COMAP) published Guidelines for Assessment and Instruction in Mathematical Modeling Education (GAIMME) (2016, 2019) (see www.siam.org), which provides a clear picture of the mathematical modeling process along with examples of how to incorporate mathematical modeling into instruction at all levels, from primary grades through undergraduate courses. SIAM, NCTM, and COMAP are continuing to collaborate on the Mathematical Modeling Hub, an online community and resource repository supporting integration of mathematical modeling into PreK–undergraduate classrooms (qubeshub.org/community/groups/mmhub).

Other CBMS societies, for example, NCTM, Mathematical Association of America (MAA) and American Mathematical Association of Two-Year Colleges (AMATYC) highlight the importance of mathematical modeling in recent curriculum recommendation publications.

Working Directly with State Educators

Briars described the upcoming CBMS Forum on High School to College Mathematics Pathways, which will bring together leadership teams from 22 states to analyze the current status of their state's mathematics education and develop a plan for policies and practices in mathematics instruction that will contribute to successful completion without reducing quality. CBMS, in partnership with the Dana Center at the University of Texas at Austin, will support the state teams for 18 months as they start implementing the plans. A follow-up for the state teams will be held in fall of 2020. See the CBMS website for information from the Forum (www.cbmsweb.org/cbms_forum_6/).

A Responsibility to Provide Support

The bottom line, Briars says, is that teachers need resources and support to effectively incorporate mathematical modeling into their instruction. Specifically, teachers should not be expected to create their own mathematical modeling materials. Instead, "teachers need to have instructional materials that give them that support."

ASSESSING MATHEMATICAL MODELING

Based on the panel presentation by Ted Coe of Achieve

BETWEEN 2010 AND 2014, FORTY-FIVE STATES adopted the Common Core State Standards in Mathematics (CCSSM, 2010), but by September 2017, twenty-four of those states had in some way revised the standards. Some good things came from the CCSSM, including features like mathematical practices; a balance of procedures, conceptual understandings, and applications; mathematical modeling in the high school curriculum; and statistics. Organizations like the Mathematical Association of America agree that these skills will be important as data science continues to increase in prominence in the next few years.

Changing Standards: How Did States Revise CCSM?

Ted Coe's organization, Achieve, looked at the 24 states that changed the standards in order to try to understand what they had changed. First, Achieve compared the standards for mathematical practice across states to those of the CCSSM. Those standards are:

- Make sense of problems and persevere in solving them
- Reason abstractly and quantitatively
- Construct viable arguments and critique the understanding of others
- Model with mathematics
- Use appropriate tools strategically
- Attend to precision
- Look for and make use of structure
- Look for and express regularity in repeated reasoning

These practices were fairly well retained, but three states de-emphasized them moderately and three states de-emphasized them severely or removed them entirely.

Achieve also looked at several specific topics in the K–8 curriculum to see if there were standards that addressed them rigorously. For example,

- Standards for Pythagorean theorem and division of fractions were retained.
- Statistical topics were also fairly well-represented, but not quite as much as earlier topics. Some would argue that the topics that had been removed could be addressed in more advanced mathematics classes in the last year of high school, but only 68% of 12th graders in lower-income schools are even offered a statistics course in high school.

	2: STRONG	1: MODERATE	0: WEAK/ABSENT
M2: The standards emphasize the importance of the mathematical practices.	The standards include something like the practices.	The standards include something like the practices but with less prominence or emphasis than should be present.	The standards severely de-emphasize or have nothing like the practices.

Presence of mathematical processes in standards

2: STRONG	1: MODERATE	0: WEAK/ABSENT
M2: The standards emphasize the importance of the mathematical practices.		

AL	AZ	AR	CA	FL	GA	ID	IN	IA	LA	MA	MS	MO	NJ	NY	NC	ND	OH	OK	PA	SC	TN	UT	WV

De-Emphasis of modeling. Modeling is one of the conceptual categories for high school in the Common Core, on the same level as number, algebra, functions, geometry, and statistics/ probability. Modeling standards were de-emphasized or absent in many of the 24 states that changed their standards. Coe found the de-emphasis in mathematical modeling discouraging.

Modeling Supports Math Literacy

Coe shared the PISA (Programme for International Student Assessment) definition of mathematical literacy:

> *Mathematical literacy is an individual's capacity to formulate, employ and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to recognise the role that mathematics plays in the world and to make the well-founded judgements and decisions needed by constructive, engaged and reflective citizens.*

As a community, mathematics educators need to take control of this conversation and have standards for what a mathematically proficient student can do and what a mathematically rigorous experience involves. The United States does not do particularly well on PISA assessments, so Coe argues it should not be de-emphasizing mathematical modeling.

Assessments

Achieve works with both standards and assessments. The CCSSO (Council for Chief State School Officers) has five criteria for assessment:

C.1 Focusing strongly on the content most needed for success in later mathematics

C.2 Assessing a balance of concepts, procedures, and applications

C.3 Connecting practice to content

C.4 Requiring a range of cognitive demand

C.5 Ensuring high-quality items and a variety of item types

Aspects of Rigor (AOR) Matrix

	The item does not involve application.	The item involves an application.
The item targets procedural skill expected by the grade level.	P	P-A
The item targets conceptual understanding and procedural skill expected by the grade level OR targets conceptual understanding but can also be answered using at least some procedural skill expected by the grade level.	P-C	P-C-A
The item targets conceptual understanding. Students may explain, strategize, evaluate, determine, compare, or classify.	C	C-A

(https://www.achieve.org/mathematics-assessment-cognitive-complexity-framework)

Achieve and Student Achievement Partners (a non-profit organization that designs tools, resources, and professional development for K–12) developed a framework called the Aspects of Rigor (AOR) matrix to help assess the assessments for criterion C.2. The AOR is designed to understand what the assessments are measuring and whether they are balanced appropriately.

	Level 1	Level 2	Level 3
Procedural Complexity	Solving the problem entails little procedural demand or procedural demand is below grade level.	Solving the problem entails common or grade-level procedure(s) with friendly numbers.	Solving the problem requires common or grade-level procedure(s) with unfriendly numbers, an unconventional combination of procedures or requires unusual perseverance or organizational skills in the execution of the procedure(s).
Application Complexity	Solving the problem entails an application of mathematics, but the required mathematics is either directly indicated or obvious.	Solving the problem entails an application of mathematics and requires an interpretation of the context to determine the procedure or concept (may include extraneous information). The mathematics is not immediately obvious. Solving the problem requires students to decide what to do.	In addition to an interpretation of the context, solving the problem requires recognizing important features, and formulating, computing, and interpreting results as part of a modeling process.
Conceptual Complexity	Solving the problem requires students to recall or recognize a grade-level concept. The student does not need to relate concepts or demonstrate a line of reasoning.	Students may need to relate multiple grade-level concepts of different types, create multiple representations or solutions, or connect concepts with procedures or strategies. The student must do some reasoning, but may not need to demonstrate a line of reasoning.	Solving the problem requires students to relate multiple grade-level concepts and to evidence reasoning, planning, analysis, judgement, and/or creative though OR work with a sophisticated (nontypical) line of reasoning.

(https://www.achieve.org/mathematics-assessment-cognitive-complexity-framework)

For category C.4, they developed three levels of procedural, conceptual, and application complexity. For example, this is the scale for application complexity:

Level 1. Solving the problem entails an application of mathematics, but the required mathematics is either directly indicated or obvious.

Level 2. Solving the problem entails an application of mathematics and requires an interpretation of the context to determine the procedure or concept (may include extraneous information). The mathematics is not immediately obvious. Solving the problem requires students to decide what to do.

In addition to an interpretation of the context, solving the problem requires important features, and formulating, computing, and interpreting results as part of a modeling process.

Leveling Up!

When Coe works with teachers, he encourages them to get out of level 2 of procedural complexity. Do they ever allow the students to get to play with problems of level-3 complexity? He gave an example of a problem to the group: "A baker has 159 cups of brown sugar and 264 cups of white sugar. How many total cups of sugar does the baker have?" This has level-1 application complexity. In contrast, the following is a level-3 problem:

A Level-3 problem

Max is organizing a trip to the airport for a party of 75 people.

He can use two types of taxi.

A small taxi costs $40 for the trip and holds up to 4 people.

A large taxi costs $63 for the trip and holds up to 7 people.

1. (a) If Max orders 6 large taxis, how many small taxis will he need?

   (b) How much will the total cost be?

2. Max can organize the journey more cheaply than this! How many taxis of each type should max order, to keep the total cost as low as possible?

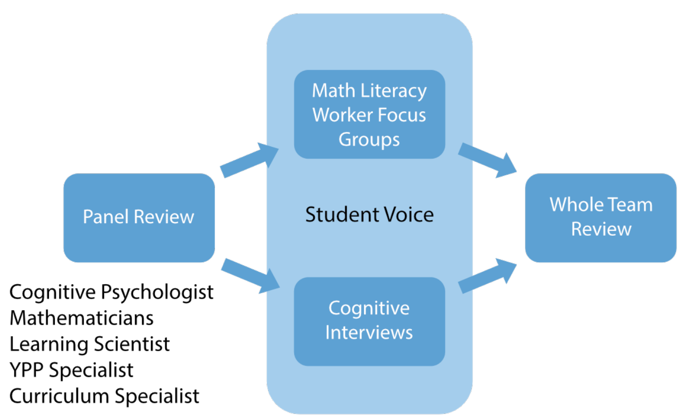
Achieve is trying to move the conversation so that when teachers are talking about assessments, they are talking about modeling.

INCORPORATING STUDENT VOICE INTO AN ASSESSMENT FOR THE FUNCTION CONCEPT

Based on the panel presentation by Edith Aurora Graf of Educational Testing Service (ETS)

EDITH AURORA GRAF SPOKE ABOUT INCORPORATING student voice into the design of an assessment. In this particular case, it was for an NSF-supported project, a joint venture of ETS, the Algebra Project, the Young People's Project, and Southern Illinois University Edwardsville, to validate the interpretation of a learning-progression-based assessment for the concept of function.

Work began with an extensive panel review of the tasks that were developed based on a learning progression. The reviewers included a cognitive psychologist, mathematicians, a learning scientist, and a curriculum specialist. After that, focus groups from the Young People's Project discussed the tasks and offered feedback. At the same time, they conducted cognitive interviews where students worked on the tasks, shared their thinking, and answered questions about the problem-solving process. The team reviewed this input and modified the tasks as needed.



Task Review and Improvement Process

Assessment Challenges

Graf described three challenges of assessing mathematical modeling tasks and some potential solutions.

1. *Developing authentic mathematical modeling tasks.* Some questions to ask about tasks is whether the situation to be modeled is culturally relevant and what restrictions exist on the conditions under which the model applies. Developers can meet this challenge by reviewing tasks carefully with experts from several disciplines as well as students. Cognitive interviews with students as they work on the tasks can be valuable.

2. *The iterative nature of mathematical modeling.* Mathematical modeling often requires several cycles of model development, in which models are refined over time. The process, rather than the product, is often the goal. Often, the time allotted in an assessment is limited, so it can be difficult to assess the full process. A solution is to assess components—things like assumptions, the model itself, and the way it is communicated—separately.

3. *Mathematical modeling involves teamwork, and most assessments are individual.* One solution is to create a situation where an individual plays a role on the team and to assess their approach to modeling with respect to that role.

Following the workshop, the project team provided further reflections. The student voice components provided important insights into students' mathematical thinking and monitoring processes. These insights helped to improve task design and support validity, and raised broader questions about the potential value of the student voice components for learning, teaching, and instructional strategies. In the future, they would also like to explore how to assess team collaboration on rich mathematical tasks.

This project was supported by NSF Grant No. 1621117.

MATHEMATICAL MODELING LEARNING PROGRESSION

Based on the panel presentation by Leslie Nabors Oláh of Educational Testing Service (ETS)

LESLIE NABORS OLÁH ENCOURAGED THE AUDIENCE to consider what they mean by "standardized testing." She believes the term is not as well defined as some think. Over the course of the workshop, several people shared their dissatisfaction with standardized testing, but what were they really dissatisfied with?

Nabors Oláh notes that there are several common objections to standardized testing:

- It has high stakes, for both students and teachers.
- It is timed, so mathematical modeling tasks can be harder to fit in.
- Exams are multiple-choice, and mathematical modeling is difficult to assess with multiple-choice questions.
- It is externally-mandated, giving teachers little discretion in what is tested or how.
- It is inauthentic.

A Learning Progression to Serve as a Basis for Assessment

When Nabors Oláh worked for the School District of Philadelphia, she was dissatisfied with standardized testing because she thought that the scores did not communicate much information. Later, when she went to ETS and started working on assessments for mathematical modeling, she wanted to develop a learning progression for mathematical modeling that teachers could use. The progression involves assessing students' readiness to do mathematical modeling, the model they build, how they present the model, and their ability to compare their model to others and critique models.

Readiness	Building	Presentation	Comparison & Critique
Level 2/3	Level 3	Level 3	Level 3
Level 1	Level 2	Level 2	Level 2
	Level1	Level1	Level1

Mathematical Modeling Learning Progression

She and her colleagues are currently working on designing a series of tasks that can be given within a class period (35–45 minutes), partially machine-scored and partially human-scored, that would be conducive to productive assessment of mathematical modeling.

THE NAEP FRAMEWORK AND MATHEMATICAL MODELING

Based on the panel presentation by Shandy Hauk of WestEd and San Francisco State University

WESTED PARTNERED WITH THE GOVERNING BOARD to support updating the framework for the 2025 National Assessment of Educational Progress (NAEP) in mathematics. The updated framework is likely to guide NAEP Mathematics development through 2040 and is driven by a visioning panel of 30 experts in mathematics education and mathematics. A subset of 15 people (called the development panel) is working closely with WestEd to create the update about what mathematics students should know and be able to do. Results are contextualized in the Nation's Report Card (www.nationsreportcard.gov).

Context of the Assessment Framework

The NAEP is not designed to give students individualized feedback about their learning. It is rather a snapshot of 120,000 to 150,000 students at a given grade level (4, 8, or 12) in a given year and the mathematics they have had an opportunity to learn.

Leadership Learning Context

Much of the 2019 CIME workshop was focused on the inner yellow disk in the figure above, which represents the relationship of teachers and students to the material in schools. The

NAEP assessment takes place in the larger blue context, of leader and policy development, which interacts with professional learning of teachers (green disk), which, in turn, interacts with what happens in the yellow disk at the classroom level.

Because the group is trying to develop assessments for exams through 2040, it has to look further than the Common Core Standards, which may have transformed a great deal by then. The effort to update the framework for 2025 includes revisions to the descriptions of the mathematics objectives as well as the addition of attention to five mathematical practices (representing; justifying and proving; abstracting and generalizing; mathematical modeling; and collaborative mathematical activity). Because it is a framework for designing an assessment, not a course or curriculum, the update also includes descriptions of achievement levels; priorities for surveying students, teachers, and administrators on important mathematics-specific contextual variables related to students' opportunities to learn; and item specifications. The NAEP assessment has more than just multiple choice or constructed response items; the recommendation for 2025 and beyond is to include scenario-based tasks.

Hauk sought public feedback from teachers and mathematicians in attendance on the updates to the framework and shared a web link for accessing the draft documents during the public feedback period, from April to June 2019.

AN ASPIRATIONAL VISION FOR TEACHER TRAINING IN MATHEMATICAL MODELING

Based on the panel presentation by Mike Steele, University of Wisconsin-Milwaukee and Association of Mathematics Teacher Educators (AMTE)

THE *STANDARDS FOR PREPARING TEACHERS OF MATHEMATICS* is an AMTE document that lays out an aspirational vision for what the initial preparation of all teachers, pre-K through 12th grade, who teach mathematics. While the phrase "teachers of mathematics" is clunky, it was deliberately chosen to emphasize that the standards are for everybody who has a stake in students' mathematical learning. So this includes not only middle school and high school teachers who only teach mathematics, but it also includes generalist teachers, mathematics specialists, special education teachers, and interventionists.

Meeting Standards: Teachers

In his presentation, Mike Steele focused on two particular aspects of the document. First, there is a set of standards that teacher candidates should know. Those standards are:

- C.1 Mathematics concepts, practices, and curriculum
- C.2 Pedagogical knowledge and practices for teaching mathematics
- C.3 Students as learners of mathematics
- C.4 Social contexts of mathematics teaching and learning

How do these standards intersect with mathematical modeling? There are some indicators within the standards that are particularly relevant to mathematical modeling. A few of those are:

- C.1.2 Demonstrate mathematical practices and processes
- C.1.3 Exhibit productive mathematical dispositions

C.2.5 Enhance teaching through collaboration with colleagues, families, and community members

C.3.3 Anticipate and attend to students' mathematical dispositions

C.4.2 Cultivate positive mathematical identities

C.4.3 Draw on students' mathematical strengths

C.4.4 Understand power and privilege in the history of mathematics education

C.4.5 Enact ethical practice for advocacy

The last four items, which relate to the social contexts of mathematics teaching and learning, are especially connected to ideas from the CIME workshop.

Meeting Standards: Teacher Preparation

Steele shared some standards for the characteristics that mathematics education programs should have:

P.1. Partnerships

P.2. Opportunities to learn mathematics

P.3. Opportunities to learn to teach mathematics

P.4. Opportunities to learn in clinical settings

P.5. Recruitment and retention of teacher candidates

The associated indicators that seem especially relevant include:

P.1.1 Engage all partners productively

P.2.2 Build mathematical practices and processes

P.3.3 Address the social contexts of teaching and learning

P.4.1 Collaboratively develop and enact clinical experiences

P.5.2 Address diverse community needs

Necessary Actions

Steele ended with a call to action:

- Support implementation of the AMTE standards as a means to argue for stronger mathematical modeling teacher preparation
- Work across all stakeholders and contexts (content, pedagogy, field) to infuse mathematics modeling teacher education
- Integrate mathematics modeling for all teachers of mathematics
- Develop and share mathematics modeling integration across contexts
- Engage practicing teachers in sustained, generative professional development around modeling

PART 6

EMERGING CRITICAL ISSUES AND CLOSING THOUGHTS

In discussions and breakout work that followed each panel session, panelists and participants identified critical issues and questions that should motivate continuing work and improvements in the field

EMERGING CRITICAL ISSUES

IN DISCUSSIONS AND BREAKOUT WORK that followed each panel session, the CIME panelists and participants identified critical issues and questions that should motivate continuing work and improvements in the field. These emerging critical issues are listed below and range from very local, personal issues in the classroom all the way to large, structural issues.

One theme of the conference, which was the basis for the final panel session, was *How can organizations and CIME participants work together?* This kind of cooperation is of course required to address all the topics below; issues that are particularly connected to organizational cooperation are marked with a starred bullet.

Defining Mathematical Modeling Education

- How do mathematicians, educators, curriculum developers, and other stakeholders come to a ***shared definition of mathematical modeling***? What is gained or lost by choosing a narrower or broader definition?
- What counts as ***"real-world" mathematics***? For example, marginal tax rates are much less real to most students than abstract polygons are, but tax rates would be considered a real-world application of mathematics. How can we define mathematical modeling so students' reality and cultural contexts are respected?

In the Classroom: Pedagogy and Teacher Support

- How can teachers effectively ***integrate writing and presentation skills*** into mathematical modeling classes?
- Once a teacher has decided on a mathematical modeling task to use in the classroom, how do they ***establish classroom norms*** that support students as they learn to do modeling?

- What ***classroom routines*** can teachers use to help students learn to mathematize the world?
- Teachers may have a goal of using authentic data sets with students. How can they ***ensure the data sets they find are reliable*** and then, if necessary, make them accessible and relatable, especially to young children?
- ⊛ Humanizing mathematics means that sometimes a teacher's focus is not on the mathematics but on other aspects of a classroom, such as the way they are communicating and whether their lessons are engaging students' backgrounds, cultures, and lives. How should educators think about the ***non-content lessons*** they are teaching students?
- ⊛ How can organizations and teams of teachers ***support individual teachers*** in finding and implementing the pedagogical tools that are most important in teaching mathematical modeling?
- ⊛ Teachers should not have to build all of their mathematical modeling lessons from the ground up. These tasks should be included in their instructional material. How, then, can organizations or individuals ***advocate for the inclusion of good mathematical modeling lessons*** in instructional materials available to teachers?

Equity and Social Justice

- How can educators ***include disability*** as a focus of their mathematical modeling questions as well as gender, race, and ethnicity?
- How can teachers be prepared to ***learn about social justice issues*** that could be used as mathematical modeling projects in their classrooms?
- Has research similar to Jo Boaler's research on gender equity in mathematics competitions been done on other groups of interest, particularly race and ethnicity? ***Broader research into equity*** in current mathematical modeling practices is needed.
- How can teachers and mathematics competition organizers ***address barriers to student participation*** in mathematics competitions; that is, what mechanisms can be employed to help students with less extensive backgrounds get up to speed so they can reap the benefits of participating in mathematical modeling competitions?
- ⊛ New ways of teaching that center social justice and mathematical modeling have the potential to alienate people who were comfortable with the status quo. How can educators ***deal with this pushback***?
- ⊛ It is important for teachers to adapt to their students' experiences and cultures, which is difficult to do with a one-size-fits-all curriculum. How can organizations and publishers ***support teachers in developing***

intercultural competence that will allow them to find the right balance and adapt tasks appropriately for their students?

Non-STEM and Professional Pathways

- How can university mathematics departments get more mathematical modeling into not only upper-level courses but ***into all math courses***, including those that will be taken by pre-service teachers?
- What is the relationship between mathematics and computer science courses in mathematical modeling? How can departments prepare students so ***programming will not be a barrier*** to taking mathematical modeling courses?
- What ***assessment models*** are most fair and effective in undergraduate mathematical modeling courses?
- ⊛ Are there ways for university-level mathematical modeling classes to work with city and state governments on ***problems that are important in their communities***?
- ⊛ Can mathematicians and educators find support for ***adapting mathematical modeling problems*** from programs like the Mathematical Problems in Industry program ***to high school level projects*** on a large scale and share them broadly?

Cooperation with Organizations

- ⊛ How can participants in this CIME workshop, and others interested in the same issues, bring their ideas ***outside of their own academic circles*** and actually ***influence policymakers*** at the highest levels?
- ⊛ Teachers spend a lot of time developing tasks, sometimes on their own and sometimes in groups. They experiment to find out what works for them and their students. How can these ***insights be scaled up*** so more teachers can benefit from them?
- ⊛ What role does a funding agency like the National Science Foundation (NSF) play in helping to ***disseminate these insights*** to a broader group?
- ⊛ How can professional organizations ***make mathematical modeling more visible*** to more teachers and teacher educators?
- ⊛ How can professional organizations that work at different levels (for example, state or national levels), or at different points in students' education, ***collaborate with each other*** to support teachers in using mathematical modeling?

CLOSING THOUGHTS

Based on closing comments by lead organizers Cynthia Anhalt and Rachel Levy

CYNTHIA ANHALT thanked CIME participants for their energy, dedication, and interest in mathematical modeling education for everyone. She shared some thoughts from the workshop that resonate for her:

- Mathematical modeling is exciting and beautiful.
- There is power and authenticity in mathematical modeling problems.
- Modeling is interactive, rigorous, and stimulating.
- Mathematical modeling provides access for more students to participate in more mathematics.
- Mathematical modeling problems can have high ceilings, low floors, and wide walls.
- "If a student can say the word 'calculus,' they are allowed in my modeling class." (A quote from Ricardo Cortez from the first day of the workshop.)
- Mathematical modeling provides an opportunity for real critical social issues to be explored.
- Mathematical modeling is never-ending. You find a solution to a question, you get an answer, and you get more questions.

RACHEL LEVY closed the conference with a poem which was inspired and written in between sessions:

Mindful modelers
Interacting
Making mathematical choices
Passionate discussers
Simplifying complexity
Solving meaningful problems
Considering community and cultural contexts
Engaging ethical implications
Empowered
Whole
Belonging
Math-doers

READING LIST FOR TEACHERS: ADVOCACY AND UNDERSTANDING

Jennifer A. Czocher shared this list of research articles about mathematical modeling in order to help them advocate for including modeling in curricula and understand how to teach modeling effectively.

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Manouchehri, A., & Lewis, S. T. (2017). Reconciling Intuitions and Conventional Knowledge: The Challenge of Teaching and Learning Mathematical Modeling. In G. A. Stillman, W. Blum, & G. Kaiser (Eds.), *Mathematical Modelling and Applications: International Perspectives on the teaching and Learning of Mathematical Modelling* (pp. 107–116). Cham: Springer.

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