Collaborating to Teach Teachers: Mathematicians & Educators Team Up

By Julie Rehmeyer

Mathematical Sciences Research Institute

ACKNOWLEDGEMENTS

Using Partnerships to Strengthen Elementary Mathematics Teacher Education

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University of Nebraska

William McCallum
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Julie Rehmeyer may be reached at: jrehmeyer@gmail.com

MSRI
Mathematical Sciences Research Institute

Main Office 510-642-0143 • Fax 510-642-8609
Shiing-Shen Chern Hall • 17 Gauss Way • Berkeley, CA 94720-5070
www.msri.org

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By Julie Rehmeyer

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At the University of Michigan, educator Deborah Ball and mathematician Hyman Bass lead a group identifying the mathematical knowledge and skill that teachers need in their everyday work in the classroom. They developed an assessment of teachers' mathematical knowledge for teaching. Problems adapted from that test are sprinkled throughout this document.

Problem numbers referenced are from http://sitemaker.umich.edu/lmt/files/LMT_sample_items.pdf

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WHY COLLABORATE?

Reasons Mathematicians Need to Collaborate with Educators

- An educator can help connect mathematics to prospective teachers' experience in the classroom.
- Students tend to doubt that a mathematician has anything to contribute to their preparation to be an elementary teacher. But if they see the educator respecting the mathematician's contribution, they'll follow suit.
- An educator is steeped in the culture of teaching and can help a mathematician relate to the students.

If you are a mathematician or an educator who wants to improve mathematics education, one of the best ways to do it is to collaborate.

Such a partnership is a highly effective way to strengthen the education of university students preparing to teach elementary school. Obviously, prospective elementary school teachers need to learn how to teach. Perhaps less obviously, though, prospective teachers also need to learn a significant amount of math beyond what they learned in high school. And even more, they need to learn how to use that mathematical knowledge to serve their students in the classroom.

Although the mathematics elementary school children are learning may not seem very sophisticated, the skills required of teachers are formidable. Teachers need a deep, flexible, and intuitive understanding of basic mathematics, and they have to be able to translate that understanding into words and images that a young child can understand, while maintaining mathematical accuracy. They need to be able to deduce how a child's reasoning went wrong on a math problem, and they then need to be able to explain the mistake to the child. They need to be able to write good problems that will support the learning of particular concepts. They need to be able to work fluidly with multiple models for a single mathematical concept (multiplication, for example, can be modeled as repeated addition or as a way of computing area), with a strong understanding of when each model is useful.

Mathematicians are precisely the people who know how to think deeply about mathematics, including the fundamental mathematics taught in elementary school. And usually, prospective teachers take math courses taught by mathematicians. Unfortunately, though, most mathematicians know little about the particular demands elementary school teaching presents, and they often find it difficult to communicate effectively with prospective teachers.

In the meantime, prospective teachers take classes in pedagogy from educators who are usually working in isolation from the mathematicians. The reality, though, is that mathematics and pedagogy are inextricably interwoven in teaching practice. To prepare their students for the demands they'll face in the classroom, educators need to be able to identify the mathematical issues raised in teaching. But doing this skillfully requires an enormous depth of understanding of mathematics that non-mathematicians can't be expected to have.

So both mathematicians and educators have much to contribute that is critical to effectively teach future teachers. Furthermore, classes that integrate mathematics content and mathematics teaching are essential for prospective teachers to learn to apply their mathematical knowledge to everyday problems in the classroom.

It's easy for mathematicians to feel that their obligations end with their own research and the particular classes they choose to teach, but the profession of mathematics carries a responsibility to educate elementary school teachers to teach math effectively. One reason is simply money: Math courses for prospective teachers help finance math departments. Furthermore, mathematicians want well-prepared students in all their classes, and university students' mathematical skills are strongly influenced by the quality of their elementary education. Finally, teachers are the primary ambassadors for the field of mathematics, because public understanding of the nature of mathematics itself is strongly influenced by how it's taught in elementary schools. So mathematicians have both self-interest and a responsibility to educate teachers, and they can do that most effectively by working with educators.

Currently, though, collaborations between educators and mathematicians are rare, and many more are needed. The Mathematical Sciences Research Institute has been working to increase the number of these partnerships though a several-year-long program of workshops and conferences, with the support of the S.D. Bechtel, Jr. Foundation. This document aims to distill some of the lessons learned by pioneering partnerships in order to inspire others to get involved and to help them figure out how to get started.

At the University of Michigan, mathematician Hyman Bass and mathematics educator Deborah Loewenberg Ball work together both on curriculum development and on research into teacher knowledge and teacher education. At the University of Nebraska, mathematician Jim Lewis and math educator Ruth Heaton have designed an integrated program of math instruction for prospective mathematics teachers. At Sonoma State University, a group of mathematicians and mathematics educators including Rick Marks, Ben Ford, Brigitte Lahme, Edith Prentice Mendez, and Kathy Morris collaborate within the mathematics department. And mathematician Barbara Li Santi and educator Ruth Cossey have formed a collaboration teaching mathematics to prospective elementary school teachers at Mills College.

The members of these partnerships have been extraordinarily dedicated, but ordinary mathematicians and educators with many conflicting demands on their time can find ways of getting involved without abandoning their other career goals. The way to start is by learning from the experiences of these pioneers.

Reasons Educators Need to Collaborate with Mathematicians

- Usually, educators don't have time to teach the math content courses themselves, so they rely on mathematicians to do so, with or without an active collaboration.
- Many of the pedagogical challenges teachers will face in the classroom are really mathematical tasks. A mathematician can recognize those moments, enriching an educator's view of their own pedagogical work.
- Math is more than a set of skills; it's a culture. Mathematicians are steeped in that culture and way of thinking, and so they are best able to convey that to students.

UNIVERSITY OF MICHIGAN

Deborah Loewenberg Ball

At the University of Michigan, educator Deborah Loewenberg Ball and mathematician Hyman Bass are collaborating on two levels simultaneously: in the teacher education classroom and in their research. Learning to teach mathematics to young children, they argue, demands not only knowledge but also skills that take practice to acquire, just as in gymnastics or surgery or music. Sure, mathematics teachers need to know mathematics, but even more, they need to know how to use their mathematical knowledge in teaching. Ball and Bass, with their research teams, are building both a theory and a set of practical tools to support that kind of practice.

Ball began work many years ago to identify the mathematical demands on teachers by studying the practice of teaching. She assembled an enormous quantity of records of real-life teaching, including videotaping every lesson in an entire year of a third grade class. Shortly after Ball and Bass met, Ball recruited Bass to mathematically analyze those data, asking him to identify all of the mathematically significant events, including mathematical tasks, student thinking, teacher responses and decisions, etc. Ball figured that a mathematician might well see different things in the videos than an educator would.

One of the things Bass noticed was that many of the problems teachers face in the classroom have to do with their care in the use of mathematical language. Informal language is often useful to make the mathematics more accessible, yet mathematics relies on precise use of terms. Classroom teachers have to be able to balance the precision of the discipline with the complex work of making mathematics accessible. This task is made more difficult because even many curriculum materials use vague language that is ambiguous or incorrect.

For example, in common parlance, an even number may be explained as one that can be divided into two equal parts. But this definition is too loose, since the number 7 can be divided into 3_ and 3_. Going too far the other direction is equally problematic: Defining an even number as "an integer multiple of 2" will hardly help a six-year-old who doesn't know what an integer or a multiple is. Changing the first definition to say an even number is two times another whole number may seem to solve the problem, but this definition excludes negative multiples of 2. Students shouldn't have to unlearn an earlier definition when they learn about negative numbers. Changing the definition to, "a whole number is even if it is 2 times another whole number," is mathematically honest while remaining accessible to a young child. This kind of careful, precise thinking is natural for a mathematician, but prospective teachers need training to learn it.

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1. Mr. Foster's class is learning to compare and order fractions. While his students know how to compare fractions using common denominators, Mr. Foster also wants them to develop a variety of other intuitive methods.

   Which of the following lists of fractions would be best for helping students learn to develop several different strategies for comparing fractions?

   a) $\frac{3}{4}$ $\frac{1}{20}$ $\frac{1}{19}$ $\frac{1}{2}$ $\frac{1}{10}$

   b) $\frac{4}{13}$ $\frac{3}{11}$ $\frac{6}{20}$ $\frac{1}{3}$ $\frac{2}{4}$

   c) $\frac{5}{6}$ $\frac{3}{8}$ $\frac{2}{3}$ $\frac{3}{7}$ $\frac{1}{12}$

   d) Any of these would work equally well for this purpose.

Source: page 17 #21 of "Mathematical Knowledge for Teaching (MKT) Measures," from http://sitemaker.umich.edu/lmt/files/LMT_sample_items.pdf

Hyman Bass

Bass found the work so fascinating that he ended up moving from Columbia University, where he'd taught and done math research for 40 years, to join Ball at Michigan in a joint appointment between the education school and the mathematics department.

Once Ball, Bass and their collaborators had identified the mathematical knowledge they believed was relevant for teaching, they put their ideas to the test. With research scientist Heather Hill, they developed tests to evaluate teachers' "mathematical knowledge for teaching" (MKT). And indeed, they found that if a teacher scored well on the tests, their students tended to do better in math.

Ball and her team are now working to design teacher education curricula with a focus not only on knowledge, but on the use of that knowledge in practice. This means that prospective teachers are helped not just to know mathematics deeply for teaching, but to use their mathematical skills in classroom contexts.

To illustrate this, consider the teaching of multi-digit subtraction. Teachers must have a solid understanding of place value as a foundation for teaching computational methods for subtraction, which they should be prepared to teach. Typically, teacher education programs make sure that teachers understand place value and the algorithms for subtraction themselves, and stop there.

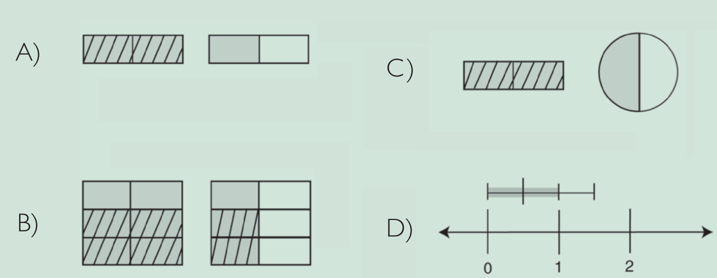
But teachers need to be shown how to unpack the algorithms and make them concrete for children, Bass and Ball say. These algorithms for arithmetic are highly efficient, sophisticated methods for calculating, even with very large numbers, that were distilled from millennia of computation. The cost for that efficiency, though, is that the methods are very encrypted and abstract, so it's easy for kids to lose track of what they mean and why they work.

Ball and Bass teach their prospective teachers to respond to this by going back to the familiar and concrete. For example, the teachers can use "base-10 blocks" – cubes to represent units, rods of ten cubes to represent ten, and 10 x 10 flats of one hundred cubes to represent a hundred, etc. The teacher can then use those manipulatives to translate an abstract problem to a physical model and show what trading means.

Take, for example, the problem 342 – 5. The teacher could show how the number 342 corresponds to three flats, four rods, and two cubes, and the number 5 corresponds to five cubes. In taking away 5 from 342, there aren't enough single cubes to take away five, so the student can trade one of the rods for ten cubes. The number 342 is now represented by three flats, three rods, and twelve cubes. Then it's easy to take away five cubes from twelve cubes,

2. At a professional development workshop, teachers were learning about different ways to represent multiplication of fractions problems. The leader also helped them to become aware of examples that do not represent multiplication of fractions appropriately.

Which model below cannot be used to show that $1\frac{1}{2} \times \frac{2}{3} = 1$?



Ibid, page 7 #6.

leaving seven cubes. That leaves three flats, three rods and seven cubes, or 337.

For children, this way of carrying out the subtraction is visible and makes sense. The problem, of course, is that manipulating blocks is not a practical way to do subtraction in general. So the teacher then needs to make a precise linkage from the moves in this more transparent physical model of subtraction to the symbolic moves of the subtraction algorithm. For example, the trade above is recorded in the subtraction algorithm by crossing out the 4 in 342, replacing it by 3, and placing a 1 before the 2, thus converting it to 12. This correspondence provides a way for kids to make sense of the otherwise abstract and opaque symbolic moves in the algorithm. Prospective teachers who don't know how to do this – even if they can perform the algorithms correctly themselves, and even if they understand the meaning of those algorithms thoroughly – aren't fully prepared for the classroom.

The team uses their methods classes as a laboratory in which to design curricula using their theory. Ball formed the "Mathematics Methods Planning Group" in 1998, initially just to gather the current instructors to organize different sections of the course. The group has evolved into a stable environment for their practice-based research, consisting of the current semester's instructors, those who are preparing to teach the course in the future, and other interested scholars, graduate students, and teachers. One of the great advantages of the large group is that instructors can move in and out without disrupting the stability and continuity. This allows mathematicians to get involved for a while, learn from the accumulated knowledge of the group, contribute their own knowledge, and then move on to other projects.

The group meets at the beginning of the week to plan the lesson and assignments. Everyone then attends the lead section of the class and they debrief it afterwards. Finally, the rest of the instructors teach their own sections. While this is time-consuming, most class preparation gets completed during this process, so participants say the time commitment is manageable.

Over the years of teaching these courses, the team has developed an array of materials that all sections of the class use. Slides, in-class tasks, questions, assignments, and tests are all common. Just having a common syllabus or a common final, Ball says, is not nearly enough to coordinate classes.

These materials provide the foundation for materials development projects the group is in the midst of in order to make their ideas and methods available to groups at other universities. One of the projects currently has two segments available, one on using definitions and one on fractions. These materials are available at http://mod4.soe.umich.edu/mod4/home/.

In addition, each summer they run the Elementary Mathematics Laboratory, a two-week program in which Ball teaches a group of rising fifth-graders in a "fishbowl" format, observed by a group of teachers, researchers, teacher educators, prospective teachers, and mathematicians. In 2009, Bass is also offering a concurrent workshop on Mathematical Knowledge for Teaching Teachers. Information on the workshops is available at http://sitemaker.umich.edu/eml2009.

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3. As Mr. Callahan was reviewing his students' work from the day's lesson on multiplication, he noticed that Todd had invented an algorithm that was different from the one taught in class. Todd's work looked like this:

$$\begin{array}{r} 983 \\ \times \quad 6 \\ \hline 488 \\ +5410 \\ \hline 5898 \end{array}$$

What is Todd doing here? Will Todd's method always work? How would you explain why or why not?

Ibid, page 9 #8.

Every semester, Jim Lewis used to read student evaluations for all mathematics courses taught at the University of Nebraska-Lincoln as part of his job as department chair. And every semester, he saw the same thing: Instructors in the mathematics courses for elementary school teachers received terrible evaluations. "This course is irrelevant for my work as a teacher," the prospective teachers would say. "Why do I have to learn this stuff?"

At the same time, he heard from his instructors how weak the prospective teachers' mathematical preparation was, even about basic issues like place value that are clearly essential for elementary school teachers to understand.

So Lewis decided to teach the mathematics classes for elementary school teachers himself. To make sure he didn't fall into the same difficulties, he decided to collaborate with a math educator. He knew that in order to reach the prospective teachers and get them engaged in the work of his class, he'd have to connect the mathematics to teachers' everyday work in the classroom. Not having taught elementary school himself, he needed help to do that. He also knew that the prospective teachers didn't have a lot of respect for what mathematicians had to contribute to their education, so he needed someone whose opinion they valued to back him up and convince them that the mathematical work was important.

So in 1999, he approached Ruth Heaton, a junior professor in UNL's Department of Teaching, Learning, and Teacher Education. Heaton had been noticing the same problem, but she didn't have the time to teach the mathematics herself. Furthermore, she recognized that a mathematician might be able to enrich her own teaching of mathematical pedagogy.

Lewis and Heaton identified and began working with a group of highly motivated students who were determined to become good teachers, but mostly began with a rigid, algorithmic understanding of mathematics. They usually knew only one way to solve a problem and couldn't even imagine that there might be multiple ways of going about it. Many of them were afraid of mathematics. Others figured teaching mathematics would be a breeze – after all, they said, mathematics is just a matter of following rules.

Jim Lewis (left) and Ruth Heaton

To create good teachers, the pair realized, they needed to radically change their students' conception of mathematics. They wanted their new teachers to have rich mathematical habits of mind: to understand which tools are appropriate when solving a particular problem, to be flexible in their thinking, to use precise mathematical definitions, to know how to develop strategies to solve a problem, to be able to explain their solutions to others, to be persistent in pursuing a problem. Their current math content courses fell far short of accomplishing these goals.

4. Mrs. Jackson is getting ready for the state assessment, and is planning mini-lessons for students focused on particular difficulties that they are having with adding columns of numbers. To target her instruction more effectively, she wants to work with groups of students who are making the same kind of error, so she looks at a recent quiz to see what they tend to do. She sees the following three student mistakes:

	I)	II)	III)
	1	1	1
	38	45	32
	49	37	14
+	65	29	19
	142	101	64

Which have the same kind of error?

Ibid, page 11 #13.

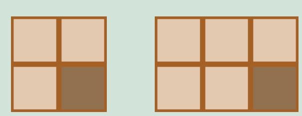
So they were going to have to reinvent the math curriculum for prospective teachers. But since their efforts were nascent and experimental, they wanted to do so in a way that would not elicit their colleagues' resistance, would have the potential to create a sustainable model, and would increase the chances that the first experience would be positive. So they decided to begin by working within the existing courses and by recruiting motivated volunteers for their pilot courses.

Together with Patience Fisher, a colleague in teacher education, they applied for and were awarded a National Science Foundation grant to support the work. Heaton was the principal investigator and the grant bought her out of some of her other teaching duties, giving her additional time for the new classes. Furthermore, bringing money into the department won Heaton credibility with her senior colleagues.

Still, the grant didn't solve all the practical hurdles. To integrate the curriculum, Lewis and Heaton needed their two courses to meet back-to-back in a three-hour block. Merely finding a room, when the courses were offered by different departments within the university, was a struggle. Fortunately, Lewis was department chair, so they could use a room that was controlled by the math department.

The pair spent a year planning before they started teaching together, which was a critical period for them to get to know and respect one another and learn about the substantial cultural differences between mathematics and education. These cultural differences, they found, were significant. For example, they once distributed a questionnaire to their colleagues asking about their beliefs about

5. Takeem's teacher asks him to make a drawing to compare $\frac{3}{4}$ and $\frac{5}{6}$. He draws the following:



and claims that $\frac{3}{4}$ and $\frac{5}{6}$ are the same amount. What is the most likely explanation for Takeem's answer?

a) Takeem is noticing that each figure leaves one square unshaded.

b) Takeem has not yet learned the procedure for finding common denominators.

c) Takeem is adding 2 to both the numerator and denominator of $\frac{3}{4}$, and he sees that that equals $\frac{5}{6}$.

Ibid, page 14 #16.

An assignment for prospective teachers

*Jim Lewis of the University of Nebraska-Lincoln received the following letter:**

Dear Math Professors,

We are 1st and 2nd graders at Saunders County District 50 School in Ithaca, Nebraska. We love to work with big numbers and have been doing it all year! Every time we read something with a big number in it we try to write it. Then our teacher explains how to write it. We are getting pretty good at writing millions and billions!

We have a problem that we need your help with. We were reading amazing "Super Mom" facts in a Kid City magazine. It told how many eggs some animals could lay. We came across a number that we don't know. It had a 2 and then a 1 followed by 105 zeros!! We wrote the number out and it stretches clear across our classroom! We know about a googol. We looked it up in the dictionary. A googol has 100 zeros. Then what do you call a number if it has more than 100 zeros? Is there a name for it?

** A copy of this is, with original formatting, is contained in slide 44 at: http://jessica2.msri.org:80/attachments/13508 pages/135080044.htm.*

learning. While the two groups agreed significantly, they also had some dramatic disagreements. The mathematicians believed that algorithms are best learned through repeated drill and practice; the educators didn't. The educators believed that teachers should let children work from their own assumptions when solving problems; the mathematicians disagreed. The educators thought it was effective for children to use key words to solve word problems; the mathematicians didn't. Another basic issue involved grading: the average grade in an undergraduate mathematics class was 2.53, whereas in an education class, it was 3.64.

These differences have played out for Lewis and Heaton in the classroom, and they've significantly influenced one another's views. Lewis tended to lecture, while Heaton almost never did; Lewis lectures less now. Lewis relied on exams and Heaton never gave them; Heaton now gives a final. Mutual respect was critical to navigating these disagreements. Both members of the partnership came in believing they had things to learn as well as things to teach. During the period of their grant, they met regularly, giving them the time to iron out these issues. Even when they didn't come to agreement, they saw their different viewpoints as a strength of their partnership, because it showed their students how people could disagree and still work together with mutual respect.

They got an unexpected boost for their pilot program after it had finished: In courses the following semester focused on teaching reading, the students who had been in their mathematics program shined. The skills Heaton and Lewis had taught them, it turned out, appeared relevant far beyond mathematics teaching and learning. This was critical in persuading the rest of the faculty to bring their project to a larger scale after two years of their pilot program.

6. Ms. Williams plans to give the following problem to her class:

Baker Joe is making apple tarts. If he uses $\frac{3}{4}$ of an apple for each tart, how many tarts can he make with 15 apples?

Because it has been a while since the class has worked with fractions, she decides to prepare her students by first giving them a simpler version of this same type of problem. Which of the following would be most useful for preparing the class to work on this problem?

I. Baker Ted is making pumpkin pies. He has 8 pumpkins in his basket. If he uses $\frac{1}{4}$ of his pumpkins per pie, how many pumpkins does he use in each pie?

II. Baker Ted is making pumpkin pies. If he uses $\frac{1}{4}$ of a pumpkin for each pie, how many pies can he make with 9 pumpkins?

III. Baker Ted is making pumpkin pies. If he uses $\frac{3}{4}$ of a pumpkin for each pie, how many pies can he make with 10 pumpkins?

Ibid, page 19 #23.

Another problem is that we learned about using commas in large numbers. In the magazine article they used no commas when writing this large number. That confused us. Also, if you write a "googol" with 100 zeros, how do you put the commas in? It doesn't divide evenly into groups of 3 zeros. There will be one left over. We would appreciate any help you can give us solving this "big" problem. Thank you for your time.

Sincerely,

Mrs. Teetor's 1st & 2nd graders

Mallory Bergwell, David Wegman, David Swetson, Brian Murray

Enclosed with the letter was the clipping from Kid City magazine, which said this:

The tiny female apple aphid is a champ as an egg-layer. The insect can lay as many as 21000000000000

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0000 eggs in 10 months.

Heaton and Lewis give their prospective teachers an assignment to write a letter responding to the children.

They now have an "immersion semester" required of all students preparing to be elementary school teachers in which they are focused on the teaching and learning of mathematics. Heaton and Lewis designed a block of four courses, for a total of ten credit hours, all with an emphasis on mathematics teaching and learning: a mathematics content class, a mathematics pedagogy class, a field experience that involved working in an elementary classroom two days each week, and a class with master teachers on creating learner-centered classrooms. The pair worked to integrate the classes, creating a common syllabus and, when possible, common assignments.

This required significant cooperation from Heaton's colleagues. Heaton and Lewis's students ordinarily entered the teacher education program as juniors, at which point they'd often already completed their mathematics requirements. Requiring students to take mathematics classes as they moved through the teacher education program meant they had less time for other courses and had to rearrange their schedules. So to create this integrated mathematics program, students had to work closely with advisors and Heaton's colleagues had to be willing to adjust other classes.

In designing their program, Lewis and Heaton realized that it was not financially practical for them to co-teach a single course, because the university would then be paying two instructors for a single class. So instead, they designed a program where they each teach their own course, but students are required to take both courses the same semester and are the courses are offered back-to-back in the same room. As often as is possible, Heaton and Lewis sit in on one another's classes, participating in the other's class when appropriate. Because they can often do other work while observing the other person's class, they haven't found this to be an excessive burden.

Students who have finished Lewis's and Heaton's classes have gotten better evaluations of their student teaching in all subjects, not just in mathematics. Another indication that they are having an impact on their students is that Lewis gets an extraordinary number of requests for letters of recommendation from students who have been in his classes (sometimes as many as two-thirds of the students in a class). Heaton gets even more requests for recommendations. Lewis notes that this is particularly surprising because even students who haven't gotten very high grades often ask. The students say they value Heaton and Lewis's recommendations because they're proud they survived the class, they believe they really learned something, or they particularly enjoyed it.

And unlike in previous versions of the course, student evaluations are excellent.

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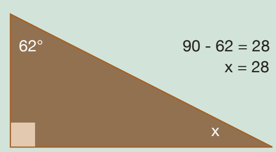
7. Ms. Donaldson's class was working on an assignment where they had to find the measures of unknown angles in triangles. One student consistently found the measures of unknown angles in right triangles by subtracting the known angle from 90.

For example:



Ms. Donaldson was concerned that this student might run into difficulty when trying to find the measures of unknown angles in more general triangles. Which of the following questions would be best to ask the student in order to help clarify this issue?

a) "What do you get when you add 90 + 62 + 28?"
b) "Why does subtracting 62 from 90 give you the measure of the unknown angle?"
c) "How could you find the missing angle in an isosceles triangle?"
d) "How did you know that this was a right triangle?"
e) "What if this angle measured 17° instead of 62°?"

Ibid, page 21 #25

SONOMA STATE UNIVERSITY

Unlike the other schools in the project, Sonoma State has long had both mathematicians and mathematics educators within the mathematics department. Such collaborations are far more common at state universities which focus on teaching and teacher education programs than at research universities, the Sonoma State professors say. Since the majority of elementary school teachers are educated at state universities, these collaborations are particularly influential for the elementary school teaching profession.

The Sonoma State professors have found two powerful ways to collaborate. One is through co-teaching courses. When new faculty come into the department with an interest in math education, the administration makes a particular effort to arrange an opportunity for them to co-teach with an experienced faculty member. Ideally, grant-funded release time makes this possible, but when that has failed, instructors have split credit for the course (though, they acknowledge, the partial teaching credit doesn't cover the full amount of time such a collaboration requires).

The other method of collaboration they've used is "lesson study," a technique adapted from Japanese grade school teachers to closely scrutinize and improve their own instruction. A team of five professors in the mathematics and education departments (Rick Marks, Edith Prentice Mendez, Kathy Morris, Ben Ford and Brigitte Lahme) used the Bechtel grant as an opportunity to do a lesson study of a two-and-a-half-week unit they had used successfully for many years to get students to understand place value and base ten numbering systems through inventing the base five system for themselves.

Typically, the instructors of this mathematics content course meet use a common syllabus and common expectations and stay in close contact informally over the course of the semester. They have not, however, closely coordinated how they've taught each section. For the period of the lesson study, they planned their classes jointly in great detail, attended one another's classes, videotaped many classes and joint discussions, and met between each section of the class to discuss it and adjust their plan.

In this segment of the course, they began with a story adapted from Tom Bassarear's "Mathematics for Elementary School Teachers" about a prehistoric tribe that counted using the letters A to Z, which corresponded to 1 through 26. Beyond 26, they just said "many." The tribe had begun to need to count higher, and the students were assigned the task of inventing a new system that used just the symbols A through D and a new one, 0.

The common difficulty was that students would devise variations on a Roman numeral system rather than a base five system. The team realized from past experience that students who did manage to develop a base five system for themselves, even with extensive hints, did better for the rest of the semester. So while in the past they'd been content if at least one group came up with base five that was then adopted by the whole class, they made it their goal for all the groups in the class to develop the idea on their own. The team devised various strategies to nudge the students away from the Roman numerals, including using manipulatives (a block, a rod of five blocks, and a flat of 25 blocks) and giving additional clues (like that A still meant 1 and B still meant 2). They tested their new techniques on each subsequent section of the class, refining them until they were effective. This greater directiveness produced far better results.

The instructors had used this unit for years, and they'd always considered it to be pretty successful. So they were surprised to end up making such substantial changes, and they saw that as testimony to the power of the lesson study technique.

They also used the lesson study as an opportunity to examine their process of collaboration. The big-picture view of mathematicians, Ford says, helped to draw out which details were most mathematically significant. Some aspects of an algorithm, for example, are essential and unavoidable, while some are artifacts of choices that could have been made differently. The training of the mathematics educators made them attuned to the demands of a classroom, so they were able to connect the material to teachers' future work.

The experience of doing the lesson study was "exhilarating and exhausting," the instructors say. During the two-week period of the study, they had to put in many hours of work more than usual, but the intensive collaboration and the clear results of their work were extremely satisfying. The lesson study was so successful that they've applied the technique more broadly. The team is now planning to use it for the section of their math course for prospective teachers on fractions. The mathematicians have taken it outside of the context of teacher education to improve a segment of their real analysis class.

MILLS COLLEGE

As a result of the Bechtel grant, educator Ruth Cossey and mathematician Barbara Li Santi of Mills College co-taught a math course for prospective elementary school teachers for the first time. They had met decades earlier when they were both volunteering at an Expanding Your Horizons conference, which is designed to encourage girls' interest in math and science. They had never previously taught together, though.

Cossey and Li Santi had two great advantages in bringing their collaboration about: Li Santi was head of the Mathematics and Computer Science department, and the president of the college was a supporter of the work of the grant. So they got immediate approval for their plan to co-teach a class in spring 2007.

The pair faced an immediate challenge, though. Because of regulatory changes, Mills no longer had an undergraduate teacher education program. Still, some of their undergraduates planned to become elementary school teachers, so the pair went ahead and offered a math class for prospective elementary school teachers as an elective.

Only three students signed up, and that wasn't enough students to run their course. So Cossey and Li Santi took another approach: they turned Math 001, a basic math course students usually took to satisfy their general education requirements, into a math content class for prospective elementary school teachers. To argue that the course was generally applicable for all students, not just future teachers, they pointed out in their course description that the skills taught would be relevant for parenting.

They ran the course with a very strong emphasis on equity, in the spirit of the civil rights movement. These students, they noted, had met with a lot of damage to their mathematical identities, which the pair sought to mend. They had the students write mathematical autobiographies. The students discussed what kind of learning environment they preferred, the kinds of things they particularly did or didn't want to hear while they were doing mathematics. That discussion led to an agreed-upon set of norms for the classroom, which Cossey and Li Santi light-heartedly enforced over the course of the semester. The pair made a particular effort to express appreciation for the students' work.

They also made the students responsible for one another's success. The class involved a lot of group problem solving, homework with partners, and a

8. Ms. James' class was investigating patterns in whole-number addition. Her students noticed that whenever they added an even number and an odd number the sum was an odd number. Ms. James asked her students to explain why this claim is true for all whole numbers. After giving the class time to work, she asked Susan to present her explanation:

I can split the even number into two equal groups, and I can split the odd number into two equal groups with one left over. When I add them together I get an odd number, which means I can split the sum into two equal groups with one left over.

Which of the foliowing best characterizes Susan's explanation.

a) It provides general and efficient basis for the claim.
b) It is correct, but it would be more efficient to examine the units digit of the sum to see if it is 1, 3, 5, 7, or 9.
c) It only shows that the claim is true for one example, rather than establishing that it is true in general.
d) It assumes what it is trying to show, rather than establishing why the sum is odd.

Ibid page 4 #5

final project. To make sure that each student had acquired personal competence, midterms were individual – but they were also take-home, and the students could do them over if they chose to.

The course included work on traditional concepts like place value, but it also included more fundamental material often not included in a teacher preparation course. Cossey and Li Santi had learned through experience that these students had a hard time following a mathematical argument, so they also worked on basic logic. One way they did this was through cooperative logic puzzles, where the students were given clues to a puzzle they solved as a group. Then they challenged the students to explain why they had reasoned the way they had.

This process of translating mathematics to and from language was a key component of the course. When students offered an answer in class, Cossey and Li Santi would ask "Are you sure? Why?," regardless of whether the answer was right or wrong. They used the same approach with one another. Li Santi found that Cossey's constant questioning often pushed her to come up with deeper explanations herself. Mathematicians, like any other expert, tend to store their knowledge in compartments, and they don't necessarily immediately know why the things they believe are true. Li Santi found that explaining those "obvious" things required delving down deep into things she thought she already understood.

Because Cossey and Li Santi were building on a decades-long friendship and shared a lot of common ideals, particularly around equity, they didn't have many issues around cultural differences between mathematics and mathematics education. They used their relationship as a model for their students, encouraging the students to behave respectfully through their respectful treatment of one another.

Cossey has continued to teach the course frequently since then using the methods the pair developed together, and Li Santi was able to join in part of the

Barbara Li Santi (left) and Ruth Cossey

course one other semester. Regular co-teaching, however, is difficult to arrange logistically. Because Cossey is a professor in the School of Education at Mills and does not hold a Ph.D. in Mathematics, she would not ordinarily have been assigned to teach a course offered by the Mathematics and Computer Science Department. As department chair, Li Santi was able to make it possible, and it's now become routine.

Mills has also recently restored its undergraduate teacher preparation program, through a "4 + 1" program in which the students earn a four-year undergraduate degree and a one-year master's degree, plus a teaching credential.

9. Create a story problem which illustrates the expression:

$$1\frac{1}{4} \div \frac{1}{2}$$

GETTING STARTED YOURSELF

There is no standard method for collaboration. Each institution has a different internal organization and different structures for educating teachers, so each collaboration will have to take on a different form. Lessons can be distilled, though, from the experiences of these pioneer collaborators. Below are some of their suggestions for how to get started forming a collaboration.

To begin, take advantage of existing efforts in your area. The National Science Foundation has a program on Math and Science Partnerships (www.nsf.gov/funding/pgm_summ.jsp?pims_id=5756), which is a good place to look.

Math education conferences are another good opportunity. The Mathematics Association of America offers math education sessions at all of its conferences including the Joint Mathematics Meetings, and the Association of Mathematics Teacher Educators has an annual meeting. The Institute for Mathematics and Education at the University of Arizona offers workshops bringing mathematicians and mathematics educators together. MSRI also offers a workshop each spring in its "Critical Issues in Mathematics Education" series.

Getting involved in other math education activities offers an opportunity to meet like-minded, interested people in other disciplines. Volunteering for extracurricular math education activities is one approach, like offering a workshop at an Expanding Your Horizons conference (www.expandingyourhorizons.org) or teaching a session at a local math circle (www.mathcircles.org). There are also math circles specifically for teachers, and volunteering to teach one of those is a way to impact teacher education without making a big commitment.

If you're a mathematician who has never taught prospective teachers before, the next step might be to volunteer to teach a math class for elementary teachers and to find someone in the education school to work with. Go through the web pages of the people in the education school to find faculty members with a particular interest in mathematics. Direct or indirect personal connections help. One good approach might be to ask the educator to come make a presentation in your classroom. If nothing else, that will help your students take you more seriously.

Mathematicians often experience some early frustration as they learn how to connect with a very different group of students. Focus on learning a bit about your students and coming to respect their willingness to learn and their determination to do their job well. If possible, spend a bit of time volunteering in an elementary school classroom. You don't have to become an expert on education, but having a feel for the world that your students will be entering will help.

If you're an educator who would like to collaborate with a mathematician, look for mathematicians who teach the math content courses for teachers frequently. You could start by asking to sit in on some math classes. Some instructors may not be receptive to such a suggestion, but those who aren't probably aren't good candidates for collaboration anyway. You could also ask a mathematician to give a presentation to your pedagogy class on a mathematical topic of particular relevance to pedagogy, like definitions, for example.

Both mathematicians and math educators should be aware that attitudes between departments vary widely, from warmth to outright hostility. Take time to learn a bit about the culture in your university first. Be respectful both of expertise within a discipline and across disciplines: educators often know a surprising amount about mathematics, and mathematicians often know a surprising amount about education.

Keep your initial efforts modest, working within the existing structure of courses. On the other hand, don't necessarily stick with the curriculum that has been used in the past, since many math content courses for prospective teachers don't work well. The assessment of mathematical knowledge for teaching that Ball and Bass's research group has created is very useful for gauging students' knowledge base at the beginning of the course.

Early on, solicit support from those in leadership positions in the university. Administrative gridlock can magically ease with the grease of high-level support.

GETTING STARTED YOURSELF CONTINUED

Professional jealousies that can torpedo worthwhile projects can be cut off by a skillful and committed administrator who simply moves on to the next agenda item. Small amounts of money or resources at the critical moment can make the difference.

Once a mathematician and math educator have managed to form a promising initial collaboration, they can start planning ways to make the collaboration deeper. Co-teaching a combined course on pedagogy and math content is powerful if the logistics can be arranged. A lesson study of a segment of a course requires less long-term commitment and can be very illuminating. Winning a grant can provide release time to make more ambitious goals practical, and participants cite the National Science Foundation, the Mathematics Association of America, state and local education agencies, and the university itself as possible sources.

The Mathematics and Science Partnerships program at the NSF is a valuable source of funding.

Collaborations need political and administrative support at the departmental level. Department chairs and heads of schools can do much to encourage partnerships by starting conversations between the departments, matching appropriate faculty members together, making funding available for release time, valuing educational research as scholarly work, and freeing classrooms for long joint classes. Best of all is if department chairs lead by forming partnerships of their own.

The point of these suggestions is that there are hundreds of ways for mathematics educators and mathematicians to work together to produce more mathematically competent teachers. It's just a matter of making the commitment and getting started.

10. Students sometimes remember only part of a rule. They might say, for instance, "two negatives make a positive." For each operation listed, decide whether the statement "two negatives make a positive" sometimes works, always works, or never works.

	Sometimes works	Always works	Never works	I'm not sure
a) Addition	1	2	3	4
b) Subtraction	1	2	3	4
c) Multiplication	1	2	3	4
d) Division	1	2	3	4

Ibid, page 24 #29.

APPENDIX: RESOURCES

The literature on mathematics education is, of course, vast. This is a small selection of books, websites and projects that members of these partnerships have found particularly useful and which might offer guidance to those starting up new partnerships.

Ma, Liping. 1999. *Knowing and Teaching Elementary Mathematics: Teachers' Understanding of Fundamental Mathematics in China and the United States.* Lawrence Erlbaum.

Ma argues that the most effective mathematics teachers have a "profound understanding of fundamental mathematics." In this book, she creates a compelling picture of what that means by doing interviews with U.S. and Chinese teachers. The Chinese teachers, despite having less formal education, generally have much greater understanding of mathematics, she finds.

Conference Board of the Mathematical Sciences. (2001). *The Mathematical Education of Teachers.* Providence RI and Washington DC: American Mathematical Society and Mathematical Association of America.

This report offers recommendations on mathematics curriculum and instruction for future teachers and describes the need for cooperation among different parties involved in educating teachers. The document is available in short form (51 pages) or as a full book (145 pages). It would be an excellent place to start for a mathematician just venturing into the education of teachers. It is available for download at www.cbmsweb.org/MET_Document/index.htm or it can be purchased in paper form from the MAA or AMS.

Conference Board of Mathematical Sciences Issues in Math Education book series

The CBMS publishes a series of high-quality books on math education: **http://www.cbmsweb.org/Issues/issues_books.htm**

Algebra Project

The Algebra Project brings mathematicians and math educators together to work on curriculum projects that are designed to support the mathematics education of underserved students: **www.algebraproject.org**

American Institute of Mathematics

AIM has a project linking math circles for teachers across the country: **http://mathteacherscircle.org/**

Association of Mathematics Teacher Educators

The AMTE holds a valuable conference each winter: **www.amte.net**

Association for Women in Mathematics Teacher Partnership

The AWM has a program which can help link mathematicians with mathematics teachers: **http://www.awm-math.org/teacherpartnership.html**

Institute for Mathematics and Education

The IME is designed to get educators and mathematicians working together. It offers many workshops, including ones designed to introduce mathematicians to key issues in math education. **www.ime.math.arizona.edu**

Mathematics Association of America and the American Mathematics Society

The MAA and AMS offer many math education sessions at their conferences: **www.maa.org and www.ams.org**

The MAA also has a special interest group on math circles for students and teachers.

Mathematical Sciences Research Institute

MSRI frequently holds conferences on math education, including one every spring on Critical Issues in Math Education: **www.msri.org**

Math Forum's Mathematicians in Math Education website

This site documents and encourages the involvement of mathematicians in grade school education and suggests a variety of low- or no-budget ways to dip a toe in the water: **www.mathforum.org.**

Lesson Study Group at Mills College

This site has good information about the techniques used in lesson study, the method the Sonoma State mathematicians and math educators used: