

Critical Issues in Mathematics Education Series, Volume 5

Teaching Undergraduates Mathematics



MSRI 
Mathematical Sciences Research Institute

May 2009 Workshop Report

Prepared by Yvonne Lai

A Workshop Publication Sponsored by MSRI with Generous Support from

*National Science Foundation
Math for America
Noyce Foundation*

Workshop Organizing Committee

<i>William McCallum</i>	<i>Deborah Loewenberg Ball</i>
<i>Rikki Blair</i>	<i>David Bressoud</i>
<i>Amy Cohen-Corwin</i>	<i>Don Goldberg</i>
<i>Jim Lewis</i>	<i>Robert Megginson</i>
<i>Robert Moses</i>	<i>James Donaldson</i>

Mathematical Sciences Research Institute

*17 Gauss Way, Berkeley, CA 94720-5070
510-642-0143 ◦ FAX 510-642-8609 ◦ www.msri.org*

Contents

Preface	iii
List of Mathematics (and Physics) Problems	v
List of Figures	vi
List of Tables	vii
1 Demands balanced in mathematics teaching	1
1.1 Articulation between high school and college	1
1.2 Relationships with other disciplines	6
Summary and further reading	13
2 What does the literature suggest?	15
2.1 Describing mathematical understanding	15
2.2 Describing problem solving	19
2.3 Knowledge for teaching	23
Summary and further reading	27
3 Portraits of teaching and mathematics	29
3.1 Teaching with inquiry, action, and consequence	29
3.2 Advanced mathematics from an elementary standpoint	31
3.3 Inquiry-oriented differential equations	33
Summary and further reading	35
4 Assessments	37
4.1 Force Concept Inventory and related diagnostic tests	38
4.2 Basic Skills Diagnostic Test and Calculus Concept Inventory	41
4.3 The Good Questions Project	45
Summary and further reading	48
5 End Notes	49
Bibliography	51

Preface

This booklet, based upon a May 2009 workshop at MSRI titled *Teaching Undergraduates Mathematics*, is written for undergraduate mathematics instructors who are curious what resources and research may support the philosophy and practice of their mathematics teaching.

Ten years ago the AMS report *Towards Excellence* argued that “to ensure their institution’s commitment to excellence in mathematics research, doctoral departments must pursue excellence in their instructional programs.” Mathematicians in all collegiate institutions share the common mission of teaching mathematics to undergraduate students, and the common problem that transitions from high school to college and from 2-year to 4-year college are challenging for many students. The success of a mathematics program depends on habits of learning and quality of instruction.



The following questions guided the workshop:

Research. What does research tell us about how undergraduate students learn mathematics? Are we listening to and learning from that research?

Curriculum. How do considerations of design and assessment of courses and programs enhance the success of our teaching? What works at different types of institution (community colleges, four-year liberal arts colleges, comprehensive universities, and research intensive universities) and different student audiences (mathematics majors, engineers, scientists, elementary teachers, business majors)?

Pedagogy. How does the way we teach influence our ability to recruit students to mathematically intensive disciplines or to retain the students we have? Can research experiences play an important role in exciting students to learn mathematics? How can technology be harnessed to help undergraduates learn mathematics and to help departments deliver instruction efficiently?

Articulation with High Schools. What mathematical knowledge, ability, and habits does a high school graduate need for success in mathematics in college? Do AP and concurrent enrollment courses lead to the same learning as their traditional on-campus counterparts? Is there a need for greater articulation of high school and collegiate mathematics? What mathematical and cultural problems do students have in their transition from high school to college, and what programs should colleges have that address these problems?



The audience for the workshop included mathematicians, mathematics educators, classroom teachers and education researchers who are concerned with improving the teaching and learning of mathematics in our undergraduate classrooms. The workshop showcased courses, programs and materials whose goal is to increase students’ knowledge of mathematics, with an emphasis on those that show promise of being broadly replicable.

Acknowledgements

I thank the organizers for putting together a lively workshop, and the numerous presenters for sharing their work. I also thank David Bressoud, Ed Dubinsky, Jerome Epstein, Karen Marrongelle, and William McCallum for providing insightful feedback, and David Auckly and Amy Cohen-Corwin for support in preparing this for publication. In writing this report, I have drawn significantly from workshop conversations and events, especially the presentations of David Bressoud, Marilyn Carlson, Bill Crombie, Wade Ellis, Jerome Epstein, Deborah Hughes-Hallett, John Jungck, Karen Rhea, Natasha Speer, Maria Terrell, and Joseph Wagner. I gratefully acknowledge the workshop presenters and participants for inspiring the contents of this report.

Structure of this booklet

Chapter 1 discusses demands balanced in mathematics teaching, in particular, the articulation between high school and college mathematics teaching and learning, and the relationship of mathematics to partner disciplines. The content focus of this chapter is calculus, which plays a central role in many collegiate programs.

Chapter 2 summarizes some findings from the mathematics education literature: ways to observe mathematical understanding, phases of mathematical problem solving, and what is entailed in teaching mathematics in the K-20 setting. This chapter showcases ideas that help us describe teaching and learning activities, so that we can better see and hear our students and ourselves.

Chapter 3 provides snapshots of teaching: using inquiry to understand algebraic concepts, teaching calculus concepts well before a formal introduction to calculus, and using inquiry to structure an ordinary differential equations class. The purpose of this chapter is to provide glimpses of teaching in action.

Chapter 4 discusses several assessment projects: the Force Concept Inventory and related diagnostic tests from physics, which inspired the creation of the Basic Skills Diagnostic Test and Calculus Concept Inventory, as well as the Good Questions Project. This chapter highlights findings from studies using these instruments.

It is often easier to understand ideas through examples. Throughout this booklet are sample problems from the projects and assessments discussed. The following page contains a list of these mathematics and physics problems.

List of Mathematics (and Physics) Problems

2.1	Bottle Problem	19
2.2	Temperature Problem	19
2.3	The Ladder Problem	19
2.4	Paper Folding Problem	21
2.5	The Mirror Number Problem	22
2.6	Pólya Problem	22
2.7	Car Problem	22
2.8	Slicing with Planes	22
2.9	Motivating Induction	23
2.10	Mathematical Knowledge for Teaching Subtraction	23
3.1	Area Problem	32
4.1	Mechanics Diagnostic Problem	38
4.2	ConcepTest (Blood Platelets)	41
4.3	Place Value	42
4.4	Proportional Reasoning (Piaget’s Shadow Problem)	43
4.5	Exponential Reasoning	43
4.6	Proportional Reasoning (Numbers Close to Zero)	44
4.7	Height Problem	46
4.8	Repeating 9’s	47
4.9	Adding Irrationals	47

List of Figures

1.1	Timeline of AP Calculus events, 1950-1987.	2
1.2	High school mathematics course enrollments, 1982-2004.	3
1.3	Fall mathematics course enrollments in 2-year colleges, 1985-2005.	5
1.4	Percentage of students in 4-year programs enrolled in calculus or above, 1985-2005.	5
1.5	Fall mathematics course enrollments in 4-year programs, 1985-2005.	5
1.6	Fall enrollments in Calculus I vs. AP Calculus exams taken, 1979-2009.	5
1.7	Fall enrollment in Calculus II, 1990-2005.	6
1.8	Fall enrollment in Calculus III and IV, 1990-2005.	6
1.9	Intended and actual engineering majors, 1980-2008.	7
1.10	Prospective engineers vs. total fall calculus enrollments, 1980-2008.	7
1.11	Intended and actual majors in selected STEM fields, 1980-2006.	8
1.14	Quantitative concepts for undergraduate biology students.	12
2.2	A geometric representation of inverse as process.	16
3.1	Set-up for inquiry questions on slope.	30
3.2	Set-up for inquiry questions about angle.	31
3.3	Diagram for the Area Problem.	32
3.4	Defining problems of elementary calculus.	32
3.5	Sequence of regions used to solve the Area Problem.	32

List of Tables

1.12	Some biological phenomena and their associated curves.	10
1.13	Biological phenomena associated to graphs.	10
2.1	Action and process understandings of function, with examples.	17
2.3	Mental actions during covariational reasoning.	20
2.4	Questioning strategies for the mental actions composing the covariational framework.	20
2.5	Approaches to the Paper Folding Problem.	21
4.1	Average Mechanics Diagnostic Test results by course and professor.	40
4.2	Force Concept Inventory pre- and post-test data.	40
4.3	Peer Instruction results using the Force Concept Inventory and Mechanics Baseline Test	42
4.4	Data from Basic Skills Diagnostic Test	44
4.5	Results of the Good Questions Project.	47

Who we teach & what we teach: Demands balanced in mathematics teaching

Teaching entails many demands. Good instruction, in addition to conveying mathematics with integrity, also

- responds to students' mathematical backgrounds, and
- serves students well for their future, inside and outside of mathematics.

Integrity, responsiveness, and service are competing principles. For example, what constitutes a good mathematical explanation depends on a students' background – the most economical or elegant explanation is not always the most accessible. Instructors of prerequisite service courses may need to negotiate mathematical coherence with the skills, habits, and dispositions needed by their students' for their future courses.

Because of its place in the curriculum, calculus is central to discussions about knowledge of students and the mathematics they know. Calculus is both an area with rich mathematical foundations as well as a course prerequisite to a host of disciplines: a calculus instructor must balance integrity, responsiveness, and service.

Section 1.1 proposes a possible agenda for improving calculus instruction. Investigating

what happens in high school calculus classrooms, as well as the motivation for taking calculus, will give perspective on the mathematical background and needs of entering college students.

Among the students we teach are future representatives of various disciplines. Section 1.2 discusses ways that mathematics and mathematics classes interact with partner disciplines in science, technology, and engineering.

1.1 *Articulation between high school and college teaching and learning*

Workshop presenter David Bressoud, then president of the Mathematical Association of America, proposed in [5] that to serve their students better, the mathematics community must:

- **Get more and better information about students who study calculus in high school:** What leads high school students to take calculus, and what are the benefits and risks to future mathematical success of having taken high school calculus classes?

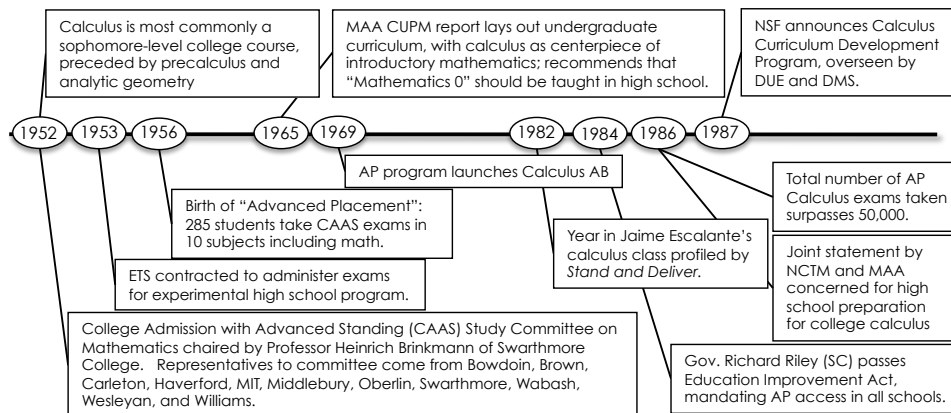


Figure 1.1. Timeline of AP Calculus events, 1950-1987. For more details about these events, see Bressoud's articles [2][3][5][4][6][7].

- **Play a role in the design, support, and enforcement of guidelines for high-school programs offering calculus:** High school calculus classes must be designed to give students a solid mathematical preparation for college mathematics.
- **Re-examine first-year college mathematics:** There must be appropriate next courses that work with and build upon the skills and knowledge that students carry with them to college, whether or not each student is ready for college freshman calculus.

This section is based upon Bressoud's articles [2][4][5][6][7], which analyze the history of calculus as a course in this country.

Section 1.1.1 summarizes how accountability, along with two complementary and at times conflicting ideals – individual enrichment and wide access – contributed to the disarticulation between high school Advanced Placement (AP) and college calculus classes.

From a demographic perspective, high school calculus enrollments have risen exponentially since the first Advanced Placement Calculus exam more than 50 years ago, while college calculus enrollments have remained steady. Section 1.1.2 discusses two NSF-sponsored studies, one with Bressoud as a Principal Investigator, which address Bressoud's above proposed agenda by identifying features of successful high school and college mathematics experiences.

Many students take AP Calculus – more than 300,000 as of 2009; and calculus is foundational in college curricula. Knowing more about AP Calculus experiences and their impact on college learning are promising ways to understand better the mathematical backgrounds and needs of entering college students.

1.1.1 *Disarticulation between high school and college calculus*

Enrollments in high school and college calculus courses are expressions of three ideals: accountability, enrichment, and access. The tensions across these ideals have contributed to the disarticulation between high school and college classes.

History and enrollment of calculus courses. The Advanced Placement (AP) programs began more than fifty years ago, when calculus was typically a college course for sophomores. At this time, some leading collegiate institutions formed the College Admission with Advanced Standing (CAAS) committee, which piloted year-long programs aimed to enrich students in selected strong high schools. The program included end-of-year exams written by what is now known as the College Board and administered through the Educational Testing Service. These programs eventually became

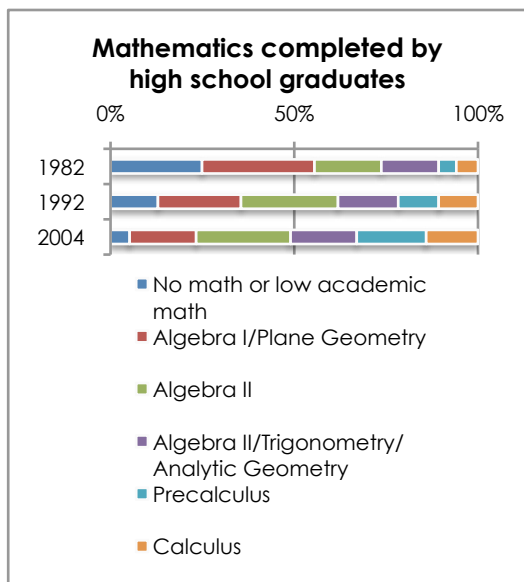


Figure 1.2. Percentage of high school graduates who completed different levels of mathematics courses in 1982, 1992, 2004. Note that in 2004, more than 75% of graduates had completed Algebra II or a more advanced course, and more than 33% of graduates had completed Precalculus or a more advanced course. Data from Dalton, Ingels, Downing, and Bozick [14, p. 13].

what is now known as Advanced Placement (AP).

Both exam taking and mathematics course enrollment have increased (see Figure 1.2). Since its inception, the number of AP Calculus exams taken has increased by several magnitudes of order. Given this dramatic shift, college mathematics course enrollments are strangely close to stagnant (see Figures 1.3-1.6) and may potentially drop (as Section 1.2 discusses).

In 2-year programs, total mathematics enrollment during the fall term has remained at roughly 25% of total enrollment in these colleges. But the percentage of mathematics enrollment in precollege mathematics has increased from 48% in 1980 to 57% in 2005 while the percentage of mathematics enrollment in calculus

and above has decreased from 9% to 6%. In 4-year undergraduate programs, total mathematics enrollment during the fall term has dropped from 20% of total undergraduate enrollment in 1980 to 15% in 2005. In 1980, 10% of all students were taking a mathematics course at the level of calculus or above in the fall term. By 2005, that was down to 6%.

Thus, across all students, enrollment increase in calculus and above has seen a modest increase, but it is close to the increase in total college enrollments.

Accountability, enrichment, and access. What might explain the simultaneous secondary expansion and tertiary stagnation?

The CAAS formed the Advanced Placement program in the 1950's to enrich students in high schools already known for intellectual strength. But, starting approximately twenty years later, the public perceived the AP program as a vehicle to find and help talented students regardless of background. (The 1982 blockbuster *Stand and Deliver* profiled Jaime Escalante's AP Calculus class.)

In 1986, the National Council of Teachers of Mathematics (NCTM) and MAA issued a joint statement warning students against taking calculus in high school with the expectation of re-taking it in college, entreating them instead to spend time mastering the prerequisites of calculus. Whether the NCTM and MAA interpreted the data accurately in the 1980's, there seems to be little effect from AP Calculus exam taking on college mathematics enrollments.

One possible explanation for this contrast is that accountability exacerbated the tension between enrichment and access. It is certainly desirable to improve access to challenging, interesting mathematics. However, AP Calculus was not designed for mass expansion. Based on conversations with students, Bressoud suspects that many students take AP Calculus and college calculus not for the mathematics, but as a step toward future employment. This suggests that calculus is viewed as a course culminating in a one-time test, rather than an opportunity for mathematics to influence lifetime learning.

Some of Bressoud's students arrived unprepared for college-level calculus and its applications. Some remaining students, despite content mastery, arrived with visceral distaste for mathematical study. Both cases are problematic.

The AP Calculus program strives to articulate with college calculus. As part of regular maintenance of the AP curriculum, the College Board periodically surveys the calculus curricula of the 300 tertiary institutions receiving the most AP Calculus scores. However, history suggests that topic lists alone cannot effect preparedness in or appreciation of mathematics.

1.1.2 Articulation between high school and college mathematics

Two studies, currently underway, support Bressoud's proposed agenda (see the beginning of Section 1.1). The Characteristics of Successful College Calculus Programs (CSCCP), an NSF-sponsored project headed by Bressoud, Marilyn Carlson, Michael Pearson, and Chris Rasmussen will examine collegiate data via a survey conducted in Fall 2010; and Factors Influencing College Success in Mathematics (FICS-Math), a study out of Harvard, will examine secondary data collected in Fall 2009.

Knowing students better. College mathematics instructors must help students overcome distaste and mischaracterization of mathematical study. A dangerous temptation is to treat students as blank slates. However, personal dispositions are not easily dislodged, even after hearing the statement of a better alternative (e.g., Confrey [10]).

Instructional interventions must be finely targeted, addressing clearly described problems with well-defined goals. The CSCCP and FICS-Math studies will give insight into college mathematics students as a whole. However, individual instructors should still engage in conversation with their own students about their motivations and background. Knowing their students better will help instructors support mathematical learning, therefore supporting students' mathematical trajectory through college.

Guidelines for calculus. History suggests that successful articulation between high school

and college calculus must go beyond lists of topics. After all, instruction does not consist of a collection of topics: it also includes interactions between students and the topics, as well as between the students and the teachers. The CSCCP and FICS-Math studies will shed light on these interactions, and how these may inform worthwhile guidelines for the design of calculus in college and high school.

Re-examining first-year college mathematics. College calculus is where mathematics departments interact with the most number and variety of students. Moreover, it is most commonly a foundation for future study or a capstone. In both cases, calculus should be an opportunity to influence the mathematical knowledge and dispositions of undergraduate students. To do so, instructors must better know their students, and the content must also be better suited to the mathematical backgrounds and needs of the students. The CSCCP and FICS-Math studies can inform the design of courses to supplement or build upon calculus that will be mathematically profitable for students.

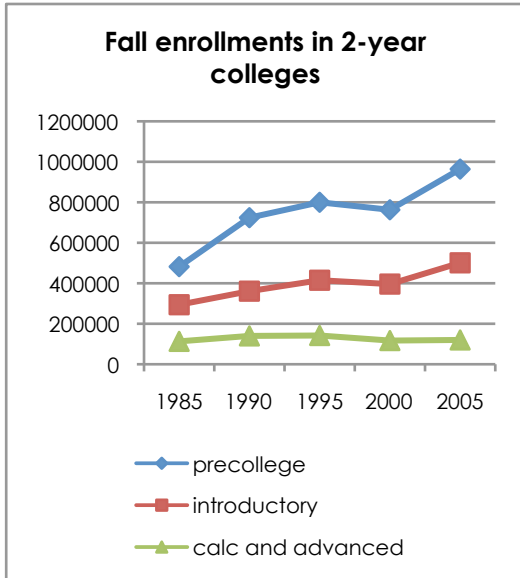


Figure 1.3. Data compiled by Bres-soud from CBMS data.

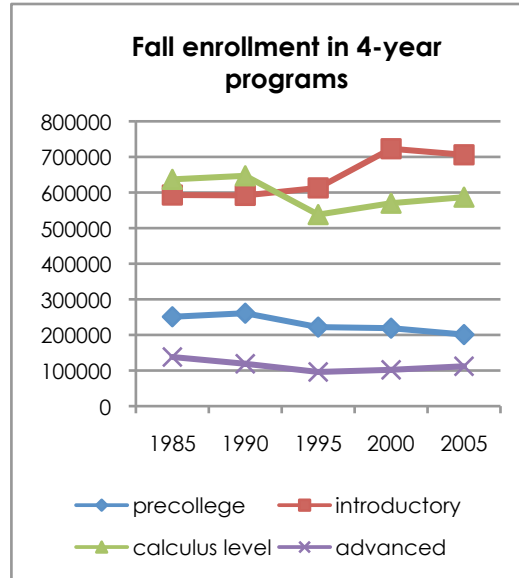


Figure 1.5. Data compiled by Bres-soud from CBMS data.

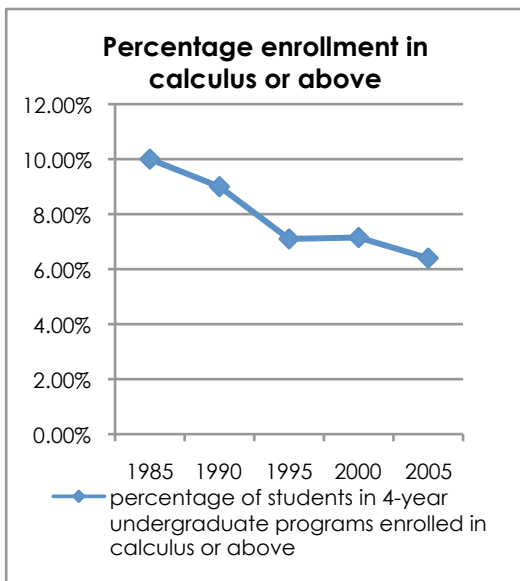


Figure 1.4. Slight drop in advanced course taking. Data compiled by Bres-soud from CBMS data.

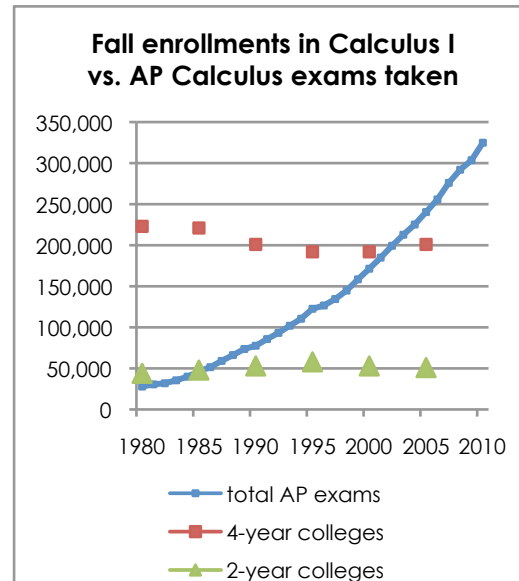


Figure 1.6. Nearly constant enrollments vs. approximately exponential exam taking. Data compiled by Bres-soud from CBMS and ETS data.

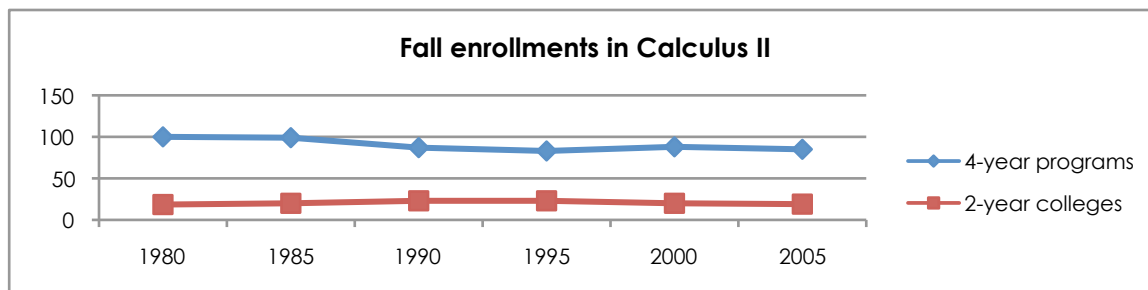


Figure 1.7. Fall enrollment in Calculus II, 1990-2005. Since 1995, there has been a 22% decrease in the number of students taking Calculus II in the Fall term in 2-year and 4-year programs.

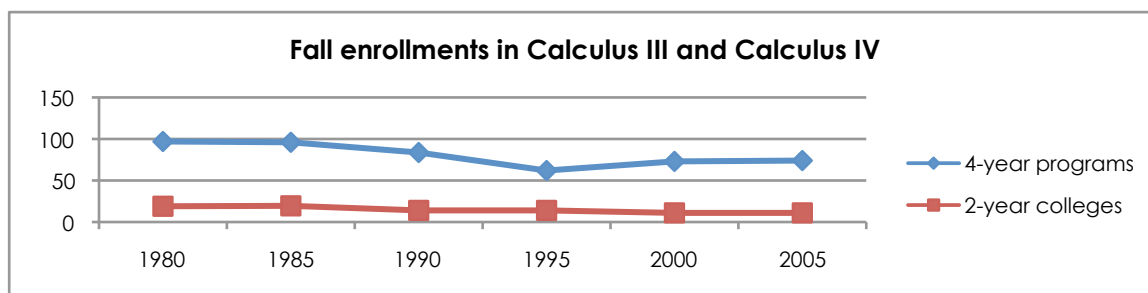


Figure 1.8. Fall enrollment in Calculus III and IV, 1990-2005.

1.2 *The role of mathematics courses: relationships with mathematics and other disciplines*

“Most of our students,” Deborah Hughes-Hallett opened her presentation, “will not go on in mathematics. Most of our students are in our classes because someone sent them there – usually not themselves.”

Calculus is a pre-requisite for the STEM fields of Engineering, Physics, Chemistry, and Mathematics. It is sometimes a pre-requisite for Computer Science, and occasionally for Economics and Biology. The data strongly suggest that the number of prospective engineering majors predicts fall calculus enrollments (see Figures 1.9 and 1.10), and this population is percentage-wise on the decline. If this trend

continues, the mathematics community should expect dropping calculus enrollment.

At the same time, over the past twenty years, prospective biological sciences majors are on the rise (see Figure 1.11). Biology undergraduate programs do not consistently require mathematics classes beyond calculus I for their majors, even though biological work uses mathematics found in Calculus I, Calculus II, and Ordinary Differential Equations.

In serving the needs of other disciplines, mathematics instructors face a disadvantage. The majority of our students are in their first two years of college, before they have taken the courses that apply the mathematics found in our courses, leaving our mathematics contextless. The students in our classes may not be able to provide feedback on how to accomplish this mission. However, by conversing with professors of their future courses, we may be able to find out more. We highlight two

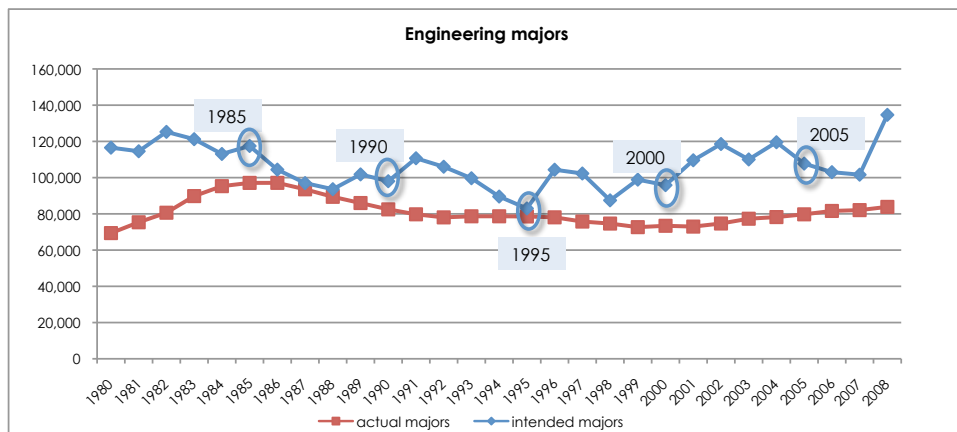


Figure 1.9. Data compiled by Bressoud for the workshop from The American Freshman and NCES data.

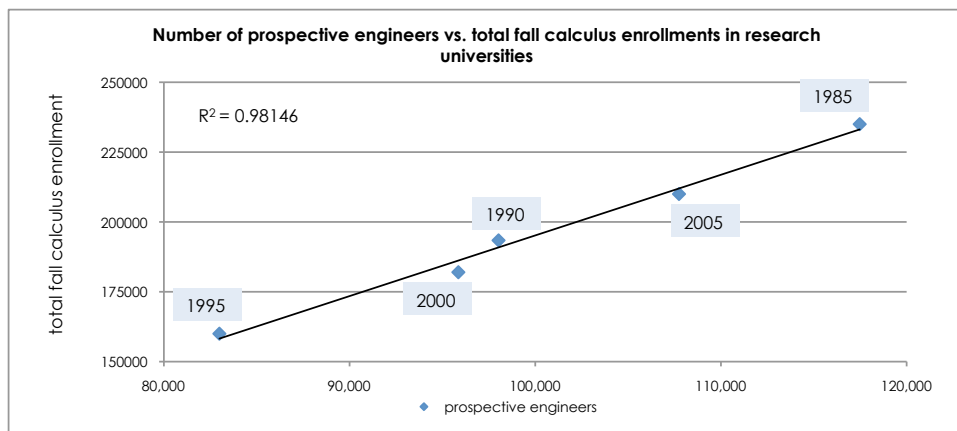


Figure 1.10. Data compiled by Bressoud for the workshop from CBMS and CIRP data.

talks, one by Deborah Hughes-Hallett on the MAA-CRAFTY (Curriculum Renewal Across the First Two Years) project, discussed by Deborah Hughes-Hallett; and one on curriculum reform efforts, by John Jungck, one of the founders of the BioQUEST Curriculum Consortium (Quality Undergraduate Education Simulations and Tools).

1.2.1 CRAFTY: Reports of conversations with partner disciplines

An “asymmetry” lies between mathematics and other disciplines. Math majors may have

taken a chemistry or physics course or two in high school, but students in these fields may well have been required to take two or more semesters of mathematics courses – in college. In general, math majors are not required to take more courses in any other particular scientific field than members of that field are required to take of mathematics courses. Thus, whether or not other disciplines have an understanding of mathematics in a way that we would characterize as accurate, it remains that they know our courses in a way that we do not know theirs.

Reflecting upon conversations with colleagues, Hughes-Hallett recommends, “The

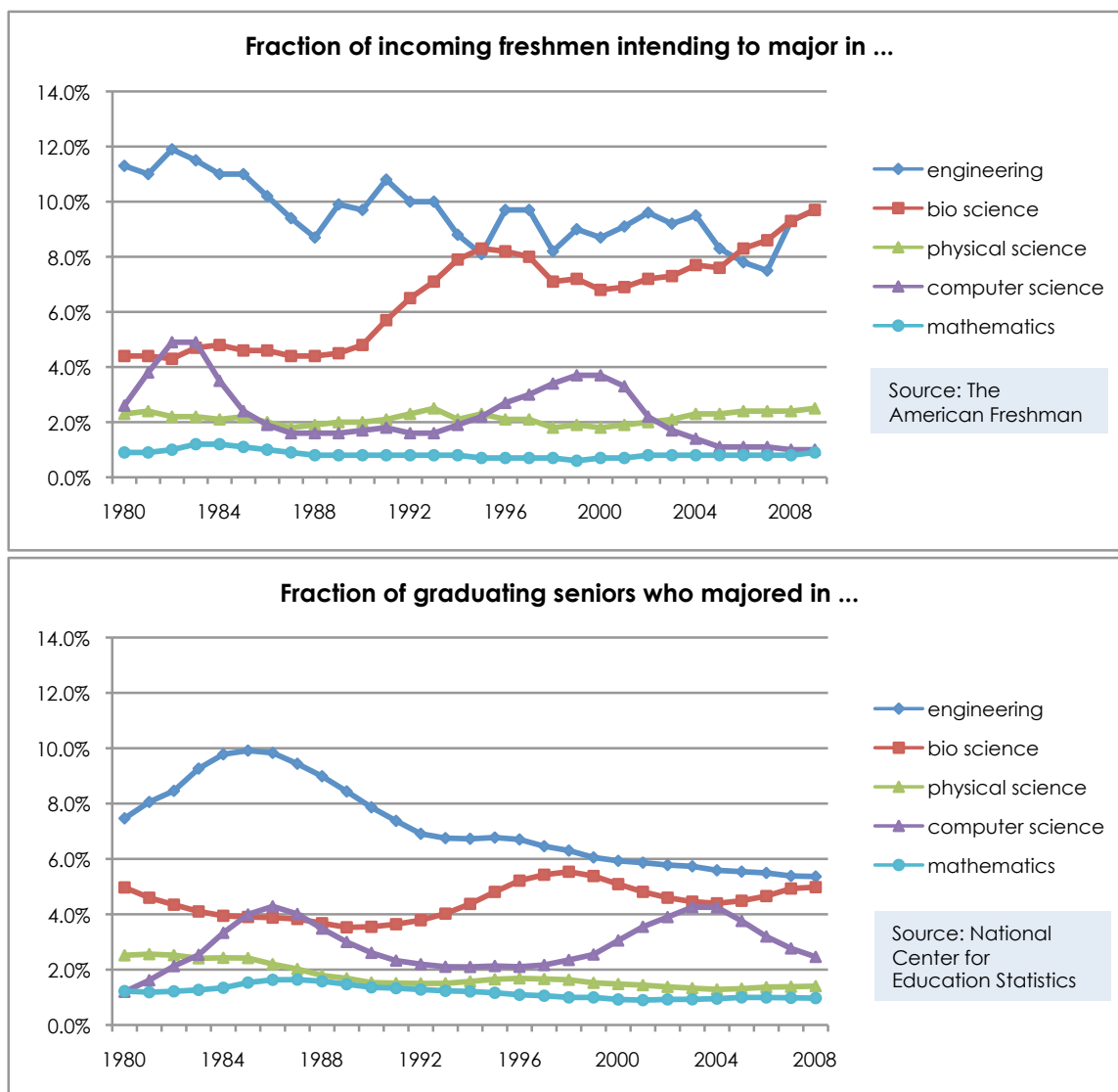


Figure 1.11. Data compiled by Bressoud for the workshop from The American Freshman and NCES data.

thing I have found most helpful is not whether they need to know this topic or that topic, because that shifts over time. Instead, what is more helpful as a common thread is to ask them what is useful about how they think about mathematics.” The MAA CRAFTY project, “Voices from the Partner Disciplines” [17], compiled reports from faculty in other disciplines on what they would like to see in mathematics

courses their students take during the first two years of college.

A few salient themes from the MAA-CRAFTY project are stances on graphing calculators and conceptual understanding.

Our partner disciplines would like to see our courses place more emphasis on approximation and estimation, and advocate spreadsheet modeling – rather than graphing calcu-

Excerpt from the CRAFTY Summary Recommendations for Understanding, Skills, and Problem Solving [17]:

Emphasize conceptual understanding:

- Focus on understanding broad concepts and ideas in all mathematics courses during the first two years.
- Emphasize development of precise, logical thinking. Require students to reason deductively from a set of assumptions to a valid conclusion.
- Present formal proofs only when they enhance understanding. Use informal arguments and well-chosen examples to illustrate mathematical structure.

There is a common belief among mathematicians that the users of mathematics (engineers, economists, etc.) care primarily about computational and manipulative skills, forcing mathematicians to cram courses full of algorithms and calculations to keep “them” happy.

Perhaps the most encouraging discovery from the Curriculum Foundations Project is that this stereotype is largely false. Though there are certainly individuals from the partner disciplines who hold the more strict algorithmic view of mathematics, the disciplinary representatives at the Curriculum Foundations workshops were unanimous in their emphasis on the overriding need to develop in students a conceptual understanding of the basic mathematical tools.

lators, which are rarely used in, for example, physics, chemistry, biology, business, engineering, or information technology. In her conversations, Hughes-Hallett has heard repeatedly that spreadsheets are consistently the “second best” technology for working on a problem, and in this way are fundamental to the toolkit of many disciplines.

As far as conceptual understanding, the skills regarded as essential by most partner disciplines include the concept of function, graphical reasoning, approximation and estimation of scale and size, basic algebraic skills, and numerical methods. For example, partner disciplines would like students to:

- “become very comfortable with the use of symbols and naming of quantities and variables” (physics),

- have an “understanding that many quantitative problems are ambiguous and uncertain” and be “comfortable taking a problem and casting it in mathematical terms” (business and management),
- “summarize data, describe it in logical terms, to draw inferences, and to make predictions” (biology),
- “formulate the model and identify variables, knowns and unknowns”, “select an appropriate solution technique and develop appropriate equations; apply the solution technique (solve the problem); and validate the solution” (civil engineering).

Thus our mathematics courses should nurture conceptual understanding, mathematical modeling, facility with applications, and fluency with symbols and graphs as a language tool.

On solution methods, almost all disciplines broached the importance of fluency in numerical solutions rather than analytical solutions. However, more intricate problems in engineering may require understanding analytical solutions so as to be able to validate numerical solutions.

Partner disciplines value computational skills. But, without a strong conceptual understanding, the computational skills become impotent. To understand this assertion, Hughes-Hallett offered a quote from her colleague Nolan Miller, a microeconomist at the Kennedy School:

“While much of the time in calculus courses is spent learning rules of differentiation and integration, what is more important for us is not that the students can take complicated derivatives, but rather that they are able to work with the abstract concept of ‘the derivative’ and understand that it represents the slope, that if $u : \mathbb{R}^2 \rightarrow \mathbb{R}$, then $-u_1/u_2$ is the slope of a level surface of the function in space.”

It may at first seem striking to separate the ability to do difficult derivatives from the ability to capture a definition as a geometric object. However, these abilities are in fact distinct. One can be quite skillful at “complicated derivatives” while lacking the ability to verbalize conceptual understanding in a precise way – and vice versa.

Table 1.12. Some biological phenomena and their associated curves. Prepared by John Jungck for this workshop.

Curve	Biological phenomena
linear	fat intake vs. cancer
log-linear	log survival vs. dose radiation
log-log	allometry
positively exponential	exercise curve vs. O ₂
negatively exponential	Newton's law of cooling
gaussian	variation
sinusoidal	heart rhythm
logistic	r, K
chaotic	tribolium
rectangular hyperbolic	Michaelis-Menten
elliptical phase	predator-prey, PV loop
hysteresis	DNA melting

Table 1.13. Biological phenomena associated to graphs. Prepared by John Jungck for this workshop.

food webs	brain circuits	metabolic pathways
pedigrees	phylogenies	fate maps
interactomes	microarray clusters	linkage maps
restriction maps	complementation maps	nucleotide sequences
protein sequences	3-D protein backbones	(or HP lattices)

1.2.2 Mathematics curricula and the biological sciences

Biological research and mathematics. As Jungck argued in his presentation, the number of biological science majors is on the rise, and mathematics and biology faculty stand to benefit from each others' expertise. Biological research depends on mathematical know-how, and mathematicians can engage students through mathematical modeling content.

Classically, understanding the *dynamics* of biological phenomena required understanding *functions* – for example, linear, exponential, chaotic, logistic functions. (See Table 1.12 for examples.) “Part of the literacy for my biology students,” Jungck observed, “is that when they see their graphs of their data coming out in these kind of forms, that they can begin to develop a simple kind of intuition. We’re not asking them to remember the equation. But these are familiar objects, an alphabet for thinking about modeling. For many mathematical biolo-

gists, having this kind of repertoire of biological examples that fit these kinds of things is kind of like a beginning kind of language.”

More recently, the mathematics relevant to biological research has had more to do with *relations* than functions, and more to do with *topology* than dynamics. Drawing a comparison with families of functions, Jungck proposed, “We can have a similar set of topologies of simple graphs that almost every biologist would immediately recognize, whether it’s a food web or a pedigree or a phylogenetic tree or a metabolic pathway, that these kinds of things are there. You have, again, an advantage. You already know our language, you already understand the topology of these kinds of systems.” (See Table 1.13 for examples.) These systems deal with relations because they often feature many-to-one and one-to-many maps, simultaneously. It is in part due to mathematics that biologists can work with this data; behind meaningful inter-

pretations of biological phenomena such as robustness or fragility are mathematics.

As early as 1996, Lou Gross proposed the option of teaching the relevant mathematics through biology departments:

It is unrealistic to expect many math faculty to have any strong desire to really learn significant applications of math that students will readily connect to their other course work, though there is a core group who might do this.

So what do we do to enhance quantitative understanding across disciplines? Below is what I say to life science faculty: Who can foster change in the quantitative skill of life science students? Only you, the biologists can do this!

Two routes:

1. *Convince the math faculty that they're letting you down*
2. *Teach the courses yourself.*

Gross [19], as quoted in Jungck [26]

The disappointment launched at mathematics faculty resulted from lack of immediate relevance of mathematics coursework to biological applications. Even if a student could in theory derive the mathematics from starting principles, it is not the ability to use basic principles that is the most critical – it is the ability to apply the mathematics after the derivations have finished. Application and derivation are distinct areas of mathematical fluency, and teaching one does not ensure expertise in the other.

BioQUEST and lessons learned. The relationship between mathematics, computer science, and biological research motivated the founding of BioQUEST (Quality Undergraduate Education Simulations and Tools), which sought deep reform of the undergraduate biology program. The BioQUEST curriculum consortium began as a collection of mathematicians, computer scientists, philosophers of science, science and math educators, biology educators, and biology researchers. In 2005, BioQUEST convened kindred programs who sought to effect change in undergraduate education, including the Harvard Calculus Consortium, Workshop Mathematics Project, Project CALC, and C*ODE*E (Consortium of ODE Experiments). At the workshop *Investigating Interdisciplinary*

Interactions: Collaboration, Community, & Connections, these programs met with others from biological sciences, computer science, statistics, and physics, among other disciplines.

John Jungck, one of the initial founders of BioQUEST, has found that discussions about an individual course or an individual department tend to be ineffective for the reform-oriented. “Frankly, if you want to change the culture to a more learner-centered student achievement, you may find your best ally in someone in a cognate discipline, and they may already be connected to a national curricular initiative. I urge you to expand your community to beyond the peers in the next-door office.” He pointed out that as partner disciplines, we write and read one another’s grants, retentions, promotions, and awards. In our academic environment, we rely on each other; our curricula and teaching should reflect this.

To borrow an idea from anthropology, popularized by Silicon Valley, we need to “cross the chasm.” Jungck advocates looking for allies in other disciplines and other schools, and to maintain a broad view. Enthusiasts must be able to work with, convince, and talk to many departments in schools of a variety of persuasions – community colleges, Research-I, liberal arts, small state schools, historically black schools, predominantly undergraduate institutions. To go beyond the “early adopters” of nuclear, local projects, and reach a national or international perspective, the earlier enthusiasts must demonstrate success in a variety of contexts.

Principles for Biology classes. Biology departments require mathematics courses, yet their coursework may not use mathematics. The National Research Council [12] supports the inclusion of more mathematics in biology courses:

Given the profound changes in the nature of biology and how biological research is performed and communicated, each institution of higher education should reexamine its current courses and teaching approaches to see if they meet the needs of today’s undergraduate biology students. Those selecting the new approaches should consider the importance of building a strong foundation in mathe-

Quantitative concepts for undergraduate biology students (Lou Gross)

Rate of change

- specific (e.g. per capita) and total
- discrete - as in difference equations
- continuous - calculus-based

Stability

- Notion of a perturbation and system response to this.
- Alternative definitions exist including not just whether a system returns to equilibrium but how it does so.
- Multiple stable states can exist - initial conditions and the nature of perturbations (history) can affect long-term dynamics

Visualizing

- there are diverse methods to display data
- Simple line and bar graphs are often not sufficient.
- Non-linear transformations can yield new insights.

Figure 1.14. Quantitative concepts used in biology (adapted from Gross [18]).

mathematics, physical, and information sciences to prepare students for research that is increasingly interdisciplinary in character. The implementation of new approaches should be accompanied by a parallel process of assessment, to verify that progress is being made toward the institutional goal of student learning.” (p. 44)

“Concepts, examples, and techniques from mathematics, and the physical and information sciences should be included in biology courses, and biological concepts and examples should be included in other science courses. Faculty in biology, mathematics, and physical sciences must work collaboratively to find ways of integrating mathematics and physical sciences into life science courses as well as providing avenues for incorporating life science examples that reflect the emerging nature of the discipline into courses taught in mathematics and physical sciences.” (pp. 47-48)

Some quantitative concepts, compiled by Lou Gross, are shown in Figure 1.14.

If the average grade of a pre-med student in a calculus class is an A, then biology classes – from lower-division to upper-division courses – should use calculus. Jungck has written that the “exclusion of equations in [biological] textbooks has three unfortunate consequences; namely, a lack of respect for, consistency with, and empowerment of students” [26, p. 13]. Without more mathematics, biology classes are guilty of the same. Using the mathematics shows respect for the discipline of mathematics as well as students’ intellectual capabilities. Currently, only upper-division courses use calculus. The lack of consistency between lower-division courses and upper-division courses causes deskilling and frustration in students. One form of empowerment is economic access, and lack of mathematics “has differential career consequences” [26, p. 13]. There is a strong, positive correlation between the amount of mathematics and computer sciences that biologists have had and their professional career opportunities and advancement (e.g., Gross [19]).

We end with a quote from Jungck.

○ ◎ ○

Go to your library and open a variety of biological journals; the diversity and richness of mathematics therein may surprise you. Why shouldn’t this literature be accessible to far more of our students?

– John Jungck , in [26].

Summary and further reading

Mathematics plays a variety of roles in the pursuit of disciplinary knowledge: it gives ways to express quantities and concepts, to approximate and estimate, to model and predict real-life phenomena, to prove, to derive, and to problem solve. Each of these domains is distinct from the rest, and expertise in one area does not guarantee expertise in the rest. Mathematics and our partner disciplines would like service courses to nurture fluency in all these domains.

Those who have been heavily invested in teaching mathematics in service courses have found that relevance and respect can help overcome mathematical fears and dislikes. Relevant material can interest students; relevant skills align with applications to the majors we serve. Respecting students must include building upon students' prior knowledge and experiences rather than ignoring or denying that students come in with ideas about content and what it means to do mathematics; respect also includes supporting a variety of future coursework in as direct a manner as possible. To respect students and teach relevant material, individuals of the mathematics community need to find out more about their students' experiences in high school, and to interact with partner disciplines at local institutions.

References and readings by presenters or recommended by presenters include the following.

○ ◎ ○

Experiences in engaging students

- *The Algebra Project*. A national, nonprofit organization that uses mathematics as an organizing tool to ensure quality public education for every child in America.
- *The Young People's Project*. Uses math and media literacy to build a network of young people who are better equipped to navigate life's circumstances, are active in their communities, and advocate for education reform in America.
- *Mathematics and Theoretical Biology Institute*. The efforts of this institute has significantly increased the national rate of production of U.S. Ph.D.'s since the inception of the institute, and recognizes the need for programmatic change

and scholarly environments which support and enhance underrepresented minority success in the mathematical sciences.

- *BioQUEST*. This project supports undergraduate biology education through collaborative development of open curricula in which students pose problems, solve problems, and engage in peer review.
- *MathForLife*. An innovative one semester terminal mathematics course intended to replace existing core or terminal courses ranging from "math-for-poets" to Finite Math whose primary audience is the undergraduate majoring in the humanities or social sciences.

Articles

- *Ten Equations that Changed Biology: Mathematics in Problem-Solving Biology Curricula*. (Article by John Jungck.)
http://papa.indstate.edu/amcbt/volume_23/
- *Meeting the Challenge of High School Calculus*. (Series by David Bressoud, as part of his online column, Launchings from the CUPM Curriculum Guide)
<http://www.macalester.edu/~bressoud/pub/launchings/>

Reports

- *BIO2010: Transforming Undergraduate Education for Future Research Biologists*. (Report by the National Research Council Committee on Undergraduate Biology Education to Prepare Research Scientists for the 21st Century.)
http://www.nap.edu/catalog.php?record_id=10497
- *Curriculum Foundations Project: Voices of the Partner Disciplines*. (CRAFTY report.)
<http://www.maa.org/cupm/crafty/>
- *Math & Bio 2010: Linking Undergraduate Disciplines*. (MAA publication, edited by Lynn Steen)
- *Quantitative Biology for the 21st Century*. (Gives concrete examples, with references, of biological research strongly influenced by mathematical and statistical sciences. Report by Alan Hastings, Peter Arzberger, Ben Bolker, Tony Ives, Norman Johnson, Margaret Palmer.)
<http://www.maa.org/mtc/Quant-Bio-report.pdf>

Teaching problem solving and understanding: What does the literature suggest?

“Procedural knowledge” versus “conceptual learning”, “teacher-directed instruction” versus “student-centered discovery”: these debates distract the community with false dichotomies and vague premises.

With this opening, Marilyn Carlson called attention back to foundational questions:

- What does it mean for students *to understand* a mathematical idea?
- What are *problem solving abilities and processes* for mathematics learners?
- What is the *nature of the knowledge that teachers need to have*?

This chapter summarizes and elaborates upon Carlson’s presentation.

A challenge to common ground on “understanding” is that many topics in mathematics have no widely accepted specification on what it means “to understand”. Promisingly, there are key topics of secondary and tertiary mathematics whose learning has been examined in detail. One such topic is (real) functions. This chapter discusses two alternative characterizations of understanding functions, Action-Process-Object-Schema (APOS) Theory and Co-

variation. In its treatment of APOS Theory, this chapter focuses on Action and Process.

With respect to the second question, various researchers and mathematicians have studied the teaching and learning of problem solving. To support problem solving in mathematics classes, this chapter describes stages of problem solving as examined by Carlson and her colleagues. This work builds upon literature by Pólya and Schoenfeld among others.

Finally, there is currently no broad consensus on the nature of the knowledge needed for teaching, which is problematic for TA training programs as well as K-12 teacher preparation programs. We discuss research on the mathematical knowledge entailed in teaching, including research on tertiary instruction presented by Natasha Speer and Joe Wagner.

2.1 Describing mathematical understanding: Functions

Algebra is a gateway class: completing mathematics beyond the level of Algebra II correlates significantly with enrollment in a four-year col-

lege and graduation from college (e.g., National Mathematics Advisory Panel, [29, p. 4]).

At the heart of school algebra are functions, especially linear, quadratic, and exponential functions. Two characterizations of understanding functions prevalent in the literature on undergraduate mathematics are *Action-Process-Object-Schema (APOS) Theory* and *Covariation*. Mathematicians may be interested in these ideas as ways to help observe and assess their students' thinking.

2.1.1 Action and process understandings

"Action" and "process" are part of Dubinsky's APOS Theory (Action-Process-Object-Schema; see Dubinsky and McDonald [15] for an introduction). There are four stages to Dubinsky's theory, inspired by Piaget's developmental theories on children's learning; this section concentrates on the first two stages, Action and Process.

Although this chapter as a whole focuses on algebraic concepts, Section 2.1.1 provides examples from exponential expressions and group theory as well, intending that a greater variety of examples will provide more leverage for readers to apply APOS Theory to their own teaching.

Examples regarding functions in the text and the tables are from Oehrtman, Carlson, and Thompson [28] and Connally, Hughes-Hallett, Gleason, et al. [11]. Examples regarding exponential expressions are from Weber [39]. Examples regarding group theory and the descriptions of action and process stages are from Dubinsky and McDonald [15].

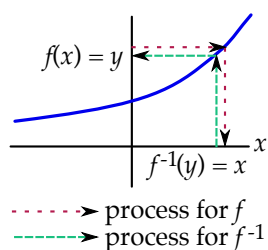


Figure 2.2. A geometric representation of inverse as process.

Action. An *action* on a set of mathematical objects is a step-by-step transformation of the objects to make another mathematical object or objects. A student in the *action stage* of understanding an object can likely, for instance, perform algorithmic computations on those objects. The student also likely needs prompting to take the action.

For example, in the action stage of understanding a particular function f or g expressed in terms of x , students can likely evaluate $f(x)$ or even $g(f(x))$ for given x . However, students may not be able compose functions whose data is given to them only through tables and graphs (e.g., see Table 2.1). As well, the understanding of functions as primarily step-by-step manipulations comes with implications for understanding of graphs, inverses, and domain and range.

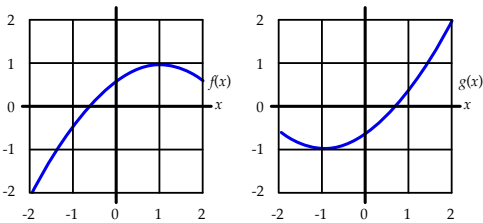
In the case of exponential expressions, students can view 2^3 as repeated multiplication of 2, but may not be able to make sense of non-integral exponents or logarithms.

In the case of group theory, students can compute the left cosets of $\{0, 4, 8, 12, 16\}$ in $\mathbb{Z}/20\mathbb{Z}$ by adding elements of the whole group to elements of subgroup. However, such students may encounter difficulty with more intricate structures, such as for cosets of D_4 , the symmetry group of a square within a permutation group such as S_4 . Students may be able to compute through brute force, but would not be likely to find efficient, holistic techniques.

Process. When a student repeats an action and reflects upon it, they internalize the action into a *process*, which may no longer need external prompting to perform. "An individual can think of performing a process without actually doing it, and therefore can think about reversing it and composing it with other processes" [15, p. 276].

In the *process stage* of understanding functions, students can likely find simple compositions from tables and graphs; as well, the concepts of injectivity, inverse function, and domain and range are more accessible. Examples are provided in Tables 2.1, and a geometric representation of inverse as process is provided in Figure 2.2.

Table 2.1. Action and process understandings of function (adapted from Oehrtman, Carlson, and Thompson [28]). Each understanding is followed by examples of the types of problems (adapted from [28] and Connally, Hughes-Hallett, Gleason, et al. [11]) that a student in that stage could likely complete.

Action understanding	Process understanding
Working with functions requires the completion of specific rules and computations.	Working with functions involves mapping a set of input values to a set of output values; it is possible to work with a space of inputs rather than just specific values.
Inverse is about algebraic manipulation, for example, solving for y after switching y and x ; or it is about reflecting across a diagonal line.	Inverse is the <i>reversal of a process</i> that defines a mapping from a set of output values to a set of input values.
Finding the domain and range is at most an algebraic manipulation problem, for example, solving for when the denominator is zero, or when radicands are negative.	Domain and range are produced by operating and thinking about the set of all possible inputs and outputs.
○ ◎ ○	○ ◎ ○
<i>Examples of problems solvable with an action understanding:</i>	<i>Examples of problems solvable with process understanding:</i>
Find $h(y)$, where $h(y) = y^2$, and $y = 5$.	Express $(f \circ g)^{-1}$ as a composition of the functions f^{-1} and g^{-1} .
Find $f(g(x))$ for $f(x) = 4x^3$, $g(x) = x + 1$, and $x = 2$.	Simplify $\cos(\arcsin t)$ using the notion that an inverse “undoes”.
Given $f(x) = \frac{2x+1}{7-x}$, find $f^{-1}(x)$.	A sunflower plant is measured every day t , for $t \geq 0$. The height, $h(t)$ centimeters, of the plant can be modeled with
Given the graph of $f(x)$, sketch a graph of $f^{-1}(x)$.	$h(t) = \frac{260}{1 + 24(0.9)^t}$
Find the domain and range of $f(x) = \frac{\sqrt{1+x}}{x+3}$.	What is the domain of this function? What is the range? What does this tell you about the sunflower’s growth? Explain your reasoning.*
If the graph of an invertible function is contained in the fourth quadrant, what quadrant is the graph of its inverse function contained in?	Use the figures below to graph the functions $f(g(x))$, $g(f(x))$, $f(f(x))$, $g(g(x))$.*
	

*These problems are adapted from Connally, Hughes-Hallett, Gleason, et al., *Functions Modeling Change: A Preparation for Calculus*, §2.2: Example 3, and §8.1: Problems 27-30, ©2006, John Wiley & Sons, Inc.

In the case of exponential expressions, a student can likely interpret b^x as “the number that is the product of x factors of b ” and $\log_b m$ as “the number of factors of b that are in the number m ” [39].

In the case of left cosets, the student can likely find at least two elements $g, h \in S_4$ not in the subgroup D_4 and so that g and h represent distinct left cosets.

Applying the notions of action and process understandings to teaching. APOS Theory can guide in-class activities, exam problems, or homework. Below are several recommendations to help students advance from action understanding to process understanding. Suggestions on teaching functions are taken from [28] unless otherwise noted.

- Ask students to explain basic function facts in terms of input and output.

Examples. (a) Ask students to explain their reasoning for whether $(f \circ g)^{-1}$ equals $f^{-1} \circ g^{-1}$ or $g^{-1} \circ f^{-1}$.

(b) In addition to questions such as “Solve for x where $f(x) = 6$ ”, ask students to “find the input value(s) for which the output of f is 6”, both algebraically and from a labelled graph of the function, and to explain their reasoning.

- Ask about the behavior of functions on entire intervals in addition to single points.

Examples. (a) Ask students to find the image of a function applied to an infinite-cardinality set (such as an interval), e.g., find the length of $f(g([1,2]))$, where $f(x) = 2x + 1$ and $g(x) = 4x - 3$.

(b) Ask students to find the preimage of an interval in the context of the definition of limit or continuity.

- Ask students to make and compare judgements about functions across multiple representations, that is, how a function is introduced or what information students are given about the function.
- Ask students to describe symbols as mathematical objects.

Examples from [39], with desired student responses given in **bold**. (a) Describe each of the exponential expressions in terms of a product

and in terms of words:

$$\begin{aligned} 4^3 &= 4 \times 4 \times 4 \\ &= \text{the number that is the product of} \\ &\quad \text{3 factors of 4} \end{aligned}$$

$$\begin{aligned} b^x &= b \times b \times b \times \dots \times b \quad (x \text{ times}) \\ &= \text{the number that is the product of} \\ &\quad x \text{ factors of } b \end{aligned}$$

(b) Simplify each of the expressions below by writing each exponential term as a product. Summarize each simplification in words.

$$b^2 b^4 = b \times b \times b \times b \times b \times b = b^6$$

The product of 2 factors of b and 4 factors of b is 6 factors of b .

$$bb^x = b \times (\underbrace{b \times b \times b \times \dots \times b}_{x \text{ times}}) = b^{x+1}$$

The product of b and x factors of b is $(x+1)$ factors of b .

- Incorporate computer software packages that help students visualize or experiment with mathematical concepts, and use computer programming to help students reflect upon actions. A description of a number of studies in which computer software and programming aided student learning can be found in Dubinsky and Tall [16]; in fact, the examples on exponential expressions, from [39], are part of a study which included MAPLE programming activities for the students.

2.1.2 Covariational reasoning

The Oxford English Dictionary defines *covariant* as, “Changing in such a way that interrelations with another simultaneously changing quantity or set of quantities remain unchanged; correlated.” In studying students’ learning of functions, Carlson has focused on helping students relate dependent quantities. This section presents some of her findings, especially from Carlson, Jacobs, Coe, Larsen, and Hsu [9] and Oerhtman, Carlson, and Thompson [28].

In [9], *covariational reasoning* is described as the “cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other”(p. 354), for example, viewing $(x, y) = (t, t^3 - 1)$ as expressing a relationship

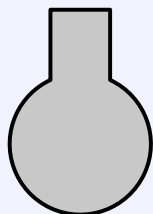
where x and y can both change over time, and changes in x may come with changes in y .

Covariational reasoning means attending to co-varying quantities in contexts such as parametric equations, physical phenomena, graphs, and rates of change. Precalculus, calculus, multivariable calculus and differential equations all feature simultaneously varying quantities.

The following are three examples of problems, from [9] and [28], whose solutions entail covariational reasoning.

Problem 2.1: Bottle Problem.

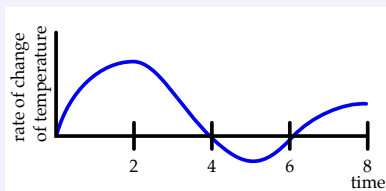
Imagine this bottle filling with water. Sketch a graph of the height as a function of the amount of water that is in the bottle.



In this case, quantities to attend to are height and volume of water. Covariation appears through applying concepts related to rate of change and convexity.

Problem 2.2: Temperature Problem.

Given the graph of the rate of change of the temperature over an 8-hour time period, construct a rough sketch of the graph of the temperature over the 8-hour time period. Assume the temperature at time $t = 0$ is zero degrees Celsius.



Here, quantities to attend to are the rate of change and the original function. Covariation

appears through the interpretation of critical points, positive slopes, and negative slopes.

Problem 2.3: The Ladder Problem.

From a vertical position against a wall, a ladder is pulled away at the bottom at a constant rate. Describe the speed of the top of the ladder as it slides down the wall. Justify your claim.

Here, quantities to attend to are the speed of the top of the ladder and the placement of the bottom of the ladder.

One way that studies in math education can serve mathematics instructors is elaborating what it means to “understand”, and how students arrive at understanding. Observations of students working on problems similar to the above suggest that covariational reasoning decomposes into five kinds of mental action; this led Carlson, Jacobs, Coe, Larsen, and Hsu to develop interventions that improved calculus students’ covariational reasoning abilities [9]. The mental actions are summarized in Tables 2.3-2.4.

Ways suggested in [9] to enhance students’ covariational reasoning may include:

- Ask for clarification of rate of change information in various contexts and representations. For example, ask students to provide interpretations about rates in real-world contexts, given algebraic or graphical information. Probe further if students do not incorporate all variables in their explanation, and the relationship between the variables. If students use phrases such as “increases at a decreasing rate”, ask them to explain what this means in more detail.
- Ask questions associated with each of the mental actions. Questioning strategies are found in Table 2.4 for discussing rates of changes, a concept foundational to calculus and differential equations.

2.2 Describing problem solving

Mathematics instructors often would like their students to be problem solvers: to celebrate mathematical tasks that are not immediately

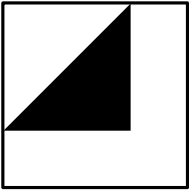
Table 2.3. Mental actions during covariational reasoning (adapted from [28, p. 163]). Behaviors are those observed in students working on the Bottle Problem.

mental action	description of mental action	behaviors
Mental Action 1 (MA1)	Coordinating the dependence of one variable on another variable	Labeling axes, verbally indicating the dependence of variables on each other (e.g., y changes with changes in x)
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable.	Constructing a monotonic graph Verbalizing an awareness of the direction of change of the output while considering changes in the input.
Mental Action 3 (MA3)	Coordinating the amount of change in one variable with changes in the other variable.	Marking particular coordinates on the graph and/or constructing secant lines. Verbalizing an awareness of the rate of change of the output while considering changes in the input.
Mental Action 4 (MA4)	Coordinating the average rate-of-change of the function with uniform increments of change in the input variable.	Marking particular coordinates on the graph and/or constructing secant lines. Verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input.
Mental Action 5 (MA5)	Coordinating the instantaneous rate-of-change of the function with continuous changes in the independent variable for the entire domain of the function.	Constructing a smooth curve with clear indications of concavity changes Verbalizing an awareness of the instantaneous changes in the rate-of-change for the entire domain of the function (direction of concavities and inflection points are correct).

Table 2.4. Questioning strategies for the mental actions composing the covariational framework (adapted from [28, p. 164]).

mental action	questioning strategy
MA1	What values are changing? What variables influence the quantity of interest?
MA2	Does the function increase or decrease if the independent variable is increased (or decreased)?
MA3	What do you think happens when the independent variable changes in constant increments? Can you draw a picture of what happens (at intervals) near this input? Can you represent that algebraically? Can you interpret this in terms of the rate of change in this problem?
MA4	Can you compute several example average rates of change, possibly using the picture to help you? What units are you working with? What is the meaning of those units?
MA5	Can you describe the rate of change of a function event as the independent variable continuously varies through the domain? Where are the inflection points? What events do they correspond to in real-world situations? How could these points be interpreted in terms of changing rate of change?

Table 2.5. Two approaches to the Paper Folding Problem.

<p style="text-align: center;">Approach 1</p> <p>Let x be the interior altitude of the black triangle. Then its area is $\frac{1}{2}x^2$, and the area of the lower white region is $3 - x^2$. The areas of the white and black regions are equal, so $3 - x^2 = \frac{1}{2}x^2$. It follows that $x = \sqrt{2}$.</p>		<p style="text-align: center;">Approach 2</p> <p>The three regions in the figure below have equal area; since the square has total area 3, the black triangle and white triangle form a square of area 2. Hence the diagonal of this square has length $2\sqrt{2}$. The length sought is half the diagonal, or $\sqrt{2}$.</p>
---	---	---

solvable, to confront these tasks with a clear head, and to think carefully and completely.

The mathematician Pólya, in his famous *How to Solve It* [30], characterized four stages:

- understanding the problem – “we have to see clearly what is required”
- developing a plan – “we have to see how the various items are connected, how the unknown is linked to the data, in order to obtain an idea of the solution, to make a plan.”
- carrying out the plan
- looking back – “we look back at the completed solution, we review it and discuss it.”

Almost all research on mathematical problem solving traces back to Pólya’s insight.

Though Pólya’s stages have much face validity to mathematically-savvy problem solvers, carrying them out in practice has remained mathematically and affectively difficult for students in general. One of the key players in mathematical problem solving research is Alan Schoenfeld, who developed a course that improved undergraduates’ problem solving abilities by focusing on metacognitive processes, especially the relationship between students’ beliefs and their practices. The emphasis on metacognitive processes was inspired by emergent artificial intelligence research of the time, and has continued to shape research since.

The previous section in this chapter, Section 2.1, addressed conceptual knowledge; the present section discusses the relationship between conceptual knowledge, affect, and problem solving, as suggested by Carlson and Bloom [8]. The purpose of this section, as with Section 2.1, is to provide description that can

help instructors sharpen observations of student thinking, and to provide language that facilitates conversation among colleagues around teaching and learning.

2.2.1 Conceptual knowledge and emotion

Carlson and Bloom observed a dozen mathematicians and PhD candidates solving a collection of elementary mathematics problems designed to evoke problem solving processes. The collection included the bottle problem (from Section 2.1.2), and the Paper Folding Problem.

Problem 2.4: Paper Folding Problem.

A square piece of paper ABCD is white on the frontside and black on the backside and has an area of 3 in². Corner A is folded over to point A' which lies on the diagonal AC such that the total visible area is half white and half black. How far is A' from the fold line?

Two approaches to the problem are summarized in Table 2.5.

Carlson and Bloom found that Pólya’s stages could delineate stages of behavior observed in their participants. So, within each stage, Carlson and Bloom examined how problem solvers used conceptual knowledge, applied heuristics, exhibited motivation and emotion, and reflected upon their own work.

Conceptual knowledge and problem solving. Mathematicians do powerful planning when problem solving. Carlson noted in her presentation that the mathematicians studied in

[8] often exhibited rapid 20-minute cycles, stopping to ask themselves about a particular line of reasoning: “Hmm ... should I do that? Maybe I should plug in some numbers. If I do that, then I will get this relationship between the triangle and the square ...” This sort of construction and evaluation expertly applies conceptual knowledge of area and triangles. When the problem solver who used Approach 2 initially went off track, his knowledge of area allowed him to get unstuck.

The detail and organization in planning reminded Carlson of mathematicians discussing precalculus material. Carlson observed in her presentation, “It’s one thing to say ‘distance’. It’s another thing to say, ‘ d is the number of miles that it takes that it takes to drive from Phoenix and Tucson.’” Whatever the mathematical level, problem solving entails thoughtful, non-linear processes that draw upon careful connections to conceptual knowledge.

Conceptual knowledge was also used to verify answers. The successful problem solvers in [8] used their understanding of areas and geometry to check results and computations, for example, by making certain that the areas of the regions totaled to 3.

Emotion and problem solving. Carlson and Bloom were struck by the intimacy of the solving process. In their participants, they observed frustration, joy, and the pursuit of elegance. Successful problem solving entails effective management of emotion, especially to persist through many false attempts.

2.2.2 Sample problems

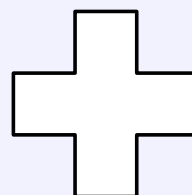
For readers’ interest, the following were other tasks used in the study reported in [8].

Problem 2.5: The Mirror Number Problem.

Two numbers are “mirrors” if one can be obtained by reversing the order of the digits (i.e., 123 and 321 are mirrors). Can you find: (a) Two mirrors whose product is 9256? (b) Two mirrors whose sum is 8768?

Problem 2.6: Pólya Problem.

Each side of the figure below is of equal length. One can cut this figure along a straight line into two pieces, then cut one of the pieces along a straight line into two pieces. The resulting three pieces can be fit together to make two identical side-by-side squares, that is a rectangle whose length is twice its width. Find the two necessary cuts.



Problem 2.7: Car Problem.

If 42% of all the vehicles on the road last year were sports-utility vehicles, and 73% of all single car rollover accidents involved sports-utility vehicles, how much more likely was it for a sports-utility vehicle to have such an accident than another vehicle?

2.2.3 Intellectual need as motivation

We end the section on problem solving with an argument for “intellectual need” and a nod to Pólya’s teaching.

In his presentation on problem solving, David Bressoud commented on materials by Pólya and Guershon Harel introducing induction. A classic problem used by Pólya in *Let us teach guessing* (now a DVD, originally on film in 1966) is the following.

Problem 2.8: Slicing with Planes.

How many regions that one plane can slice \mathbb{R}^3 into? 2 planes? 3 planes? n planes?

In this situation,

- 1 plane creates 2 regions
- 2 planes can create 4 regions
- 3 planes can create 8 regions.

Many students guess that 4 planes could create 16 regions – rather than the maximum of 15. Expectations are overthrown. At this moment, Pólya argues, learning can happen.

The “intellectual need” motivating genuine mathematical reasoning has also been discussed by Guershon Harel. In a recent MAA workshop, Harel characterized backwards teaching, which begins by generic outlines of techniques rather than a situation to motivate utility. For example, backwards teaching of induction might begin by discussing row of dominoes, and how knocking the first cascades the rest down. However, this theoretical description will be meaningless to someone who has never needed proof by induction. Only providing examples where the induction statement is explicitly in the problem statement, e.g., showing $\sum_{k=1}^n n^2 = \frac{n(n+1)(2n+1)}{6}$, exacerbates the situation. When taught this way, students tend to look for an “ n ” in the problem, and miss opportunities to use proof by induction when there is no apparent “ n ”, even in contexts where induction provides a productive approach.

Harel proposes opening with problems with implicit inductive statements, for example:

Problem 2.9: Motivating Induction.

Find an upper bound for the following sequence.

$$\sqrt{2}, \quad \sqrt{2 + \sqrt{2}}, \quad \sqrt{2 + \sqrt{2 + \sqrt{2}}}, \dots$$

The idea of building upon a previous case is something that students understand intuitively; the domino analogy only tells them something they feel they already know, without anchors to any mathematics. But providing a problem where the students must focus on *how* to build – rather than merely the fact that something is being built – serves students’ needs. This problem provides intellectual need to articulate patterns in terms of previous patterns, motivating the use of an index variable as well as recursive forms.

2.3 Knowledge for teaching

Mathematics teachers teach mathematics; and teaching entails skills beyond the content aimed to students. For example, consider this question, about subtraction in Thames [37]:

Problem 2.10: Mathematical Knowledge for Teaching Subtraction.

Order these subtraction problems from easiest to hardest for students learning the standard subtraction algorithm, and explain the reasons for your ordering: (a) $322 - 115$ (b) $302 - 115$ (c) $329 - 115$.

Engaging in this sort of reasoning unprompted is an instance of mathematical thinking that skillful teaching requires, and which we do not require of students.

As an example from college algebra, from [37]: what difficulties might solving the following linear system present to a student?

$$\begin{aligned} y &= 3x - 1 \\ y &= -5x + 2. \end{aligned}$$

Students often struggle with working with fractional expressions. If, as in this case, the two distances are integral and relatively prime, fractions will arise. Here, the y -intercepts have distance 3 from each other, and the slopes differ by 8. The intersection point is

$$\left(\frac{3}{8}, \frac{1}{8}\right).$$

Because the difference between y -intercepts is relatively prime to the difference between the slopes, the x -coordinate will not simplify, and so a student will need to combine fractions to solve for the y -coordinate.

On the other hand, suppose the difference between slopes is a factor of the distance between intercepts, as in

$$\begin{aligned} y &= 3x - 7 \\ y &= -5x + 9, \end{aligned}$$

whose intersection is

$$(2, -1).$$

Solving for the y -coordinate will only involve integral computation, which students generally have less difficulty with. Knowing how the relationship between distances between y -intercepts and differences slopes affects the algebraic manipulations is another instance of the mathematical knowledge entailed in teaching.

Further examples of knowledge entailed in college algebra teaching are suggested by the following questions:

1. What kinds of functions do students have difficulty finding the inverse of?
2. What are the typical difficulties that students stumble into when trying to describe the meanings of expressions such as $f^{-1}(5)$, given the verbal description of f (e.g., f is the height of a football in feet, t seconds after it was tossed)?
3. Given that students may have difficulty in expressing the meaning of expressions such as $f^{-1}(5)$, what are ways of scaffolding the construction of such descriptions?
4. What ideas about inverse functions do each of the following notions of inverse function obscure or highlight: “undoing”, “swapped inputs and outputs”, “graph reflected across the line $y = x$ ”?
5. What counterexamples should be chosen to demonstrate non-invertible functions?

Considerations for these questions may involve knowledge about symbolic manipulation – for instance, finding the inverse of f when $f(x) = \frac{2x}{x-1}$ poses algebraic difficulties for some students; using variables other than x and y , as students sometimes have difficulty conceiving of input and output in terms of non-standard variables; and situations that call for expressing f^{-1} in terms of y if students have already solved for f^{-1} in terms of another variable such as x . As well, college algebra teaching entails deciding on representations: whether to use tables, verbal description, or graphs to communicate the function; whether or not to use a continuous function; and being careful about trigonometric functions, which have defined inverse functions only after taking a restriction.

Skillfulness around potential minefields is part of purposeful teaching: the more sensitive teachers are to differences in the solution steps, the more effectively they can minimize distract-

tions, choose examples to illustrate a point, or decide when to initiate discussion over procedures that are difficult for students but incidental to the main results.

2.3.1 *Knowing a discipline for teaching: several perspectives*

The notion that there is a distinction between the ways that teachers must know their discipline and the ways that other practitioners must know the discipline can be traced back to Lee Shulman.

Although he does not directly address mathematics, his writings have influenced the work on mathematics teacher education, and his ideas have analogues within mathematics. For example, Shulman [33] discusses three views on biology: as a science of molecules, as a science of ecological systems, and as a science of biological organisms. He argues that the “well-prepared biology teacher will recognize these and alternative forms of organization and the pedagogical grounds for selecting one under some circumstances and others under different circumstances”(p. 9). Examples from college algebra of specialized knowledge for teaching might include knowing how to demonstrate that an inverse function “undoes” an original function in algebraic, tabular, and graphical examples; or identifying the functions implicit in a verbal description of a physical phenomenon.

Shulman’s work was one inspiration for the work of Deborah Ball on Mathematical Knowledge for Teaching (MKT). In the past three decades, Ball and her colleagues have conducted research on Mathematical Knowledge for Teaching (MKT), the mathematics entailed in the work of teaching (for a more detailed description of MKT, see, e.g., Ball, Hill, and Bass [1]). Ball has focused her work on the elementary level, and proficiency with MKT has been demonstrated to be measurable and correlated with student achievement gains for first and third graders (see Hill, Rowan, and Ball [25]). One focus of MKT is distinctively mathematical work of teaching – for example, knowing a variety of ways a concept could be depicted mathematically, analyzing student errors and non-

standard methods, or selecting and sequencing problems to teach a particular idea.

Patrick Thompson, whose work has primarily examined secondary and collegiate teaching, has proposed a complementary line of research, focusing on mathematical understandings that carry through a sequence of mathematics courses. Thompson motivated this viewpoint with three observations: that improving and sustaining students' high-quality mathematical learning is the reason that mathematics educators exist; that most students rarely experience "significant" mathematics – "ideas that carry through an instructional sequence, that are foundational for learning other ideas, and that play into a network of ideas"; and that mathematics teachers tend to spend too little time attending to meaning, wanting to rush to "condense rich reasoning into translucent symbolism" and get to the methods and main results of the day (see Silverman and Thompson [34], or Thompson [38]).

As an example of significant mathematical ideas, Thompson used base ten numeration. A student who is fluent with the meaning behind base ten and place value should be able to, without prior practice on similar problems, reason through a question such as "How many hundreds are there in 35821?" and – whether they arrive at 358 or 358.21 – explain their answer.

Thompson [38] argued, "The issue of coherence is always present in any discussion of ideas. Ideas entail meanings, meanings overlap, and incoherence in meanings quickly reveals itself" (p. 46). The meaning of a mathematical idea is not simply the definition – for example, explaining *average speed* as "distance divided by time" to someone who does not already understand the concept may not help them recognize average speed in context. The meaning of average speed must include the following:

- It involves a complete trip or the anticipation of a complete trip (i.e., having a start and an end).
- The trip takes or will take a path which involves moving a definite distance in a definite amount of time.
- The average speed for that trip is the constant speed at which someone must travel to cover the same distance in the same amount of time.

Understanding why this definition translates to distance "divided by time" requires powerful understanding of constant rate. "When students understand the ideas of average rate of change and constant rate of change with the meanings described here they see immediately the relationships among average rate of change, constant rate of change, slope, secant to a graph, tangent to a graph, and the derivative of a function. They are related by virtue of their common reliance on meanings of average rate of change and constant rate of change." [38, p. 52].

2.3.2 *Mathematical knowledge for leading collegiate mathematical discussions*

On the college level, Natasha Speer has devoted her research to the professional development of mathematics instructors and teaching assistants. Speer and Joseph Wagner presented recent research on the mathematical knowledge entailed in leading discussion-oriented classrooms, reported in their work [35]. Two types of scaffolding that occur in these settings are *social scaffolding* to keep a discussion moving, and *analytic scaffolding*, which keeps a discussion moving toward a mathematically productive direction. Ineffective instruction arises from an inappropriate balance of the two types. A overemphasis on the social scaffolding can leave students buzzing with ideas, but not knowing where they have gone. On the other hand, blind pursuit of a particular mathematical direction can lead students to a correct answer that they do not appreciate or understand. Speer and Wagner focused on the ways of knowing mathematics to carry out *analytic scaffolding*.

Speer and Wagner studied the question, "What does a teacher need to do and know to be able to recognize a student's contribution as productive?" The setting for their research was a university class of 21 students. The professor, who is a mathematician, used the inquiry-oriented curriculum developed by Chris Rasmussen, which emphasizes conceptual understanding through group activities, problem solving, and extensive discussion to formalize the methods and the

ory in the class. The mathematician who taught the class has seen all the data and participated in the analysis. The data collected includes video recordings of almost all classes and audio recordings of de-briefing sessions held after almost every class with the mathematician, Speer, and Wagner.

As an example of the sort of knowledge needed in practice, consider the following episode from the professor's class. The class is discussing the equation

$$\frac{dP}{dt} = 2 - \frac{P}{10+t},$$

which the professor has rewritten as

$$(10+t)\frac{dP}{dt} + P = 2(10+t). \quad (2.1)$$

By this time, the class had solved problems by separation of variables, but had not yet seen integrating factors.¹ The goal of the discussion around the problem was to recognize that the left hand side of Equation 2.1 was equivalent to $\frac{d}{dt}((10+t)P)$. The following exchange occurs:

- Tony: I was thinking initially, like, to me it looked like a chain rule kind of thing.
 Professor: Chain rule? ...
 Tony: But I couldn't get anywhere with that, though.
 Ron: Yeah. ...
 Dan: Differentiation by parts?

In any class, and especially in discussion-oriented classroom, a teacher must in real-time make sense of what students say. Then, a teacher must decide whether or not to open a class conversation around the student's contribution. There are many judgement calls to be made.

In the first episode, a professor must decide whether to pursue the idea of chain rule, differentiation by parts, both, or neither. In this case, the student Dan who brought up differentiation

by parts was recorded by the his table microphone as explaining to his group, "like $u dv$ and $v du$ and that whole deal." Indeed the point of rewriting the equation is that the left hand side of the equation at hand has a form that can be derived from the form $u dv + v du$. Ultimately, it turned out that the student who brought up chain rule had the product rule in mind, but misremembered the name.

The knowing of differential equations and calculus entailed in interpreting this situation involves the subject matter knowledge of relating $u dv + v du$ to the differential equation at hand; and the pedagogical content knowledge that students might conflate the chain rule and product rule as situations involving the differentiation of multiple functions.

In general, analytic scaffolding requires that the instructor:

- Recognize and make sense of students' mathematical (correct and incorrect) reasoning.
- Recognize or figure out how students' ideas have the potential to contribute to the mathematical goals of the discussion.
- Recognize or figure out how students' ideas are relevant to the development of students' understanding of mathematics; and finding a way for the reasoning to propel their own understanding or their peers' understanding.
- Prudently select which contributions to pursue from among all those available.

While discussing this class featuring the above episode, the professor observed:

"I just don't understand and haven't thought enough about differential equations as a subject to be taught so that I feel any flexibility at all."

Being able to recognize or figure out the potential utility of students' contributions requires course-specific pedagogical content knowledge and subject matter knowledge. Understanding something "as a subject to be taught" takes effort and experience.

¹ $M(t)$ is an *integrating factor* for a differential equation $\frac{dP}{dt} + Q(t)P = N(t)$ if it satisfies $MQ = \frac{dM}{dt}$. This is useful for solving the equation, as multiplying both sides by M yields $M\frac{dP}{dt} + MQP = \frac{d}{dt}(MP) = MN$. Then we can solve for P by integrating both sides.

Summary and further reading

The professor's experiences in Speer and Wagner's study corroborate previous analyses of elementary and secondary content that common content knowledge is not enough. Mathematics instructors need to know their discipline "as a subject to be taught." The message from the work on mathematical knowledge for teaching is that we need greater understanding about the knowledge needed to support various instructional practices, and to design professional development opportunities to help teachers learn all the types of knowledge needed for teaching. At the same time, we need to refine our notion of what it means for students to comprehend a mathematical idea.

We have laid out several strands of research on undergraduate teaching and learning. These lines of work draw inspiration from various sources and methods of observation. The Action-Process framework was influenced by developmental psychology, especially the work of Piaget. The covariation framework was constructed through careful analysis of cognitive interviews. Work on problem solving largely traces back to Pólya's writings. Mathematical knowledge for teaching has its predecessors in the thinking of Shulman, among others.

Below we have listed references for further reading on these topics. The thinking in these works can guide us as we grapple with how to balance the demands of teaching.

○ ◎ ○

Articles

- *Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide?* (Expository article by Deborah Ball, Heather Hill, and Hyman Bass on MKT).

- *Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education.* (Paper by Patrick Thompson analyzing several foundational ideas in the undergraduate curriculum, including angle, trigonometry, and rate of change.)
- *Beyond mathematical content knowledge: A mathematicians knowledge needed for teaching an inquiry-oriented differential equations course.* (Paper by Joseph Wagner, Natasha Speer, and Bernd Rossa, *Journal of Mathematical Behavior* 26 (2007) 247-266. See especially Section 3 (pp. 253-263), which describe Rossa's personal struggles and goals in teaching this differential equations curriculum.)
- *Critical variables in mathematics education: Findings from a survey of the empirical literature.* (Article by E. G. Begle from 1979 on the statistical impact of various characteristics on student achievement. Published by the Mathematical Association of American and the National Council of Teachers of Mathematics.)
- *Applying the Science of Learning to the University and Beyond: Teaching for Long-Term Retention and Transfer.* (By Diane Halpern and Milton Hakel in *Change* 35(4): 36-41, Jul-Aug 2003. ERIC abstract: Discusses why experts from different areas of the learning sciences conclude that higher education's primary goals—enhancing long-term retention and the transfer of knowledge—may result from applying known principles of human learning.)

Books

- *How to Solve It: A New Aspect of Mathematical Method.* (A classic, by George Pólya.)
- *Making the Connection: Research and Teaching in Undergraduate Mathematics Education.* (An MAA publication, edited by Marilyn Carlson and Chris Rasmussen, which brings together practical pedagogical examples from precalculus to group theory.)

Media

- *Let us teaching guessing.* (DVD of materials for problem solving, by George Pólya.)

Portraits of teaching and mathematics

Suppose an instructor decides to change teaching style from primarily lecture to one driven by student dialogue. During a lesson following this decision, this instructor may grapple with determining when to tell students a piece of mathematics— as well as how, what and why. It is often helpful to see concrete examples of what others' pedagogical techniques.

In this section, we sketch portraits of teaching: inquiry questions with “action”, “consequence”, and “reflection”; “advanced mathematics from an elementary standpoint” as used by the Algebra Project; and leading discussions in an inquiry-oriented differential equations course.

3.1 *Teaching with inquiry, action, and consequence – Wade Ellis*

3.1.1 *Overview*

Wade Ellis began by presenting four tenets of instruction: (1) Students learn by doing; (2) Focused time on task is important; (3) Students remember what they think about; (4) Context and relevance help student learn. These three tenets fit well with the “Action-Consequence-Reflection Principle”: students should act on mathematical objects, transparently observe the consequences of their actions, and then reflect on the mathematical meaning of those conse-

quences. By using technology or physical manipulatives to examine mathematical objects, students are doing; the observations focus the task; reflecting on the mathematical meaning helps students remember; and context and relevance is created by the action and observation, when they are doing something with the mathematics. As Ellis observed, “Students find relevant what they do. One of the things that comes to my mind is the handshake problem.¹ Once they begin working on it, it becomes relevant to them, because they are very interested in the answer, even though it is not directly influential to their long-term life goals.”

To carry out the Action-Consequence-Reflection Principle, Ellis has used “inquiry questions” that extend mathematical environments so that students can understand underlying mathematics through their own reasoning and reflection. The inquiry questions help to create a classroom setting where students are confident in answering the questions and ultimately posing their own. Before class, he brainstorms questions, and then selects and sequences them. The questions ask students to

- compare and contrast phenomena;
- predict forward and backward: What is an action that can result in ... / Given this action, ...;

¹There are 20 people in a room. If everyone shakes hands with everyone else, how many handshakes will take place?

- analyze a relationship: This happens when ...;
- make a conjecture;
- provide mathematical reasoning, justify a conjecture, or prove a conjecture.

3.1.2 Snapshot

Ellis has developed several pieces of calculator software. He demonstrated two interactive systems, one for lines and another for angles.

The system for lines featured two movable points, p_1 and p_2 , and the line through them. In his experience, students naturally begin moving around the points, observing that the line moves with the points. Inquiry questions for

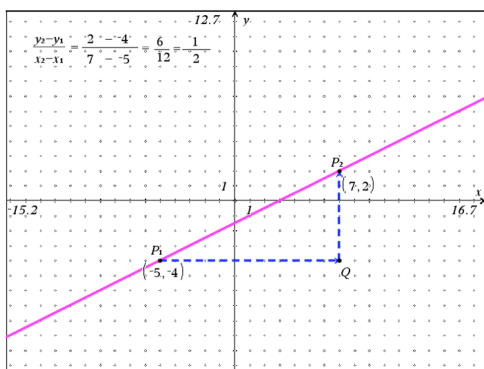


Figure 3.1. Set-up for inquiry questions on slope.

this software might include:

- How do you move p_2 to get a negative slope? (prediction)
This is a good question because it involves something that students can do; and false starts can generate interesting conversation.
- How do you move p_2 to get “no slope”? (prediction)
This question, which was phrased by a mathematician at a previous workshop Ellis ran, is interesting because it piques students with provocative language while simultaneously raising a need to improve vague but compelling language. This question can provide an opportunity for the students to prompt for precision, by asking what “no” slope means.
- When does p_1 below p_2 make the slope negative? (relationship)

This question could be scaffolded, “Move p_1 below p_2 make the slope negative”, then asking if moving p_1 below p_2 always makes the slope negative.

- What happens when you move p_1 to p_2 ? (prediction and compare/contrast)

Though this is a prediction question, it could lead to a host of compare and contrast questions that could be used to set up reasoning about infinitesimal quantities.

- Given a point q [a point such that p_1p_2q is a right triangle with legs parallel to the axes], try to move p_2 so that the length of p_2q is 10. ...so that the length of p_2q is 0.75. Is this possible? (prediction)

If p_1 and p_2 are constrained to a grid, the possible lengths of p_2q that the students can create depend on the grid’s specifications. Students can reason about the grid properties.

- Move below the x -axis in a rigid transformation. What happens to the coordinates? Why? (relationship, make a conjecture, provide reasoning)
This question could be a setting for discussing the invariance of length under rigid transformations, or the relationship between coordinates and the axes.
- What is a slope? Why doesn’t it change with the aspect ratio of the screen? (provide reasoning)

Ellis’s second demonstration featured a circle with a highlighted arc and the corresponding length of the arc on a number line. As he prompted the audience for questions, a silence fell across the room. Ellis quipped, “That’s the problem. Boy, we think, it’s so much fun, we can move it around! But if there’s no math, it’s not going to go anywhere.”

Inquiry questions for this demonstration might include:

- Does the angle measure change when you change r ? Explain. (compare and contrast)
- What are all the angles that have a sin of 1? (prediction)
- What are all the angles that have a sin of 2? (prediction)
- What angles have the same sin? cos? (conjecture)
- How many r ’s are there in the arc? (relationship)

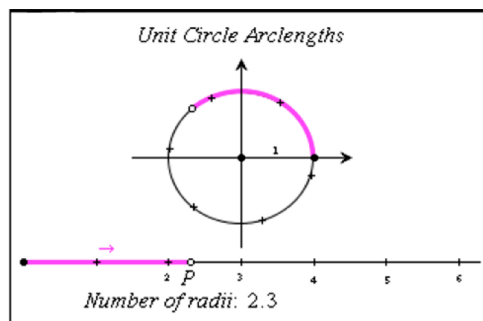


Figure 3.2. Set-up for inquiry questions about angle.

Ellis closed his presentation with a graphing calculator of “What’s my rule?” Typically, teachers use “What’s my rule?” to introduce functions: students are given a list of pairs of numbers, where the first number is related to the second by a function that the students are asked to figure out and describe. Ellis translated this to an activity on transformations: as a student moves around a point p , a second point, which is always related to p via a rigid transformation, moves correspondingly. Students attempt to figure out and describe the transformation.

3.2 *Advanced mathematics from an elementary standpoint – Bill Crombie and Bob Moses*

3.2.1 *Overview*

Bob Moses comes at math literacy from the point of view of the black civil rights movement from the 1960’s, and trying to work the “demand side” of the problem. The Algebra Project, founded by Moses, is focused on the bottom quartile of the nation’s students, asking whether or not these students can be *accelerated* rather than remediated.

The Algebra Project began its work at Lanier High School in Jackson, MS, in 2002. By state standards, Lanier was near the bottom of schools in Mississippi; and by national standards, Mississippi is at the bottom of the coun-

try. They asked students there to do math with us for 90 minutes a day, for all four years, and to set several expectations: to graduate high school, to conquer the state standards and the ACT, and to enter college ready to do college math. This cohort of students graduated in 2006. During this time period, the Algebra Project began working with research mathematicians to spend time in the classroom as well as on curricular materials.

Bob Moses observed, “You know, you can’t make the kids do all this math. It’s a coalition of the willing. So if I look at the Algebra Project and working the demand side, I think our biggest success has been developing young people who take on this idea of math literacy. We have to understand that we have no choice. We are faced with this transition of technology from industrial age to technology age. This brings with it this new literacy: quantitative literacy. My interest in all of this is that I would like the standard for all this to be for these kids to leave high school, and not be fodder for the criminal justice system.”

Bill Crombie, an Algebra Project developer, has been looking at the transition from algebra to calculus. Casting his inspiration in the philosopher Søren Kierkegaard, who wrote “Life is to be understood backwards, but it is lived forwards”², Crombie is interested in what he calls “backwards curriculum design”.

The learning trajectory, for students preparing for college in the US, must proceed from Algebra, Geometry, and Trigonometry to Calculus. Many times, designers consider the skills necessary from previous classes to be successful in a capstone class, usually calculus. Yet algebra is not enough for the understanding of calculus. The Algebra Project interprets “backward curriculum design” as not just asking about skills and related concepts, but also asking how much of that actual content – from the destination of calculus – can be brought back in a coherent, true, adequate fashion?

The Algebra Project is interested in a learning trajectory from high school to college, with the goal that students – including those in the

²“Livet skal forstås baglaens, men leves forlaens.”

bottom quartile – are able to make their own career choices. There are many reasons for an individual to pursue or choose another career, but mathematical under-preparation should not be one of those reasons.

3.2.2 Snapshot

As an example, Crombie presented the following problem, which he has used in teaching high school students in Geometry and in Algebra II.

Problem 3.1: Area Problem.

Given a parabola $y = x^2$ and a displacement on the x -axis, determine the rectangle with area equal to the area between the parabola and the x -axis across the given displacement.

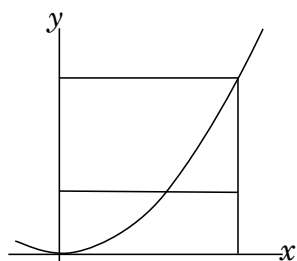


Figure 3.3. Diagram for the Area Problem.

Behind this problem is a characterization of a discipline – in this case, the study of calculus – via the questions that it asks, rather than by the techniques it uses. Crombie characterized the problems intrinsic to the study of calculus using the diagram in Figure 3.4.

To tackle the Area Problem, a teacher asks students to find relationships between a sequence of various regions derived from the picture of the parabola, on their graphing calculators (see Figure 3.5).

As the students talk through the similarities they find between the regions, they might notice that some areas, such as in pictures D and B, are complementary (relative to the area bounded by the axes and lines parallel to the axes and going through the marked point of the parabola).

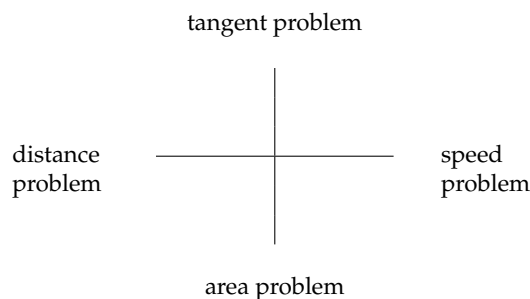


Figure 3.4. Defining problems of elementary calculus.

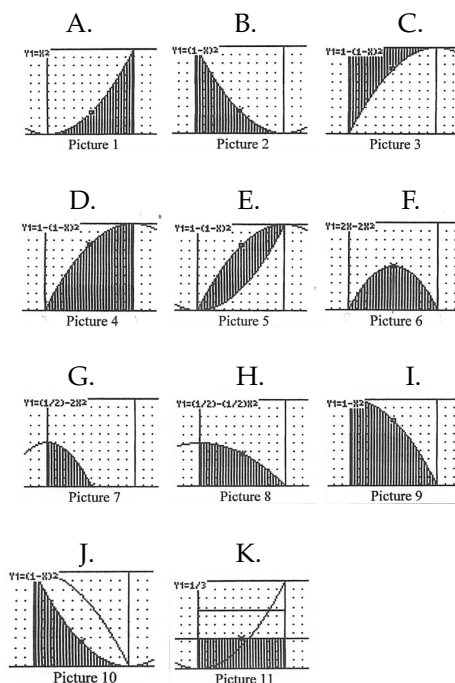


Figure 3.5. Sequence of regions used to solve the Area Problem.

How is one picture related to the next? How are the shaded regions and its area related to the next? Some of the pictures, such as D and B, are complementary. The area in picture 6 is equal to the area in picture 5; sometimes students argue this by saying that pencil-thin slices of E can “fall down” to the x -axis to create region F. Noting the equivalence of the areas in 5 and 6 is the keystone of this activity. As the students move

through the rest of the pictures, they eventually piece together that the area under the parabola is one third the cube of the displacement.

Formally, an argument could be written: Let d be the displacement, and let A be the area of the first region, B be the area of the second region, etc. Note that $A = B = C$, so $D = d^3 - C = d^3 - A$ and $E = d^3 - 2A$. From the equation for the boundary of the picture in 6, namely $y = -2(x - d)^2$, we may deduce that $\frac{1}{2}D = d^3 - 2A$. The sequence of pictures G through J show that $d^3 - E = A$, whence $A = \frac{d^3}{3}$.

The goal is not for the students to communicate the argument in this formal way; the above argument was provided simply as a reference for mathematicians. The point is to bring an area problem into a context where students could use Algebra II skills to reason about the notion of areas under a curve, and therefore to set up foundations for calculus.

3.3 Inquiry-Oriented Differential Equations – Chris Rasmussen

3.3.1 Overview

Much of the literature on undergraduate mathematics education has focused on the construction of reasoning or proofs rather than theorem usage. The inquiry-oriented differential equations curriculum developed and studied by Chris Rasmussen provides a context for students to use classical theorems, such as the existence and uniqueness of solutions, in a meaningful way.

3.3.2 Snapshot

Rasmussen and Ruan [32] describe the interactions between class members as they grapple with the solutions of the equations

$$\frac{dP}{dt} = 0.3P\left(1 - \frac{P}{12.5}\right) \quad (3.1)$$

which had arisen as a model for a deer population, and the equations

$$\frac{dh}{dt} = -h \quad (3.2)$$

and

$$\frac{dh}{dt} = -h^{\frac{1}{3}}, \quad (3.3)$$

which had arisen as models for airplane descent.

One goal of this unit was for students to understand how and why the following existence and uniqueness theorem was significant:

If $f(t, y)$ is continuous on $t \in (a, b)$, $y \in (c, d)$, and there exists a constant L such that $|f(t, y) - f(t, z)| \leq L|y - z|$ for all $y, z \in (c, d)$, then the initial value problem $\frac{dy}{dt} = f(t, y)$ with $y(t_0) = y_0$ where $t_0 \in (a, b)$ and $y_0 \in (c, d)$ has at most one solution for all $t \in (a, b)$ and $y(t) \in (c, d)$.

To motivate this theorem, the students needed to have some intellectual need for finding solution curves, and to debate different proposals for solution curves. To generalize their ideas, students needed to have a sense of how their reasoning could apply to other situations. To harness physical intuition from the model context, students needed to be comfortable with the distinction between a model and precise prediction of reality. Finally, students needed to examine situations both with and without unique solutions.

A key tool for exploring these equations was a computer system that plotted slope fields, but *not solution curves*. As Rasmussen emphasizes, this feature of the program was critical to the success of the pedagogy. The students could use the slope fields to bolster or argue against claims made about limiting behavior, but had to rely upon the equation to argue about the uniqueness of solution curves around a given point. The reader may note that of the three equations above, only Equation 3.3 does not satisfy the conditions for existence and uniqueness – its solutions have the form $h(t) = 0$ or $h(t) = \frac{2\sqrt{2}}{3\sqrt{3}}(C - t)^{\frac{3}{2}}$, so for any point on the line $h = 0$, there are two solution curves going through it. All three equations have an equilibrium solution at 0, though it is unstable in Equation 3.1 and stable in Equation 3.2.

In Equation 3.1, the following features prompted discussion from students:

- the slope field around equilibria: whether the solution curves around the $P = 12.5$ seemed to tend toward 12.5, and in what manner.
- the equilibrium at $P = 12.5$, a non-integral value.

Some students initially thought that the deer population would oscillate around $P = 12.5$, because “you can’t have half a deer running around”. This opened up a conversation in which students eventually reasoned about the relationship between the solution curves of a model for population and the actual population behavior. As the conversations progressed, students turned to the mathematical relationship in Equation 3.1, reasoning directly about the rate of change to find that $P = 12.5$ was a stable equilibrium, and that no other solution curve would intersect it. The deer population model was the first of many discussions that honed students’ ability to reason using a combination of empirical arguments (using the slope field or the real world situation) and mathematical reasons.

Later in the semester, the students discussed the Equations 3.2 and 3.3. A feature that prompted discussion from students included:

- differences in how the solution curves moved toward 0, according to exploring the slope field software, especially how suddenly the vectors “snapped” to 0 in the slope field for Equation 3.3.

This observation motivated students to solve the equation analytically and discuss the “rate of change of the rate of change.” They talked about the “snapping” of the curves in terms of changes in the rate of change of the height over time. Thus the comparison between Equations 3.2 and 3.3 opened conceptual conversation about the quantity $\frac{d}{dt}f(t, y)$ in the conditions of the uniqueness and existence theorem, giving a concrete context for whether or not the theorem holds.

A conceptual understanding of the existence and uniqueness theorem motivated these discussions. Two pieces of evidence suggest success in this goal: a post-class survey of the students’ work found instances where students cited the theorem even when the problem did

not specifically ask for the theorem; and a quantitative study by Rasmussen, Kwon, Allen, Marongelle, and Burtch [31] found that students who were taught differential equations using the inquiry-oriented curriculum were significantly more likely to use the theorem. This study, which looked at approximately 130 students in three sites (the midwest and the northwest of the U.S., and Korea), where each site hosted a traditional ODE class and a inquiry-oriented ODE class, found no significant differences in the students’ ability to solve routine problems and an improvement in the IO-DE students’ ability to reason conceptually.

Summary and further reading

It is sometimes easier to implement ideas when we hear about how others have implemented theirs. We have sketched out three ideas that were implemented in a variety of contexts. Each case depends on the teacher's preparation on how to use mathematical features and student ideas, and what responses were likely to arise from students. By thinking carefully about how to listen to students and stoke students' mathematical reasoning, we can continue to refine our personal teaching practices.

○ ◎ ○

Books

- *Radical Equations: Civil Rights from Mississippi to the Algebra Project.* (By Robert Moses and Charles E. Cobb)
- *Making the Connection: Research and Teaching in Undergraduate Mathematics Education.* (An MAA publication, edited by Marilyn Carlson and Chris Rasmussen, which brings together practical pedagogical examples from precalculus to group theory.)

Articles

- *Research Sampler.* (An occasional online MAA column on research on undergraduate mathematics education. Includes summaries of research and pedagogical interventions on proof, ordinary differential equations, and problem solving.)
http://www.maa.org/t_and_l/sampler/research_sampler.html
- *Capitalizing on advances in mathematics and k-12 mathematics education in undergraduate mathematics: An inquiry-oriented approach to differential equations.* (Comparison study of inquiry-oriented differential equations course with traditional differential equations course, by Chris Rasmussen, Oh Nam Kwon, Karen Allen, Karen Marrongelle and Mark Burtch, in the Asia Pacific Educational Review 7(1): 85-93. (2006))
<http://www.springerlink.com/content/k444461382116983/>

Media

- *Let us teaching guessing.* (DVD of materials for problem solving, by George Pólya.)

Assessments

Assessment is a perennial responsibility: how can we pinpoint our students' backgrounds and dispositions, whether or what our students have learned, or how they compare with other students?

Broadly speaking, the two most common flavors of assessment are “formative” and “summative”. *Formative* assessment is diagnostic, tends to take place during the term, and gives information for planning subsequent activities to help *form* the students as intellectuals, problem solvers, or towards a instructors' learning goals. Formative assessment is often contrasted with *summative* assessment, which takes place at the end of a unit, and gives information for *summarizing* what students have learned. In this sense, a midterm examination could be seen as both formative and summative: it may help an instructor recalibrate goals for the second half of the term as well as lay out what students learned in the first half of the term. On the other hand, questions asked with clickers during a particular lesson might be more formative than summative – the students' collective answers serve more to structure the ensuing discussion than as a final report of the students' mastery of the subject. The point is that whether assessment is formative or summative depends on how an instructor uses the information – and that no matter what form the assessment takes, it should match the instructor's ultimate goals.

Whenever using an assessment instrument, a primary concern is whether or not the results accurately represent what the instrument set out to evaluate. An unintentionally misleading phrase may throw students off for reasons other than their mathematical skills. In this case, the students' collective answers would not be very informative as either formative or summative assessment. In tension to the concern of validity is the pragmatic issue of scalability: realistically, an assessment instrument should be easy to use, with minimized time investment on the part of the instructor. The quickest assessments use multiple choice items; but when we only know whether students answered (a) or (b) or (c), we must be sure that these answers accurately represent the misconceptions we think they capture. Using written exams ameliorates this to some extent – reading student explanations gives more insight into how students were thinking. However, most instructors do not have time to read through hundreds of written answers. The usual way that this is handled by the developers of assessments is through pilot testing. In these trial runs, students are either interviewed or asked to elaborate upon why they chose or eliminated particular answers. Based on the results of this in-depth information, developers revise assessment items. Each project discussed in this chapter went through such a vetting process.

Below, we highlight the assessment endeavors of Maria Terrell, who heads the Good Questions Project, and of Jerome Epstein, who is known for his work on the Basic Skills Diagnostic Test and the Calculus Concept Inventory Exam. Before describing their work, we begin outside of mathematics, in physics. The work associated to the Force Concept Inventory has had wide influence on the physics reform movements, and inspired both Terrell and Epstein.

In each section, we give the personal or intellectual context in which the projects were conceived, example assessment questions where possible, and selected findings.

4.1 *The Force Concept Inventory and related diagnostic tests*

4.1.1 *Initial development*

In the mid 1980's, Ibrahim Halloun and David Hestenes were interested to see how students' common sense theories of physics influenced their ability to learn physics. These observations prompted them to create the Mechanics Diagnostic Test, an multiple-choice instrument to assess students' common sense concepts about motion. Halloun and Hestenes wrote their instrument to reflect two general categories: principles of motion, corresponding to Newton's Laws of Motion; and influences on motion, corresponding to specific laws of force in Newtonian mechanics. In the 1990's, Hestenes worked with Wells [23] to create the Mechanics Baseline Test, and with Wells and Swackhamer [24] to extend the Mechanics Diagnostic Test to the now well-known Force Concept Inventory (FCI). They focused on the notion of force as a foundational concept for learning physics for which students come in with many erroneous ways of thinking. A report of the concepts in the inventory can be found in [24]. Here, we focus on the findings by Halloun and Hestenes, as they set in motion much of physics education reform at the college level today.

Halloun and Hestenes [22][21] found that students may believe that the trajectory of a rocket firing its engine is similar to the parabolic trajectory of a ball, not recognizing that the signature trajectory of a ball is due to constant force. Students may also believe that an object subjected to constant force will move at constant speed, or that the time interval needed to travel a particular distance under a particular constant force is exactly inversely proportional to the magnitude of the force. Each of the above examples points to concepts that are covered in almost any standard physics course in mechanics – so one might optimistically predict that after such a course, most students would apply the ideas learned from class to analyze these situations rather than relying on incorrect common sense theories.

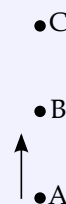
Here is an example item from Halloun and Hestenes' work (emphasis as in the original):

Problem 4.1: Mechanics Diagnostic Problem.

*The accompanying figure shows a ball thrown vertically **upwards** from point A. The ball reaches a point **higher** than C. B is a point halfway between A and C (i.e., $AB = BC$). **Ignoring air resistance:***

*What is the **speed** of the ball as it passes point C compared to its speed as it passes point B?*

- (a) *Half its speed at point B*
- (b) *Smaller than that speed, but not necessarily half of it*
- (c) *Equal its speed at point B*
- (d) *Twice its speed at point B*
- (e) *Greater than that speed, but not twice as great.*



(Reprinted with permission from Halloun, I.A. & Hestenes, D. American Journal of Physics 53(11), pp. 1043-1055 (November 1985). ©1985, American Association of Physics Teachers.)

Halloun and Hestenes [22] found that, indeed, “(1) Common sense beliefs about motion are generally incompatible with Newtonian theory. Consequently, there is a tendency for students to systematically misinterpret material in introductory physics courses.” However, contrary to the hope that an introductory course might correct students’ misconceptions about the physical world, “(2) Common sense beliefs are very stable, and conventional physics instruction does little to change them.” These findings corroborate the conclusions of prior work done by physics education researchers; the work of Halloun and Hestenes is distinguished by its large sample size and use of a massively-scalable instrument.

Halloun and Hestenes collected pre- and post-test performance from nearly 1500 students in introductory level college physics classes at Arizona State University, and 80 students beginning physics at a nearby high school. They additionally collected information on a mathematics pretest performance. In analysing correlations between their data, Halloun and Hestenes found that “pretest scores are *consistent* across different student populations”, “mechanics and mathematics pretest assess *independent* components of a student’s initial knowledge state”, and “the two pretests have higher predictive *validity* for student course performance than all other documented variables” including gender, age, academic major, and background courses in science and mathematics. Students come in with strong misconceptions about physics that are not easily dislodged.

4.1.2 Interactive Engagement methods

With scalable tests and compelling results by Hestenes and his colleagues, a natural next research question was: is there an instructional method that might help students overcome erroneous common sense theories about physics? In the 1990’s, Richard Hake used the Mechanics Baseline Test, the Mechanics Diagnostic Test and the Force Concept Inventory to study the effectiveness of “interactive engagement” methods.

Hake was interested in methods “designed at least in part to promote conceptual understanding through interactive engagement of students in heads-on (always) and hands-on (usually) activities which yield immediate feedback through discussion and peers and/or instructors, all as judged by their literature descriptions”, in contrast to those “relying primarily on passive-student lectures, recipe labs, and algorithmic-problem exams”, which he terms “traditional”. As examples of interactive engagement, Hake considers courses at The Ohio State University, Harvard University, and Indiana University. These universities used a combination of Overview Case Studies, Active Learning Problem Sets, ConcepTests, SDI labs, cooperative group problem solving, and Minute papers (see Hake [20] for references to these materials).

To measure the effectiveness of interactive engagement methods against traditional methods, Hake defines the *average normalized gain* g for a course as the ratio of average percentage gain G to the maximum possible average percentage gain G_{\max} :

$$g = \frac{G}{G_{\max}} = \frac{S_f - S_i}{100 - S_i} \quad (4.1)$$

where S_f and S_i refer to the percentage scored on the final and initial assessment.

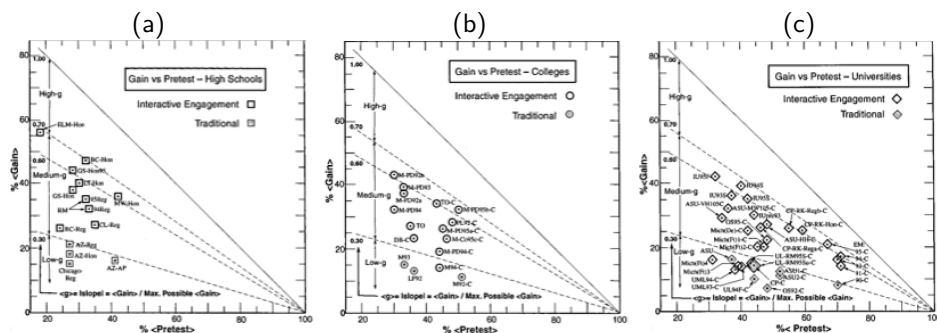
Hake analyzed the average normalized gain g for 62 schools who replied to his requests for pre- and post-FCI test data. He classified courses as using interactive engagement methods or traditional methods via survey responses on activities of students and teaching methods. The resulting graphs of gains versus pretest scores strongly suggested that interactive engagement methods influence students’ reasoning in physics more so than traditional methods do. This picture is corroborated by the collective average gains. Of the 62 schools reporting data to Hake, 14 were classified as traditional and 48 as using interactive-engagement methods. The 14 traditional courses exhibited an average gain of 0.23 ± 0.04 , in sharp contrast to the 48 interactive engagement courses’ average gain of 0.48 ± 0.14 – a nearly two-standard deviation difference.

Table 4.1. Average diagnostic test results by course and professor. Maximum scores: 36 (physics), 33 (mathematics), including five calculus items which were omitted in College Physics.

Professor	# Students	Math Pretest Mean (s.d.)	Physics		
			Pre-test	Post-test	Gain
University Physics					
A	97	17.25 (5.37) [52%]	18.47 (5.29) [51%]	23.23 (4.94) [65%]	4.76 [13%]
B	192	16.80 (6.21)	18.39 (5.14)	23.13 (4.81)	4.74 [13%]
C	70	19.56 (5.81) [59%]	18.06 (5.95) [50%]	22.91 (5.81) [64%]	4.85 [13%]
D	119	17.45 (6.37) [53%]	19.10 (6.26) [53%]	22.92 (6.57) [64%]	3.82 [11%]
College Physics					
E	82	10.48 (4.58) [37%]	13.48 (5.00) [37%]	19.00 (5.16) [53%]	5.52 [15%]
E	196	10.19 (4.51) [36%]	13.33 (5.09) [37%]	Not Available	
F	127	9.75 (4.38) [35%]	14.43 (5.16) [40%]	Not Available	
High School Physics					
G	24 (honors)		10.96 (3.28) [30%]	18.88 (5.02) [52%]	7.92 [22%]
G	25 (general)		10.83 (3.85) [30%]	15.80 (4.34) [44%]	4.97 [14%]

(Reprinted with permission from Halloun, I.A. & Hestenes, D. *American Journal of Physics* 53(11), pp. 1043-1055 (November 1985). ©1985, American Association of Physics Teachers.)

Table 4.2. Force Concept Inventory pre- and post-test data. (a) %⟨Gain⟩ vs. %⟨Pre-test⟩ score on the conceptual Mechanics Diagnostic (MD) or Force Concept Inventory (FCI) tests for 14 *high-school* courses enrolling a total of $N = 1113$ students. (b) %⟨Gain⟩ vs. %⟨Pre-test⟩ score on the conceptual MD or FCI tests for 16 *college* courses enrolling a total of $N = 597$ students. (c) %⟨Gain⟩ vs. %⟨Pre-test⟩ score on the conceptual MD or FCI tests for 32 university courses enrolling a total $N = 4832$ students.



(Reprinted with permission from Hake, R.R. *American Journal of Physics* 66(1), pp. 66-74 (January 1998). ©1998, American Association of Physics Teachers.)

4.1.3 Peer instruction methods

In recent years, Eric Mazur's work has become recognized as an exemplar in the scholarship of teaching and learning. His model of "Peer Instruction" is well-specified, his students show gains in qualitative and quantitative reasoning about physical phenomena, and his instructional methods evaluate positively.

In his speech "Confessions of a Converted Lecturer", Mazur relates the chain of events that an article on the Force Concept Inventory set in motion. The article he read had concluded no matter who the teacher was – even if they were teaching award winners – there was essentially no gain. The students were not overcoming their initial misconceptions. He reflects,

"At that time, we were doing rotational dynamics, the students had to do triple integrals to calculate dynamics. There was no comparison between what we were doing and the Force Concept Inventory, they were way beyond it. . . ."

"I was worried that my students would be offended by the simplicity of this test once they would start on it. Oh boy, were my worries quickly dispelled. Hardly had the first group of students taken their seats in the classroom when one student raised her hand, she said, 'Professor Mazur, how should we answer these questions? According to the way you have taught us, or how we usually think?' I had no idea how to answer that question."

After a damning performance by those students on the Force Concept Inventory, Mazur began to develop a method he now calls Peer Instruction, which partitions a class into a series of short presentations each centered around a physical idea and followed by:

- a conceptual question ("ConcepTest" question) related to the idea
- a one-minute or two-minute period in which students prepare individual answers to share with the instructor (via clickers, flashcards, or other method which is more visible to the instructor than to fellow students)
- a two to four minute period in which students share their answers and reasoning with peers
- a poll of students' final answers to the question, followed by an explanation by the instructor of the answer.

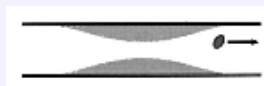
As support for this basic structure, Crouch and Mazur [13] describe gradual refinements to

their pedagogy that they developed in the first decade of using Peer Instruction.

Here is an example ConcepTest:

Problem 4.2: ConcepTest (Blood Platelets).

A blood platelet drifts along with the flow of blood through an artery that is partially blocked by deposits.



As the platelet moves from the narrow region to the wider region, its speed: (a) increases (b) remains the same (c) decreases.

(Reprinted with permission from Crouch, C.H. & Mazur, E., American Journal of Physics 69(9), pp. 970-977 (September 2001). ©2001, American Association of Physics Teachers.)

(The answer is that the speed decreases.)

Mazur found that the average normalized gain g for the Force Concept Inventory, defined as in Hake's study (Equation 4.1), doubled from 1990 to 1991, the transition between traditional instruction and the Peer Instruction method.

The Force Concept Inventory represents conceptual mastery. Mazur also recorded gains in his students' quantitative problem solving ability, represented by performance on the Mechanics Baseline Test. Mazur's results are taken from sections taught by different instructors, and are consistent with results from Hake's study [20], suggesting that the gain can be attributed to the instructor's choice of method rather than the instructor or the location.

4.2 The Basic Skills Diagnostic Test and the Calculus Concept Inventory

4.2.1 Basic Skills Diagnostic Test

Jerome Epstein developed the Basic Skills Diagnostic Test (BSDT) out of a laboratory program in 1980 with an NSF grant for working with a group of entering college students "who tested as having the mathematical and conceptual level of a 10 year old". At the time, he

Table 4.3. Peer Instruction results using the Force Concept Inventory and Mechanics Baseline Test. The FCI pretest was administered on the first day of class; in 1990 no pre-test was given, so the average of the 1991-1994 pre-test is listed. In 1995 the 30-question revised version was introduced. In 1999 no pretest was given so the average data of the 1998 and 2000 pre-test is listed. The FCI post-test was administered after two months of instruction, except in 1998 and 1999, when it was administered the first week of the following semester to all students enrolled in the second-semester course (electricity and magnetism). The MBT was administered during the last week of the semester after all mechanics instruction had been completed. For years other than 1990 and 1999, scores are reported for matched samples for FCI pre- and post-test and MBT. No data are available for 1992 (the second author was on sabbatical) and no MBT data are available for 1999.

Yr.	Method	FCI pre	FCI post	Absolute gain (post – pre)	Normalized gain (g)	MBT	MBT quant. questions	N
Calculus-based								
1990	Tradit.	(70%)	78%	8%	0.25	66%	62%	121
1991	PI	71%	85%	14%	0.49	72%	66%	177
1993	PI	70%	86%	16%	0.55	71%	68%	158
1994	PI	70%	88%	18%	0.59	76%	73%	216
1995	PI	67%	88%	21%	0.64	76%	71%	181
1996	PI	67%	89%	22%	0.68	74%	66%	153
1997	PI	67%	92%	25%	0.74	79%	73%	117
Algebra-based								
1998	PI	50%	83%	33%	0.65	68%	59%	246
1999	Tradit.	(48%)	69%	21%	0.40	129
2000	PI	(47%)	80%	33%	0.63	66%	69%	126

(Reprinted with permission from Crouch, C.H. & Mazur, E., American Journal of Physics 69(9), pp. 970-977 (September 2001). ©2001, American Association of Physics Teachers.)

believed that he was working with an outlier group. The BSDT, which examines pre-algebra and algebra concepts, has now been administered to thousands of students in high school and college. What Epstein has concluded over time, as more institutions generate data on the BSDT, is that there are many students of comparable level who are hidden from our view – and that the students he worked with in 1980 may not be as much of an exception as we might like to think.

Themes among the BSDT data suggest that areas of particular difficulty include place value, proportional reasoning, and the equivalent fractions.

We present example questions, discuss potential interpretations of the students' errors, and tabulate performance data on these items. For reasons of protecting the validity of test re-

sults, we have obscured some features of the questions. Interested readers can obtain the actual Basic Skills Diagnostic Test by contacting Jerome Epstein at jerepst@att.net and agreeing to test security conditions.

From the perspective of understanding learners, the data on these questions is interesting for the information it encodes about the sources of student ways of thinking. From the perspective of assessment item writ in, the questions are interesting in how they are crafted to detect error.

Problem 4.3: Place Value.

Put the following numbers in order from smallest to largest. n_1 n_2 \dots n_k

(In an actual test setting, n_1, n_2, \dots, n_k are replaced with a finite set of numbers.)

The numbers include an assortment of carefully chosen fractions, improper fractions, and decimals designed around common misconceptions and rote strategies.

For example, a common error is concluding that a/b is less than c/d when as $a < c$ and $b < d$. There are two misconceptions that may contribute to such an error. Students may not understand the role of the numerator and denominator of a fraction, and students may not realize that a fraction can represent a number greater than 1.

Problem 4.4: Proportional Reasoning (Piaget’s Shadow Problem).

(Piaget) Yesterday an m -foot tall man was walking home. His shadow on the ground was s_m feet long. At the same time, a tree next to him cast a shadow s_t feet long. How tall was the tree?



(In an actual test setting, m , s_m , and s_t are replaced with numbers.) The most common wrong answer is $s_t + (m - s_m)$ feet. As Piaget noted, students begin by “thinking additively” – that is, because the difference between the man’s height and his shadow was $m - s_m$ feet, students erroneously apply this difference to the relationship between the tree’s height and its shadow. This problem, which assesses whether or not a student is able to think proportionally, has an extraordinarily high correlation with success in a college algebra course. When graphing the distributions of students who answer incorrectly and who answer correctly over their level of success in a college algebra class, two Gaussians result with very little overlap. As Epstein remarked, “It is as though the students who answer this question correctly are in

a different mathematical world than those who do not.”

Another common misconception arises with linear equations with fractional coefficients and a constant term (cf. the discussion about students’ difficulties with fractions and linear systems in Section 2.3).

The above questions concern pre-algebraic and algebraic skills where faulty yet common modes of thinking can cause error. The skills represented are foundational for success in mathematics. The results of the Basic Skills Diagnostic Test are sobering, and a call to develop pedagogies that will help students overcome erroneous ways of operating with numbers and algebra, as well as the tendency to fall back upon rote procedures pursued without understanding.

4.2.2 Calculus Concept Inventory

The Calculus Concept Inventory (CCI) is directly inspired by the Force Concept Inventory. In Epstein’s words, “There is a basic level of conceptual understanding that [students] must have, or anything that they have learned for the final exam will disappear.” The CCI seeks to assess this understanding. Epstein received NSF funding in 2004-2007 for initial development and evaluation, and has collected data on the CCI ever since. The CCI has proved popular, with roughly one request per week from the beginning of the project.

Themes among the data for performance suggest that students have difficulty with reasoning exponentially as opposed to linearly. As well, the data from the CCI corroborate the BSDT’s data that students have difficulty thinking proportionally as opposed to additively. We present examples of each.

Problem 4.5: Exponential Reasoning.

We are growing a population of bacteria in a jar. At 11:00 a.m., there is one bacterium in the jar. The bacteria divide once every minute so that the population doubles every minute. At 12:00 noon, the jar is full. At what time was the jar half full? (a) 11:01 (b) 11:15 (c) 11:30 (d) 11:45 (e) 11:59.

Table 4.4. Percentages denote the percentage of students in the course obtaining the correct answer to the questions concerning place value, proportional reasoning, and linear equations.

School	Course	Place Val.	Proport.	Eqns.
Hofstra	CS-005:	63%	51%	26%
	CS-015:	84%	69%	50%
	CS-132:	85%	85%	77%
Wellesley	WRT-125:	65%	72%	87%
NYU	GenPhys:	80%	79%	69%
	Astron:	80%	71%	54%

CS-005:	Overview of Computer Science
CS-015:	Fundamentals of Computer Science I: Problem Solving and Program Design
CS-132:	Computational Modeling
WRT-125:	Introductory writing course
GenPhys:	General Physics I (for non-physics majors)
Astron:	Origins of Astronomy.

The most common wrong answer is (c), which suggests that instead of reasoning about an exponential situation, students are falling back upon familiar linear patterns: 11:30 is halfway between the start and end times of the scenario, and the question asks when the population is halfway to the end population.

Problem 4.6: Proportional Reasoning (Numbers Close to Zero).

(Terrell) A number close to 0 is divided by a number close to, but not equal to 0. The result (a) is a number close to 0, (b) is a number close to 1, (c) is a very big number, (d) could be any number, (e) is not a number.

In the performance data on this question, one wrong answer stands out: (b). It is as though students reason that the numbers are close to identical because they are “running out of room” close to 0. This is symptomatic of Piaget’s description of students who think additively, but who cannot yet think proportionally. Questions that ask students to think proportionally, and where students exhibit additive thinking, show remarkably little gain from a semester of traditional instruction.

Consistent with the hypothesis tested by Hake and Mazur, interactive-engagement sections show dramatically more gain on this question than traditional lecture instruction – ap-

proximately two standard deviations. Moreover, as we will see in the next section, this question was enormously productive as a Peer Instruction question in the Good Questions Project data.

Thus additive reasoning in calculus seems to play a similar role to erroneous common sense theories in physics (see Section 4.1.1). This analogy holds in two ways: first, in both cases, there is a tendency for students to fall back on linear patterns instead of reasoning about the situation; and second, in both cases, the only known way for a course to dislodge the misconception is through substantive discussion amongst students. The discussion must happen after engaging with a question that gives a common intellectual experience.

In Hake’s and Mazur’s studies, interactive engagement results in a two standard deviation difference above traditional lecture sections. The most dramatic result in mathematics so far has taken place at the University of Michigan, with roughly 1500 students across 55 sections of Calculus, and with normalized average gains between 0.27 and 0.40. In contrast, the typical gain in CCI data for lecture-based classes is between 0.05 and 0.25. The department at the University of Michigan uses lesson plans and classroom seating that necessitate at least minimal interactive engagement methods. Though this gain size is promising, it is still less than the gains seen in recent stud-

ies of physics courses; on the other hand, the physics movement toward interactive engagement precedes the mathematics movement by nearly two decades. The results so far indicate that substantive interaction amongst students is critical for growth in their powers of reasoning.

4.3 The Good Questions Project

The Good Questions Project, led by Maria Terrell, adapts the ideas behind Eric Mazur's ConcepTests and Peer Instruction for freshman physics, to classes in the freshman calculus for liberal arts majors. About seeing Mazur's questions for the first time, Terrell [36, p. 5] writes,

I was excited. I wondered if it would be possible to craft such questions in calculus; questions that were non-computational, that were related to students experiences, questions that were memorable, and surprising, that helped build on students partial understanding. I wondered if instructors would take time out from class to use good questions if they were attractive, if they led to active lively discussions, and if they helped students connect their intuitive understanding of the world to the concepts of calculus.

A group at Cornell resolved to write such questions, and the Good Questions Project was born.

4.3.1 Fall 2003 Study

A Good Question is intended to:

- stimulate students' interest and curiosity in mathematics
- help students monitor their understanding
- offer students frequent opportunities to make conjectures and argue about their validity
- draw on students' prior knowledge, understanding, and/or misunderstanding
- provides instructors a tool for frequent formative assessments of what their students are learning
- support instructors' efforts to foster an active learning environment.

Miller, Santana-Vega, and Terrell [27] examined the effect of peer instruction using a set of "Good Questions" in Fall 2003, after a pilot

phase the previous spring. In Fall 2003, Terrell invited instructors of a large, multi-section calculus course to use these questions. There were 17 sections of 25-30 students each, and 14 instructors. They informed the instructors through a short series of workshops of Mazur's success and how the Good Questions were intended as an analogue of the ConcepTests. As Terrell remarks of her instructors, "We knew that we couldn't tell people how to teach. Everyone has their own ideas on how students learn, and give examples, but when they close the door they will do what they want to do."

Terrell's team gathered data on the instructors use and non-use of the questions. On survey questions throughout the semester, they asked which questions the instructors used. The use of the Good Questions were facilitated by loading the questions onto a laptop that the instructors took to class, and the use of clickers, to record which questions were used as well as the distribution of answers from the students. Additionally, volunteer students were interviewed on their reaction to the Peer Instruction method and incorporation of the Good Questions.

Instructors' usage of Good Questions fell into four profiles:

- Deep: Good Questions used 1-4 days per week with peer discussion, and many Deep/Probing questions. (These questions are distinguished by their ability in the pilot trial to elicit discussions about the conditions of theorems, or the importance of precise language in mathematics.)
- Heavy plus Peer: Good Questions used 3-4 days per week with regular use of peer discussion.
- Heavy plus Low Peer: Good Questions used 3-4 days per week, with minimal or no use of peer discussion.
- Light to Nil and Low Peer: Good Questions rarely or not at all used, with no use of peer discussion.

The profiles were determined by a combination of student surveys, instructor surveys, and recorded histogram data from the laptops. The surveys asked for information about the length and nature of discussions, and to what extent Mazur's model for peer instruction was followed. Where there was a disparity between

student reports and instructor reports, Terrell's team used the student reports.

To relate the instructor's usage to performance, Terrell's team gathered data on the students' common preliminary and final exams, which included both conceptual and procedural questions.

The data suggest that different profiles affected student learning, with the students of instructors who fit the Deep profile attaining a median 10 points higher than the students of instructors who fit the Light to Nil profile. Interestingly, the median was lowest from students of instructors who fit the Heavy plus Low Peer profile. This suggests that peer discussion is a critical piece of the intervention, and that when instructors who want to engage in peer discussion of good questions are given the resources to do so, their students benefit.

A common concern about interventions focusing on conceptual problems is low ability in relatively procedural problems. However, Terrell's data did not indicate any such risk. In fact, students in lower SAT math bands gained more on their performance on procedural problems than on conceptual problems. An interviewed student suggested, "If you understand the concepts, it can help you with the numbers."

Overall, the interviewed students reacted positively to Peer Instruction. In the words of one student, "I liked it. First of all I thought it made the learning more enjoyable than just watching a lecture and then pulling out a pen maybe and finding the answer. I thought it was more valuable to look at it from a more analytical approach, and to go to take a question, and think about it for a while, and then respond to it. It was fun to talk about it, with other people, it helps you work out your own logic. It helps you figure out if there were any flaws in your logic, help you fill in a whole." Another student commented, "This method forces you to engage, forces you to think, and forces you to get into what you are doing. Because even if you put any answer, you don't have to think, well maybe no one else has my answer ... there's safety in your answer." These students felt they

benefited from the safety, the anonymity, and the expectation to articulate their conclusions.

4.3.2 Sample Good Questions

Problem 4.7: Height Problem.

*True or False: You were once **exactly** π feet tall.*

In Terrell's Fall 2003 data for sections with Peer Instruction, a typical distribution of votes was 40% of students selecting True for their first vote, and then 85% of students selecting True in the re-vote.

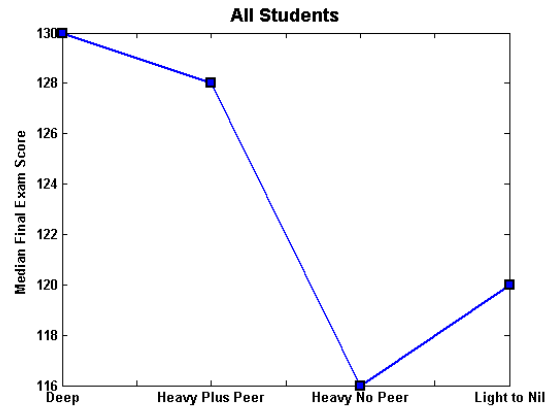
This question can incite students to defend their thinking by elaborating their assumptions. Some students will argue that because atoms are discrete, a person may skip over the height of π feet tall. Others may argue that human growth should be thought of as a continuous process. This discussion sets a context for introducing the Intermediate Value Theorem and whether or not the hypotheses of the theorem are satisfied in a particular situation.

Interestingly, when Terrell's instructors asked the same question, with π replaced by 3, the data showed very different vote distributions. Students voted 80%, then 90% for True.

As mentioned previously, another example of a Good Question is Problem 4.6 from Section 4.2.2, about division of numbers close to zero. Students' votes were initially spread across all options. At 31%, option (c) held the most votes. Only 20% opted for the correct answer of (d). Upon revoting, 98% students chose (d) and 2% choose (c), with 0% across the other options. The initial spread advantages Peer Instruction: the idea that quotient "could be any number" comes out through group discussion, when students share different examples. Discussion around this question can be used to introduce limits with indeterminate forms.

4.3.3 Sample concepts unaffected by group discussion

While the Good Questions helped dislodge student misconceptions, Terrell's team also found

Table 4.5. Results of the Good Questions Project.

examples of questions whose votes for incorrect answers seemed largely unaffected by peer discussion, such as the following.

Problem 4.8: Repeating 9's.

True or False: $0.999\cdots = 1$.

Here, 50% of students, then 45%, voted for the correct answer of True. Some problems such as this one were eliminated. However, a few problems were kept to help instructors see how deeply students held some partial understandings of decimals and limits, for example:

Problem 4.9: Adding Irrationals.

If $p = .39393939\dots$ and $q = 0.676677666777\dots$, then $p + q$ is (a) not defined because the sum of a rational and irrational number is not defined, (b) not a number because not all infinite decimals are number, (c) is defined using successively better approximations, (d) is not a number because the pattern may not be predictable indefinitely.

Initial distributions resembled re-vote distributions: each option received 8%-41% of initial votes, with approximately 20% selecting the correct answer of (c). The data from decimal problems suggest that many students have an underdeveloped sense of “equals” for limits, despite demonstrating some intuition. When

prompted, “Suppose I was your boss and I said you have to have something in 15 minutes,” students said that they would “go out really far” and calculate an approximation. They remarked that they could “always get it as close as you want it, as long as you tell me how close you want it” – thus hinting at the ϵ - δ notion of continuity. However, until students understand that when it comes to real numbers, equals is a process, it will be difficult for them to connect the definition of a limit with intuition for particular limits (cf. Section 2.1.1).

The pilot data on questions addressing number and decimal concepts suggest that students have conceptual gaps in their understanding the real numbers, and that K-20 students would benefit from a better articulation of concepts from the arithmetic of real numbers prerequisite to the study of calculus.

Summary and further reading

The Force Concept Inventory, Mechanics Baseline Test, and Mechanics Diagnostic Test sparked a wave of physics education research because their questions were created with care, the initial results provocative, and the implementation scalable. Mazur's work, which used the Force Concept Inventory and Mechanics Baseline Test as measures, is valuable for its clear pedagogical methods. Although we do not yet have published, large-scale results on the analogues of these physics education findings in mathematics, the work by Terrell and Epstein is well worth further exploration and implementation. Below are articles for further reading in the physics and mathematics education literature relevant to the material in this chapter.



Articles related to the Force Concept Inventory

- *The initial knowledge state of college physics students.* Article by Ibrahim Halloun and David Hestenes introducing the Mechanics Diagnostic Test. In *American Journal of Physics* 53(11), 1043-1048 (1985).
- *Peer Instruction: Ten years of experience and results.* Article by Catherine Crouch and Eric Mazur on pedagogical support for implementing Peer Instruction. In *American Journal of Physics* 69(9):970-977 (2001)
- *Professors as physics students: What can they teach us?* Article by Sheila Tobias and Eric Mazur, reporting on a study of 10 non-physical science professors who studied Newton's laws alongside approximately 300 students in a non-calculus-based introductory physics course. In *American Journal of Physics* 56(9): 786-794 (1988).
- *Resource letter on physics education research.* Review article by Lillian McDermott and Edward Redish on learning and teaching of physics. Contains extensive list of bibliographies on science education research regarding instructional strategies targeting conceptual understanding, analysis of assessment instruments. In *American Journal of Physics* 67(9): 755-767 (1999).

Articles on the Calculus Concept Inventory

- *The calculus concept inventory.* Article by Jerome Epstein in *Proceedings of STEM Assessment*, Washington, D.C. (2006), edited by Donald Deed and Bruce Callen.
[http://www.openwatermedia.com/downloads/STEM\(for-posting\).pdf#page=64](http://www.openwatermedia.com/downloads/STEM(for-posting).pdf#page=64)

Articles on the Good Questions Project

- *Asking good questions in the mathematics classroom.* Prepared by Maria Terrell for the AMS-MER Workshop "Excellence in Undergraduate Mathematics: Mathematics for Teachers and Mathematics for Teaching," Ithaca College, New York, March 13-16, 2003.
- *Can good questions and peer discussion improve calculus instruction?* Article by Robyn Miller, Everilis Santana-Vega, and Maria Terrell reporting on the Good Questions Project, in *PRIMUS* XVI(3): 193-200 (2006).

End Notes

Two themes that emerge from the preceding chapters are the work of *seeing* and *listening*. To refine undergraduate mathematics education, we must, like a person peering through a microscope, dial into the details within mathematical concepts and pedagogical methods. To support the work of seeing, we must be able to hear students when they reveal pieces of their ways of thinking or previous experiences. What students say gives us a context for observing the details of our own instruction and the coherence of our institutions' mathematical instruction.

This workshop launched with a comparison of collegiate and secondary course-taking patterns, presenting a stark contrast between the dramatic trend of mathematical advancement among high school students and the relatively flatlined enrollments in collegiate mathematics classes. More disturbingly, there are anecdotes of students who excel in their high school's AP Calculus class, yet place into precalculus upon entering college, or who leave with a visceral dislike of mathematics. David Bressoud pushed the mathematical community to gather more and better information about the high school students who take calculus, what happens in their classes, and the benefits and dangers to future mathematical success of taking calculus in high school. Moreover, Bressoud advocates a re-examination of first-year college mathematics, to build upon the skills and knowledge that

students carry with them to college. Such a call requires that we identify these skills and knowledge, which in turn entails careful observation of and listening to our students.

"Voices of the Partner Disciplines", a Curriculum Foundations project sponsored by the MAA, synthesizes conversations between mathematicians and colleagues from STEM fields on what they would like to see in mathematics courses their students take during the first two years of college. To see the mathematical concepts integral to other disciplines within our curricula, the Curriculum Foundations project encourages mathematics faculty to meet with faculty of fellow STEM fields at their institution. To initiate the reforms supported by the Curriculum Foundations project on a large scale, mathematical biologist John Jungck argues that we need to "cross the chasm." We must be able to work with, convince, listen and talk to departments outside our own and schools outside our own.

The Curriculum Foundations workshops were unanimous in their emphasis on conceptual understanding, in addition to computational skills. Jerome Epstein, whose work on the Basic Skills Diagnostic Test and the Calculus Concept Inventory has revealed an impoverished understanding of foundational notions such as place value and fractions on the part of some college students, echoed this point.

He urged us to continue investigations into pedagogical methods that help our students reorient their learned ways of thinking about mathematics from memorized routines to reasoned consideration of mathematical concepts. If we take a cue from the research in physics education, then such pedagogies might involve instructors setting up opportunities for students to listen to each others' thoughts on mathematical concepts. As work of Maria Terrell, the Algebra Project, and Chris Rasmussen suggest, such opportunities can be extracted from close-up views of central mathematical ideas such as variable, area, existence and uniqueness, indeterminate limits, proportional reasoning, and exponential reasoning – combined with attentiveness to students' existing ways of thinking about these ideas.

The Action-Consequence documents, the Inquiry-Oriented Differential Equations curriculum, and the Algebra Project curriculum each rely intrinsically on listening to students. The analysis by Natasha Speer and Joe Wagner, along with other work on mathematical knowledge for teaching, argues that leading mathematical discussion entails recognizing and making sense of students' mathematical reasoning, how these ideas have the potential to contribute to the mathematical goals of the discussion, and how students' ideas are relevant to the development of other students' understanding of mathematics. To make finer-grained observations and assessments of students' thinking, Marilyn Carlson advocated using frameworks from the research literature as a guide. These frameworks, which are predicated on close observations of students, may help us see the mathematics from the perspective of student learning.

With this document, we have tried to put together a resource for mathematics instructors who are interested in the ideas of other educators, within and outside of mathematics. Many of us pay attention to our teaching; as we know, there are many things to pay attention to, and sometimes these things are not precisely articulated in our minds even if we have an intuition for what they are. We have tried to articulate

these demands here. We hope that this guide will help sharpen the ways in which we listen to and observe our students and mathematics.

Bibliography

- [1] D. L. Ball, H.C Hill, and H. Bass. Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, 29(1):14–17, 20–22, 43–46, 2005. <http://deepblue.lib.umich.edu/handle/2027.42/65072>.
- [2] D. Bressoud. Disturbing trends in the CBMS data. *Launchings from the CUPM Curriculum Guide (online column)*, June 2008. http://www.maa.org/columns/launchings/launchings_06_08.html.
- [3] D. Bressoud. The new pre-med requirements. *Launchings from the CUPM Curriculum Guide (online column)*, November 2009. <http://www.macalester.edu/~bressoud/pub/launchings/index.html>.
- [4] D. Bressoud. Meeting the challenge of high school calculus: Evidence of a problem. *Launchings from the CUPM Curriculum Guide (online column)*, April 2010. http://www.macalester.edu/~bressoud/pub/launchings/launchings_04_10.html.
- [5] D. Bressoud. Meeting the challenge of high school calculus: Introduction. *Launchings from the CUPM Curriculum Guide (online column)*, March 2010. http://www.macalester.edu/~bressoud/pub/launchings/launchings_03_10.html.
- [6] D. Bressoud. Meeting the challenge of high school calculus III: Foundations. *Launchings from the CUPM Curriculum Guide (online column)*, May 2010. http://www.macalester.edu/~bressoud/pub/launchings/launchings_05_10.html.
- [7] D. Bressoud. Meeting the challenge of high school calculus IV: Recent history. *Launchings from the CUPM Curriculum Guide (online column)*, June 2010. http://www.macalester.edu/~bressoud/pub/launchings/launchings_06_10.html.
- [8] M. Carlson and I. Bloom. The cyclic nature of problem solving: An emergent multi-dimensional problem-solving framework. *Educational Studies in Mathematics*, 58:45–75, 2005.
- [9] M. Carlson, S. Jacobs, E. Coe, S. Larsen, and E. Hsu. Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, 33:352–378, 2002.
- [10] J. Confrey. Review of the research on student conceptions in mathematics, science, and programming. *Review of Research in Education*, 16:3–56, 1990.
- [11] E. Connally, D. Hughes-Hallett, A. Gleason, and et al. *Functions Modeling Change: A Preparation for Calculus, Third Edition*. John Wiley & Sons, 2006.
- [12] National Research Council. Bio2010: Transforming undergraduate education for future research biologists: Report by the committee on undergraduate biology education to prepare research scientists for the 21st century. Technical report, The National Academies Press, 2003.
- [13] C. Crouch and E. Mazur. Peer instruction: Ten years of experience and results. *American Journal of Physics*, 69(9):970–977, 2001.

- [14] B. Dalton, S. J. Ingels, J. Downing, and R. Bozick. Advanced mathematics and science coursetaking in the spring high school senior classes of 1982, 1992, and 2004. Statistical Analysis Report NCES 2007312, NCES, Washington, D.C., 2007. <http://nces.ed.gov/pubsearch/pubsinfo.asp?pubid=2007312>.
- [15] E. Dubinsky and M. McDonald. APOS: a constructivist theory of learning in undergraduate mathematics education. In D. Holton, editor, *The teaching and learning of mathematics at university level: An ICMI study*, pages 275–282. Kluwer Academic Publisher, 2001.
- [16] Ed Dubinsky and David Tall. Advanced mathematical thinking and the computer. In David Tall, editor, *Advanced Mathematical Thinking*, volume 11 of *Mathematics Education Library*, pages 231–248. Springer Netherlands, 1991.
- [17] S.L. Ganter and W. Barker. Curriculum foundations project: Voices of the partner disciplines. Mathematical Association of America, Washington, D.C., 2004.
- [18] L. J. Gross. Quantitative concepts for undergraduate biology students. <http://www.tiem.utk.edu/~gross/bio.quantitative.topics.txt>.
- [19] L. J. Gross. Quantitative training for life-science students. *BioScience*, 44(2):59, 1994.
- [20] R. Hake. Interactive-engagement versus traditional methods: a six-thousand-student survey of mechanics test data for introductory physics courses. *American Journal of Physics*, 66(1):64–74, 1998.
- [21] I. Halloun and D. Hestenes. Common sense concepts about motion. *American Journal of Physics*, 53(11):1056–1065, 1985.
- [22] I. Halloun and D. Hestenes. The initial knowledge state of college physics students. *American Journal of Physics*, 53(11):1043–1048, 1985.
- [23] D. Hestenes and M. Wells. A mechanics baseline test. *The Physics Teacher*, 30:159–166, 1992.
- [24] D. Hestenes, M. Wells, and G. Swackhamer. Force concept inventory. *The Physics Teacher*, 30:141–158, 1992.
- [25] H. Hill, B. Rowan, and D. Ball. Effects of teachers’ mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, 42(2):371–406, 2005.
- [26] J. Jungck. Ten equations that changed biology: Mathematics in problem-solving biology curricula. *Bioscene: Journal of College Biology Teaching*, 23(1):11–36, May 1997.
- [27] R.L. Miller, E. Santana-Vega, and M. Terrell. Can good questions and peer discussion improve calculus instruction? *PRIMUS*, 16(3):193–203, 2006.
- [28] M.C. Oehrtman, M.P. Carlson, and P.W. Thompson. Foundational reasoning abilities that promote coherence in students’ understandings of function. In M.P. Carlson and Rasmussen, C., editors, *Making the connection: Research and practice in undergraduate mathematics*, number 73 in MAA Notes, pages 27–42. Mathematical Association of America, Washington, D.C., 2008.
- [29] National Mathematics Advisory Panel. Foundations of success: The final report of the national mathematics advisory panel. Technical report, U.S. Department of Education, Washington, D. C., 2008.
- [30] G. Polya. *How to solve it: A new aspect of mathematical method*. Doubleday, Garden City, 2nd edition, 1957.
- [31] C. Rasmussen, O.N. Kwon, K. Allen, K. Marrongelle, and M. Burtch. Capitalizing on advances in mathematics and k-12 mathematics education in undergraduate mathematics: An inquiry-oriented approach to differential equations. *Asia Pacific Education Review*, 71:85–93, 2006.

- [32] C. Rasmussen and W. Ruan. Teaching for understanding: A case of students learning to use the uniqueness theorem as a tool in differential equations. In M.P. Carlson and C. Rasmussen, editors, *Making the connection: Research and practice in undergraduate mathematics*, number 73 in MAA Notes, pages 153–164. Mathematical Association of America, Washington, D.C., 2008.
- [33] L.S. Shulman. Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2):4–14, 1986.
- [34] J. Silverman and P.W. Thompson. Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Educator*, 11:499–511, 2008.
- [35] N.M. Speer and J.F. Wagner. Knowledge needed by a teacher to provide analytic scaffolding during undergraduate mathematics classroom discussions. *Journal for Research in Mathematics Education*, 40(5):530–562, 2009.
- [36] M. Terrell. Asking good questions in the mathematics classroom. Prepared for the AMS-MER Workshop “Excellence in Undergraduate Mathematics: Mathematics for Teachers and Mathematics for Teaching”, March 2003.
- [37] M. Thames. Using math to teach math: Mathematicians and educators investigate the mathematics needed for teaching. Mathematical Sciences Research Institute, 2006.
- [38] P.W. Thompson. Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, and A. Sépulveda, editors, *Proceedings of the Annual Meeting of the International Group for the Psychology of Mathematics Education*, volume 1, pages 45–64, Morélia, Mexico, 2008. <http://pat-thompson.net/PDFversions/2008ConceptualAnalysis.pdf>.
- [39] K. Weber. Developing students’ understanding of exponents and logarithms. In *Proceedings of the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Athens, GA: University of Georgia, 2002.

Colophon

This booklet was typeset with the \LaTeX memoir class. The chapter and section titles use the veelo style. The fonts in the booklet are \TeX Gyre Pagella, which is based on the URW Palladio family (a Palatino clone), and Latin Modern.



MSRI 
Mathematical Sciences Research Institute

www.msri.org

Main Office: 510-642-0143 • Fax: 510-642-8609
Mailing Address: 17 Gauss Way • Berkeley, CA 94720-5070