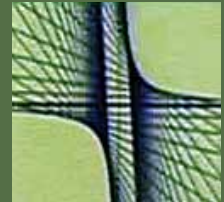


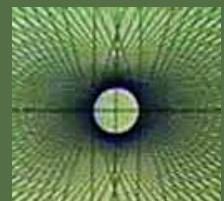
Reasoning & Sense-Making in the Math Curriculum

The Common Core State Standards • Curricula in the United States
Curricula in Other Countries • Assessment • Technology

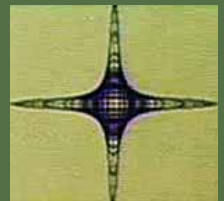
By Julie Rehmeyer



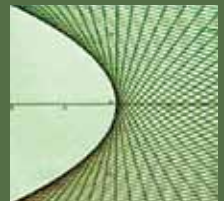
$a \cdot b = c$, with $c = 24$



$a^2 + b^2 = c^2$, with $c = 5$



$a^2 + b^2 = c^2$, with $c = 5$



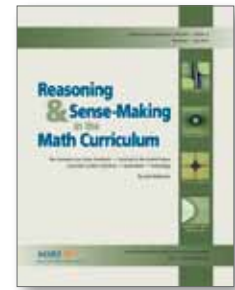
Where is $y^2 = 4x$ in
 $ax + ary = ar^2$

Reasoning & Sense-Making in the Math Curriculum

Critical Issues in Mathematics Education

Workshop 7 • June 2010

Mathematical Sciences Research Institute



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The workshop speakers were chosen for their ability to articulate widely held perspectives on mathematics education, but this choice is not meant as an endorsement of those perspectives.

Opinions, conclusions, and recommendations expressed in this booklet are those of the author and do not necessarily reflect the views of The Mathematical Sciences Research Institute, the sponsors nor the organizing committee of the workshop.

Cover images: from exercises to assess reasoning and sense-making, see page 30.

REASONING AND SENSE-MAKING IN THE MATH CURRICULUM

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INTRODUCTION

The Greek root of the word “mathematics” means “the learnable thing.” But the experience of many students is quite the opposite — that mathematics is quite difficult to learn and, in particular, that it doesn’t make sense. That shows a total breakdown of learning, because mathematics is entirely about making sense. Reasoning is the very foundation of mathematics. How is it that students so often miss this entirely, and how can the educational system help students to become facile reasoners who are able to make sense of mathematics for themselves?

That was the focus of the 2010 Critical Issues in Mathematics Education conference at the Mathematical Sciences Research Institute. The Critical Issues in Mathematics Education series of conferences are an effort to spread innovative ideas and form a community by offering those involved in all facets of mathematics education an opportunity to gather, share ideas, and work together to find new solutions in mathematics teaching.

The backdrop to the conference was that the National Council of Teachers of Mathematics (NCTM) had just released a new document, *Focus in High School Mathematics: Reasoning and Sense-Making*. Furthermore, the Council of Chief State School Officers and the National Governors Association had initiated a state-led effort to produce Common Core State Standards (CCSS), in order to move states toward national curricular coherence. The national scene was being transformed through stimulus money aimed at having states adopt common standards. It therefore was a significant time for mathematicians to weigh in for coherence and a focus on thinking, understanding, and sense-making.

The conference focused on three questions in particular: How can curricula support reasoning and sense-making? How do we assess students’ reasoning and sense-making skills? And how do we promote reasoning and sense-making by how we use technology?

THE COMMON CORE STATE STANDARDS

The advent of the Common Core State Standards (CCSS) offers an unprecedented opportunity to promote reasoning and sense-making across the United States.

Many of the arguments for common standards are practical. Kids move around a lot, and when standards vary, such moves can lead to gaps in their knowledge. Curricula based on a common standard will allow teacher education to focus specifically on what students should know and be able to do in each grade. And research can refine common standards over time, focusing resources to produce better standards than a single state could likely do on its own.

But in addition to that, the Common Core State Standards have given math education a fresh start, allowing educators to build standards designed to get students reasoning and making arguments from the very beginning.

A key aspect of that is that the CCSS are coherent: they build the mathematical concepts in a logical, orderly way, introducing new ideas only when students have had a chance to master the concepts they are built on. William Schmidt of Michigan State University showed how this structure becomes apparent when you look at the way in which the highest-scoring countries on the Trends in International Mathematics and Science Study (TIMSS) distribute topics across grade levels. The chart below shows the structure of the curriculum of mathematically high-achieving countries.

Composite of high-achieving countries

Topic	Grade 1	Grade 2	Grade 3	Grade 4	Grade 5	Grade 6	Grade 7	Grade 8 ¹
Whole Number: Meaning	j	j	j	f	f			
Whole Number: Operations	j	j	j	j	f			
Measurement Units	d	j	j	j	j	j	f	
Common Fractions			d	j	j	f		
Equations & Formulas			d	f	f	f	j	j
Data Representation & Analysis			d	d	f	f	f	d
2-D Geometry: Basics			d	f	f	f	j	j
2-D Geometry: Polygons & Circles				f	f	f	j	j
Measurement: Perimeter, Area & Volume				f	f	f	f	d
Rounding & Significant Figures				f	f			
Estimating Computations				f	f	f		
Whole Numbers: Properties of Operations				d	f			
Estimating Quantity & Size				d	d			
Decimal Fractions				f	j	f		
Relation of Common & Decimal Fractions				f	j	f		
Properties of Common & Decimal Fractions					f	f		
Percentages					f	f		
Proportionality Concepts					f	f	f	d
Proportionality Problems					f	f	j	j
2-D Geometry: Coordinate Geometry					d	d	f	f
Geometry: Transformations						f	f	f
Negative Numbers, Integers, & Their Properties						d	f	
Number Theory							f	d
Exponents, Roots & Radicals							f	f
Exponents & Orders of Magnitude							d	d
Measurement: Estimation & Errors							d	
Constructions using Straightedge & Compass							j	d
3-D Geometry							f	j
Geometry: Congruence & Similarity								j
Rational Numbers & Their Properties								d
Patterns, Relations & Functions								d
Proportionality: Slope & Trigonometry								d
Number of topics intended by at least 2/3 of A+ countries	3	3	7	15	20	17	16	18
Number of additional topics intended, on average, by A+ countries to complete their curriculum at each grade level. ²	2/4	6/7	5/8	1/1	1/2	3/5	6/10	3/7

- Mathematics topics intended at each grade by at least two thirds of A+ countries.
- A+ countries determined by looking at statistically significant differences in 8th grade scores on 1995 TIMSS.
- On average an A+ country would have 1-6 more topics per grade level in its complete curriculum.

Topics are listed on the left, and grade levels are listed along the top. "d" entries indicate topics that are intended by 4 out of the 6 top-scoring countries; "f" by all but one of the top-scoring countries (5 out of 6); and "j" by all of the countries.

Topics are introduced only once the foundational topics are sufficiently developed, they are studied for a few years, and then the curriculum moves on. This generates the clear upper-triangular structure of the above graph.

By contrast, consider the comparable graph for U.S. states, at right.

Many more boxes are filled, and the logical upper-triangular structure is gone. The graph illustrates the “mile-wide, inch-deep” phenomenon that afflicts U.S. curricula. Introducing topics too soon is damaging because students aren’t equipped to understand them. Both introducing topics too soon and holding them too long distract students from focusing on the essential ones

Mathematics topics intended at each grade by at least two-thirds of U.S. states. On average, a state would have 6-8 more topics per grade level in its complete curriculum (Schmidt, Houang, and Cogan, *American Educator*, 2005).

Below is the comparison graph for the Common Core State Standards. The shaded parts of the graph are the topics covered by the highest-scoring countries. Though the correspondence is not exact, the CCSS does have the basic upper-triangular structure, offering a far more coherent curriculum.

Composite of U.S. State Curricula

Topic	Grade	1	2	3	4	5	6	7	8
Whole Number: Meaning		j	j	j	j	f	d		
Whole Number: Operations		j	j	j	j	f	d		
Measurement Units		f	j	j	f	j	j	j	f
Common Fractions		d	f	f	f	j	f	d	d
Equations & Formulas		d	d	f	f	f	j	j	j
Data Representation & Analysis		j	j	j	j	j	j	j	f
2-D Geometry: Basics		f	f	f	f	f	f	f	f
2-D Geometry: Polygons & Circles		j	j	j	j	j	j	j	f
Measurement: Perimeter, Area & Volume			d	d	d	f	j	j	f
Rounding & Significant Figures									
Estimating Computations		d	d	f	f	f	f	f	f
Properties of Whole Number Operations		d	d	d	d				
Estimating Quantity & Size				d					
Decimal Fractions				d	f	f	f	d	d
Relation of Common & Decimal Fractions					d	d	d		
Properties of Common & Decimal Fractions									
Percentages						d	f	f	d
Proportionality Concepts							f	d	
Proportionality Problems							f	f	f
2-D Geometry: Coordinate Geometry				d	f	d	d	d	f
Geometry: Transformations		f	f	f	f	f	f	f	f
Negative Numbers, Integers, & Their Properties							d	f	d
Number Theory						f	d	d	d
Exponents, Roots & Radicals							d	d	j
Exponents & Orders of Magnitude								d	d
Measurement: Estimation & Errors		d	d	f	d	f	f	f	d
Constructions using Straightedge & Compass									
3-D Geometry		j	j	j	f	j	f	j	f
Geometry: Congruence & Similarity						d	f	f	d
Rational Numbers & Their Properties							f	f	d
Patterns, Relations & Functions		f	j	j	j	f	j	j	j
Slope & Trigonometry									
Number of topics intended by at least 2/3 of 21 states		14	15	18	18	20	25	23	22
Number of additional topics intended by states to complete a typical curriculum at each grade level.		8	8	7	8	8	5	6	6

Topics are listed on the left, and grade levels are listed along the top. “d” entries indicate topics that are intended by 4 out of the 6 top-scoring countries; “f” by all but one of the top-scoring countries (5 out of 6); and “j” by all of the countries.

Comparison of CCSS with A+ composite

Topic	1	2	3	4	5	6	7	8
Whole Number: Meaning	●	●	●	●	●			
Whole Number: Operations	●	●	●	●	●			
Measurement Units	●	●	●	●	●	●	●	●
Common Fractions	●	●	●	●	●	●	●	●
Equations & Formulas	●	●	●	●	●	●	●	●
Data Representation & Analysis	●	●	●	●	●	●	●	●
2-D Geometry: Basics	●	●	●	●	●	●	●	●
2-D Geometry: Polygons & Circles	●	●	●	●	●	●	●	●
Measurement: Perimeter, Area & Volume	●	●	●	●	●	●	●	●
Rounding & Significant Figures								
Estimating Computations	●	●	●	●	●	●	●	●
Whole Numbers: Properties of Operations	●	●	●	●	●			
Estimating Quantity & Size								
Decimal Fractions				●	●	●	●	●
Relation of Common & Decimal Fractions			●	●	●	●	●	●
Properties of Common & Decimal Fractions								
Percentages							●	●
Proportionality Concepts							●	●
Proportionality Problems							●	●
2-D Geometry: Coordinate Geometry					●	●	●	●
Geometry: Transformations				●	●	●	●	●
Negative Numbers, Integers, & Their Properties				●	●	●	●	●
Number Theory				●	●	●	●	●
Exponents, Roots & Radicals					●	●	●	●
Exponents & Orders of Magnitude								
Measurement: Estimation & Errors								
Constructions Using Straightedge & Compass								
3-D Geometry	●	●	●	●	●	●	●	●
Geometry: Congruence & Similarity							●	●
Rational Numbers & Their Properties							●	●
Patterns, Relations & Functions							●	●
Proportionality: Slope & Trigonometry								●
Topic Intended in Common Core Standards	●							
Intended in More than Half of Top Achieving Countries	●							

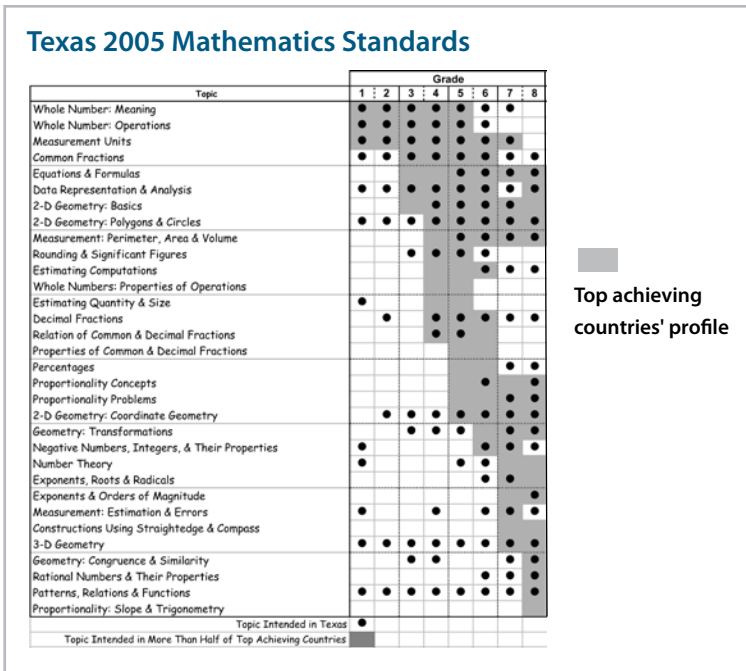
• This number of extra topics per grade level in CCSS is compatible with A+ countries.

“A curriculum should be so well set up that it makes it transparent that there’s a logic that underlies mathematics. I think that’s why a lot of countries succeed better than we do. They make sure that’s the case through their national standards.”

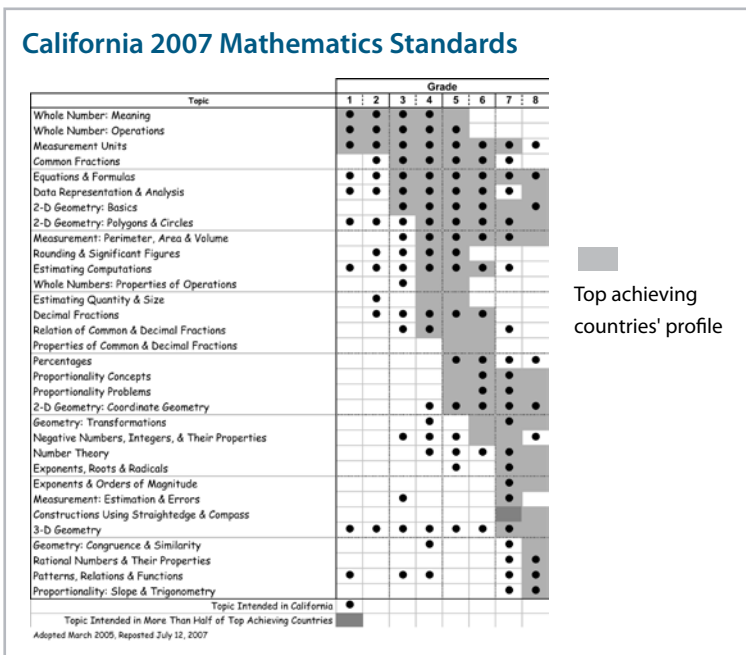
– William Schmidt
Michigan State University

To accomplish this, the designers focused largely on number and operations in the early grades. They did not include strands for data or patterns in those grades, though some work on those topics is incorporated into the primary focus on number and operations.

By comparison, the previous curricula of most states were far less structured and coherent. Texas, for example, has lots of holes:



California, which has a well-regarded curriculum, is nevertheless cluttered by dots on the left, showing topics that are introduced too soon, and some noticeable holes:



Schmidt has done a regression analysis showing that the more a state's curriculum has the "ideal" structure exhibited by the highest-scoring countries, the higher the students' achievement scores.

William McCallum pointed out at the conference that although the development of the CCSS was a remarkable and unexpected achievement, what has been done represents only perhaps 10 percent of what needs to be done. The standards need to be transformed into curricula; assessments need to be developed; teachers need to be trained. All of these are enormous tasks.

Resource: The Common Core State Standards can be downloaded from www.corestandards.org.

"During the last years I've been visiting lots of states' departments of education and hearing their incredible enthusiasm for getting involved with the CCSS. There are moments when I've scratched my head and said, 'Why?' They all have their own standards and they think they've done good work. So why are they involved in this?"

When you look at the Composite of U.S. State Curricula diagram (top of page 6), you can see their motivation to come together. They recognize the value in having some kind of common standards where you don't have everything at every grade level."

– William McCallum of the University of Arizona, a designer of the Common Core State Standards

CURRICULA IN THE UNITED STATES

Although many talks at the conference focused on ways that the standard U.S. curriculum could be improved, some illustrated how even within its constraints, good teaching could lead to students with a strong ability to reason and make sense of mathematics.

Ginger Warfield of the University of Washington described their work with teachers from rural districts of Washington in the Rural Mathematics Teaching Project. The curriculum in these high schools consists of the traditional sequence of algebra 1, geometry, algebra 2, precalculus, and then calculus. Nevertheless, their students, on average, get 26.5 on the math ACTs, when the national average is 20.1. And about half of their graduating seniors complete calculus before leaving high school.

One key to their success is that teachers in different grade levels communicate well, so that an algebra 1 teacher understands what the students will need when they get to geometry. Teachers act as facilitators, and the students collaborate with one another extensively, developing their own ideas about how to attack the problems. This means that students spend much more time in class reasoning about math for themselves and discussing their ideas with their peers.

They don't use textbooks. Instead, teacher teams develop all their own teaching materials. Time is built into the schedule for collaborations, and teachers additionally meet outside of school hours.

However, there are challenges. The mathematical knowledge of the teachers isn't always as deep and broad as is needed for these open-ended tasks, and this can prevent them from seeing the interconnections between different aspects of the curriculum. Furthermore, parents and students often don't properly understand what it means to do mathematics. Teachers also struggle to assess reasoning and sense-making: it's far easier to assess algorithmic competence.

THE INTERACTIVE MATHEMATICS PROGRAM (IMP)

Brian Lawler of the California State University, San Marcos, described an unusual curriculum developed between 1989 and 1999 by mathematics educators from the Lawrence Hall of Science and UC Berkeley, mathematicians from San Francisco State University, and teachers from Berkeley High School, Tracy High School, and San Francisco’s Mission High School.

The curriculum is built on the underlying belief that students learn important things, including math, when interacting with peers over ideas they find important. So the curriculum is organized around big, rich mathematical problems, and courses are organized by age rather than being tracked according to ability. The varied strengths and weaknesses of mathematical ability in the heterogeneous classroom is viewed as an asset, rather than a challenge to be overcome.

IMP has been quite successful: it is one of three high school math curricula identified as “exemplary” by

the U.S. Department of Education for providing convincing evidence of its effectiveness in multiple schools with diverse populations, and it was among the highest-ranked algebra textbooks in a review undertaken by the American Association for the Advancement of Science (AAAS).

Using the curriculum requires changes in how a classroom is run. The teacher has to renegotiate authority in the classroom, making student ideas the center of the work. Teachers are challenged to think differently about what mathematics is, what it means to be smart at math, and what constitutes effective teaching. It is harder work for the teacher to listen to, interpret, and act upon student ideas, than to merely explain their own mathematical understandings. Students also have to change their expectations about what math is and how math class works, as do parents. So educating the school community as a whole — including administrators, parents, the school boards, the superintendent and students — is essential to success.

“In order to present correct mathematics, two minimum requirements must be met: Textbooks have to be correct, and the teacher has to teach correct mathematics. Both of these requirements are very far from being met on a consistent basis.”

— Hung-Hsi Wu, University of California, Berkeley

THE PD³ MATH-SCIENCE PARTNERSHIP OF THE IAS/PARK CITY MATHEMATICS INSTITUTE

A national project of the IAS/Park City Mathematics Institute was aimed at support for the improvement of school districts in three sites. One of these was Seattle, where the task was to improve three urban Seattle high schools. Many students at these schools were coming into ninth grade with a very low level of mathematical competence. The project helped teachers decide to focus on what teachers can control, mainly how they teach. But it was tough to convince teachers that different results are possible with the same students given so many challenges: attitude, poor attendance, unsupportive parents, and poverty. One of the schools was never able to get past this, but the others made significant change over time.

The team started with extensive professional development using an approach called Complex Instruction. This emphasizes the impact that status in a classroom — that is, whether students are considered smart or dumb or cool or inconsequential — has on students' learning. It works to reduce the problems caused by negative status through using tasks that are rich enough to justify students working in groups

and that genuinely require the participation of all students. And it emphasizes that practice is key to learning: if students aren't participating, the tasks and textbook won't matter.

The teachers on their own chose to use a different set of textbooks, the Interactive Mathematics Program (described above). To support the change in teaching materials, teachers were provided with additional professional development. This was especially important since IMP pushed the boundaries of the teachers' mathematical knowledge, so the teachers themselves wrestled with some of the same issues that would trouble their students. In addition, the team of teachers teaching ninth-grade math had an hour a day to meet, which crucially supported the teachers in the changes they were making. There was also a monthly video club, where the teachers would analyze evidence of student learning in a short snippet of video from one of their classrooms. Though the focus was on student learning rather than teaching moves, it had a significant effect on how teachers thought about their own work.

“When we tell students that $x^2-6x=-2$, we know it is nonsense to say that a bunch of symbols is equal to a number. But we still want to maintain the façade of reasoning, so we invent a balance to explain everything. This is a very insidious move, because it seems so persuasive that it is very hard for a student to disagree. But you have to be aware that by doing this, you’re subverting mathematics. You should have said instead: ‘Suppose there’s a solution. Call it x .’ Then we can proceed, because x is now a number.

The way we teach solving an equation by regarding x as a symbol is the reason why so many students get mixed up about what the equals sign means. Once you realize this, you would know that you really don’t need to do research on students’ misunderstanding of the equal sign because you recognize that the equal sign has been so abused throughout school mathematics that students cannot help but be confused.”

— Hung-Hsi Wu, University of California, Berkeley

USING THE SINGAPORE MATH CURRICULUM IN THE U.S.

The City of Baker School System in Louisiana used to typically be in the bottom three of all Louisiana school districts in math test scores. Classroom management problems were pervasive, and the administration changed frequently.

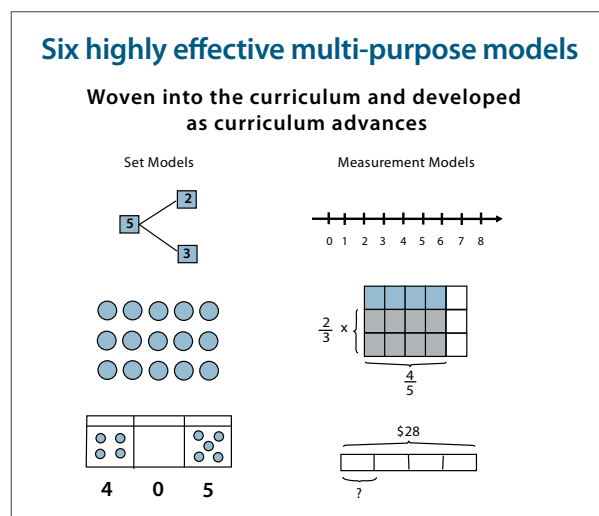
So the district decided to change to using the Singapore primary math curriculum and to invest in intensive professional development to make it work.

The program has been remarkably successful. Second-graders increased their scores between pre- and post-tests by 27 points over the course of one semester, whereas students who weren't in the program increased their score by only 12 points over the entire year. Some of the reasons the curriculum works well were described by Kimberly Basley, a teacher in the district, and Ben McCarty of Louisiana State University, which supported the project.

In addition, Robin Ramos, a math coach at Ramona Elementary in Los Angeles, reported on the successful use of the curriculum at her school, and Thomas Parker of Michigan State University described his observations of it as a mathematician.

One powerful aspect of the curriculum is that it extends old ideas into new ones, often back to back. For example, second graders learn fractions using halves and quarters of circles just before learning about telling time on a clock. Third graders learn about weight, a more challenging type of magnitude because it must be felt rather than just seen. Students start by comparing weights with a balance scale and then move to a spring scale, with an arm that swings through an arc to point at the right weight. This comes shortly before the students study measuring angles on a protractor. Each step prepares the students for the next thing in the curriculum.

The curriculum also uses just a few models that it develops through the curriculum. It focuses on just six models — simple diagrams that students can use to visualize concepts — and develops them very thoroughly over the years. For example, bar diagrams, rectangular arrays and area models are used over and over in different contexts. The pictures come to feel like old friends to the students. This judicious use of models allows the curriculum designers to show how the same reasoning applies in varied contexts.



Parker pointed out that the curriculum pays very close attention to precise language. Words like “also,” “each,” “than,” “equal groups,” “groups of,” and “altogether” all operate somewhat differently in a mathematical context than in regular language, and each one needs to be explicitly discussed in the early grades.

Los Angeles Times

Front page of the *Los Angeles Times*: March 9, 2008

Here's a little math problem: In 2005, just 45% of the fifth-graders at Ramona Elementary School in Hollywood scored at grade level on a standardized state test. In 2006, that figure rose to 76%. What was the difference?

If you answered 31 percentage points, you are correct. You could also express it as a 69% increase.

But there is another, more intriguing answer: The difference between the two years may have been Singapore math.

At the start of the 2005-06 school year, Ramona began using textbooks developed for use in Singapore, a Southeast Asian City-state whose pupils consistently rank No. 1 in international math comparisons. Ramona's math scores soared.

“It's wonderful,” said Principal Susan Arcaris. “Seven out of 10 of the students in our school are proficient or better in math, and that's pretty startling when you consider that this is an inner-city, Title 1 school.”

Ramona Elementary had so much success during the first year of using the Singapore Mathematics Curriculum that they were the subject of a front-page LA Times story.

PROBABILITY AND STATISTICS IN AMERICAN SCHOOLS

Probability and statistics have become an integral part of the U.S. mathematics curriculum. They are part of the current standards, and more people are studying the subject. For example, in 1997, the AP stats exam was taken by 7,500 people, whereas in 2008, the number jumped to 116,500.

Research indicates that a curriculum in probability and statistics is most effective if it:

1. Uses real data.
2. Clearly distinguishes concepts in statistics from concepts in mathematics.
3. Makes students active participants through technology (graphing calculators, apps, and software programs).

Anna Bargagliotti of Loyola Marymount University reviewed the statistics content in three elementary-level NSF-funded curricula: Trailblazers, Everyday Math, and Investigations. She found that there was variation among them both in content and in the level of sophistication of that content. But they were united in lacking a statistics perspective in their presentation. Probability, for example, was typically discussed in relation to the frequency of a particular outcome rather than in the context of sampling.

She pointed out that it's not difficult to shift from a more mathematical perspective to a statistical perspective.

The classic urn problem, for example, would go like this:

I have a bag with X red marbles and Y green marbles. What is the probability of randomly selecting a green marble?

To shift to a statistical perspective, the problem could be changed to this:

I have a bag with X red marbles and Y green marbles. A random sampling of Z marbles are drawn from the bag. (Here the students can either draw the samples themselves, or a table can be provided with samples.) Use the information about the random samples in order to draw conclusions about the relative values of X and Y .

Bargagliotti argued that the elementary curricula should include data gathering (mostly in the form of classroom censuses), discussion of the factors that can cause variability, and the introduction of distributions using bar graphs and plots. In the middle grades, she argued that they should include the introduction of random assignment, the comparison of distributions, the introduction of sampling error, and an acknowledgement that results may differ if data were collected a second time. At the high school level, students should formulate their own questions, collect the data needed in order to answer the questions (or have the data provided to them), analyze the data using appropriate statistical techniques, and draw conclusions from their analyses.

EXEMPLARY PROBLEM

This is a problem for first graders from an east Asian curriculum. It can be solved in two minutes, suiting the attention span of a six-year-old.



$$\underline{\quad} - \underline{\quad} = 4$$

EXEMPLARY PROBLEM

A student added 357 to a number. She added correctly and got 625. But the problem actually asked her to add 375.

What is the correct answer?

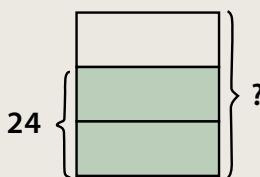
This is a third-grade problem from Korea. It is not algorithmic; it requires students to form a logical argument. It's a challenging problem, but the curriculum has given the students the tools they need to solve it.

She also discussed what teachers need to know to be prepared to teach a statistics course. At the most basic level, they should be able to articulate what statistics is, which teachers have more difficulty with than one might expect. For example, secondary teachers in a graduate course at the University of Memphis, all of whom had a master's in education and some of whom had taught AP statistics, gave answers like "It's a way to describe a data set." They couldn't explain how sampling and inference were related, and none of them had explored or manipulated data using technology.

Statistics courses for teachers, Bargagliotti argued, should include both statistical content and statistical pedagogy. They should use relevant data (educational data is especially effective with teachers, though other data sets would be more compelling to students). And they should use technology: for example, visual aids or applets to teach sampling concepts, or software that can perform statistical tests and estimate statistical models. By the end of the course, teachers should be able to formulate research questions and answer them using real data.

EXEMPLARY PROBLEM *A tank contains 24 liters of water. It's two-thirds full. What is the capacity of the tank?*

This can be done as an algebra problem, or as division of fractions. But perhaps the easiest way is to think in terms of units. If you choose units that are one-third of the tank, then two units are 24 liters, so one unit is 12, and three units are 36. So the tank holds 36 liters.



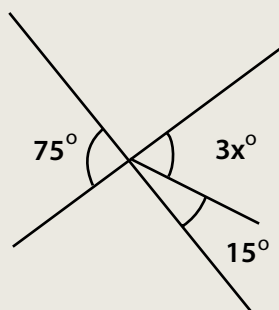
- 2 units: 24 L
- 1 unit: 12 L
- 3 units: 36 L

The tank holds 36 liters

The Singapore curriculum does an especially good job of teaching students to choose units. It is the idea underlying the Singapore bar diagram, and it's taught every single day.

EXEMPLARY PROBLEM *Study of one topic builds skill at others — leverage.*

This looks like a geometry problem, though the geometric knowledge required is minimal (just the equality of vertical angles). It's really a hidden algebra problem. Algebra is introduced through little puzzles like these, providing a motivation to find an efficient way to do it.



$$3x + 15 = 75 \quad \text{vert. } \angle\text{s.}$$

$$3x = 60$$

$$\therefore x = 20.$$

CURRICULA IN OTHER COUNTRIES

JAPAN

Thomas Judson of the Stephen F. Austin State University in Nacogdoches, Texas, described the Japanese approach to teaching mathematics.

The Japanese curriculum is structured very much like ours, with students in elementary, middle school, high school and university at the same ages as American students. However, Japan has a national curriculum that sets the number of class periods for the year, the length of the classes, the subjects that must be taught, and the content of each subject for every grade. The result is that throughout the country, most schools are teaching the same topics in the same week. This curriculum is revised every ten years, with a major revision every 50 years.

In elementary school, children learn the four operations with whole numbers in grades 1-4. In grades 5 and 6, they learn to multiply and divide decimals and fractions. They learn about measurement and the metric system; about statistical data by using percentages and circle graphs; about the concepts of area and volume and how to measure these quantities for simple geometric figures; about plane and solid geometric figures; about symmetry; and about congruence. The abacus is introduced in grade three. Most classrooms have computers but handheld calculators are relatively rare.

In middle school, students begin to prepare to compete to get into the best high schools and universities. They learn about positive and negative numbers, the meaning of equations, letters as symbols, and algebraic expressions. Algebra begins in grade 7 and continues in grades 8 and 9. Schools are supposed to teach students to use handheld calculators, but it doesn't always happen.

In high school, all students take the same math class in grade 10, learning about quadratic functions, trigonometric functions, sequences, permutations and combinations, geometry and probability. After that, students differentiate into a humanities or science track. Humanities students study no more mathematics. Science students learn about exponential functions, trigonometry, geometry, analytic geometry, and some calculus in grade 11, and calculus in grade 12.

It's a more integrated curriculum than in U.S. schools, without the division into algebra 1, geometry, and algebra 2. One of the major tasks is to prepare students for the university entrance exams. After-school education is an integral part of the educational system, and there is an expectation that students will devote significant time to their studies outside of class. The after-school *juku*, or cram schools, can be a social time for students, because that's where they often meet their friends.

An enormous difference with the U.S. is that teachers in Japan are highly trained and the profession is very competitive. In 1999, there was one opening for a high school math teacher in all of Tokyo. Teachers are nearly as respected as doctors, and they are well paid. There is also an extensive mentoring system for new teachers lasting five years. A premium is placed on carefully crafted lessons, and lesson study is common.

Still, many university professors feel that basic math knowledge has declined. Nishimura's book, *University Students Can't Do Fractions*, received widespread attention in 1999. Japan is concerned about whether it is able to keep pace with countries like Korea and Singapore.

SINGAPORE

Carmen Hoo, Sandi Kum, and Pang Soh Lian from the Raffles Girls School spoke about the mathematics curriculum used in Singapore itself. They remarked that the relationship between “Singapore math” as used in the U.S. and the way that math is taught in Singapore itself isn’t so clear.

They began by describing Singapore. It is an extraordinarily small country, smaller than the city of Los Angeles, but with more people: 5 million, whereas LA has 4 million. Singapore was founded by the English and became an independent republic in 1965. It’s a young nation whose only natural resource is its people. It has to import all its materials.

Education is the second largest chunk of the national budget, only trailing defense. It’s a passionate and emotional topic. There’s a constant pressure to evolve to respond to the needs of students, which is easier because the country is small. Diverse educational options are required to maximize individual potential.

The country has a lot of immigrants. Chinese are the largest immigrant group, but there are also many Malaysians, Indians, Americans, Australians, Vietnamese, Burmese, South Asians, Europeans, and Canadians. Two-thirds of children come from English-speaking homes. Because of the large migrant population, language can be an issue. Children from other countries pick up English very quickly.

Primary education, the first six years, is compulsory by law. Secondary education is four to five years.

More than 90% of 7-year-olds have had a year of preschool education. Many come in able to read and write. Some can’t write their names, others have read all of Harry Potter. At the end of third grade, they have a test for gifted children. If they are assessed as the top 1%, they are considered gifted and can choose to go into the gifted education program. Others do the regular curriculum until the end of fourth grade, when they have subject-based banding. So if children find math very challenging, for instance, they will take foundation math. The same child who finds English very comfortable will take standard English. Kids will often take some foundation subjects and some standard subjects. Regardless, you have to take the high-stakes primary school national leaving exam at the end of sixth grade. After that, the secondary and post-secondary education has many different pathways.

There is only one place for teacher training, the National Institute of Education. All middle and high school teachers are university graduates and, increasingly, elementary teachers are also university graduates. Teacher training at NIE is level and subject specific. Before future teachers enter NIE, they’ve already been selected to teach two specific subjects. To teach grades 11 and 12, teachers must have an honors degree in the teaching subject.

Throughout the career of a teacher in Singapore, provisions are made for pursuing masters and doctoral degrees. One hundred hours of training are required per year. Professional development workshops happen in June and December, during the holidays. Many Americans come to do professional development during this time.

Sample Assessment Task from Singapore

6 Facets of Understanding

(Application & Interpretation)

Sample Task 2:

Topic: Mensuration

Introduction



Suppose you are a chocolate manufacturer and you need to study the packaging of the existing chocolate brands in the current market. The following types of chocolates have been selected for study.

- Kit Kat
- Toblerone
- Ferrero Rocher

The task

1. Find the **total volume** and **total surface area** of each brand of chocolate.

Hence, or otherwise, determine which of them will melt in the shortest time at room temperature. Justify your answer, stating any assumptions that you may have made.

Note that students get to eat the chocolate at the end of the class!

The Raffles Girls' School was started in 1879 with 77 girls and now has 1800. It's an independent school for the top 3%. The curriculum and pedagogy are driven by the bright kids. Their students got the 98% distinction on the O-level exams. Still, their teachers wonder, do they really understand, or can they just parrot? Can they use their knowledge in a real life context? What does understanding really look like?

The school's curriculum is driven by the philosophy that math is the endeavor of man to understand, represent and process the world with precision and rigor; that math seeks truth and beauty in the elegance of patterns and logic; and that mathematics' ability to generalize and its vast applications for solving real-life problems provide a constant challenge to the inquiring mind.

The design of the curriculum begins with identifying the desired results, beginning with the end in mind. The next step is to determine acceptable evidence of learning, and the final one is to plan learning experiences and instruction.

A key principle of stage one is that less is more, that is, students have to think for themselves and uncover understanding. Essential questions need to be foremost to guide teachers in class. And central to the process is the understanding that math is a language consisting of carefully defined terms and symbols, that it involves formulation of hypotheses, conjectures, verifications, and proofs, that it is the study of patterns and relationships, and that it is a tool used to solve problems in real life.

Stage two, assessment, is guided by the belief that assessment is the heart of learning and that assessing for knowing and understanding are different. Assessment is designed around authentic tasks, and it focuses on six different facets of understanding:

- **Explanation.** Students must justify how they arrive at their answers and why they're right. How can they prove it? How does this work? Why is it so?

- **Interpretation.** Students investigate the importance of particular concepts, formulas, or theories. What does it mean? Why is this formula important? Why should it be applied here?
- **Application.** Students should ask, how or when can I use the knowledge and skills that are taught in class?
- **Perspective.** The ability to take multiple points of view. What has been assumed? Is this solution reasonable? Is it feasible in this situation? Do we accept the solution at face value, or do we need to explore further?
- **Empathy.** Seeing from someone else's point of view. What do other people see and I don't? What do I need to experience if I am to understand?
- **Self-knowledge.** Students must develop awareness of what they don't understand, or why understanding is so hard. What are the limits of my understanding? Where are my blind spots? Why am I prone to misunderstanding? Is it habit, prejudice, or style?

Assessment is done in many forms, many of which are rare in U.S. classrooms. For example, students may be sent to a mathematically interesting place and, for example, asked to figure out the slope of a ramp. At least once a year, students are given an in-depth performance task. They may be assessed through journal writing, or oral exams, or expository writing.

Stage three, designing classroom activities, is guided by the belief that learning is messy and iterative, that understanding is earned not given, and that each brain is unique. Learning activities are sequenced to help students connect ideas, make meaning of them and transfer them from one context to another. The curriculum is differentiated for students with different skill levels. And the curriculum is designed to appeal to multiple intelligences.

Sample Assessment Tasks from Singapore

Sample Task 3

Topic: Mensuration



Popcorn! Popcorn!



A new cinema operator, Silver Village, is looking for an operator to man its refreshment outlet to sell popcorn and drinks to cinema-goers.

After conducting a survey, it is revealed that as much as 70% of movie-goers feel that popcorn is a must-have while watching a movie. In order to maximize its revenue, they decided to award the operation license to the vendor who was able to design a popcorn holder with the smallest volume but uses the largest area equivalent to an A3-size vanguard sheet. In addition, the selling price and profit margin of each container of popcorn will also be taken into consideration.

This problem throws students into a new role, as a popcorn vendor and not just a moviegoer.

Viva Voce – Sample Question

Grade 10 Unit: Differentiation

Explain how you would find and verify a turning point. Hence, find the minimum gradient of $y = f(x)$

- Prompts:
 - Does the curve have a stationary point?
 - If yes, where and what is the nature? If not, why not?
 - What is the connection between the three functions, y , y^1 and y^2 ?

This problem would be used in an oral exam in grade 10, designed to get students talking about their reasoning process:

Student Work on Journal Writing

Journals can take many different forms, but here's one example. Students are given this task:

Alyssa says that $(a + b)^2$ is the same as $a^2 + b^2$.

Belinda says it's more.

Claire says it's less.

You need to settle their argument.

Explain for each whether their position is always true, never true, often true or seldom true. Provide examples where appropriate.

6 Facets of Understanding (Application)

Sample Task 1:

Topic: Similarity & Congruency

Note the shaded section



Standing by the side of Block J, you should be able to see Block K as shown in the picture on the left.

Devise an indirect method to estimate the area of the shaded section as indicated on the picture. Describe your method.

Note: You do not need to compute the actual measurement.

Students have to figure out the area of the shaded region without directly measuring it. (Students have to use the similarity between the photo and the real object.)

INDIA

Sridhar Rajagopalan is the managing director of Educational Initiatives (EI), an organization working in India to improve the assessment of student learning in order to give insights into the learning process and to provide concrete ways to improve learning. He began his presentation on the Indian educational system by describing how it currently works.

India effectively has two parallel educational systems. One consists of elite private schools with teaching usually conducted in English, and the top students at these schools can compete with the best in the world. But this system teaches only five to ten percent of the students. The rest go to public schools, and this system is currently in shambles, though an effort is being made to change that. In government schools, only a quarter of fourth grade students can do subtraction with regrouping and only half of fifth graders can read a paragraph. Half drop out by fifth grade.

Except for students opting for the International Baccalaureate program, all students take school-leaving “Board” exams in class 12, though only 20-25% of all students get to that point. These exams are extremely rote and memory-based. Few Indians are aware of how poor the levels of learning are. A lot of Indians don’t get it, partly because of the divisions in society. India focused more on higher education than on primary education until the 1990s. Since India has been growing, government spending on education has increased significantly in the last three to five years, and private spending is also significant. But no one is sure of the best way to make use of the funds.

However, India places an extremely high social value to education, much higher than in most parts of the world. Indeed, there is almost a single-minded focus on education and getting ahead. Even poor families spend a major portion of their income on education. Students who do well academically are looked up to. Students tend to be self-driven, and parents push their children. Among the middle class, engineering, management, education and medicine are in great demand, so there is lots of emphasis on math and science education. There is a very strong competitive spirit, though that also has negative implications, like a focus on grades instead of learning and high stress on students. It’s the opposite of the U.S., because the drive is so high but the management of education is very poor.

An enormous problem is the emphasis on rote learning. This real-life example sums it up: The daughter of a scientist, a fifth grader, told her father that she learned in her geography class that the difference between the equatorial and polar circumference of the Earth is 72 km. Impressed,

he asked her what she understood by “equatorial circumference.” She wasn’t sure. He asked what she understood by “polar circumference.” She had no idea. So he retreated and asked simply about circumference. “I don’t know,” she said. “What do you know then?” he asked, a little desperate.

“I know that the difference between the equatorial and polar circumference of the Earth is 72 km, that is all they are going to ask in the exam,” she replied promptly.

This is absolutely typical of the Indian system, Rajagopalan says.

Paradoxically, he says, this seems to work for the really bright students. A student who is intrinsically motivated, or from a strong family background, gets a kind of benefit from this kind of mindless rigor, though probably these students would be more creative if the system focused on understanding. A lot of the success of the Indian educational system comes from these students: It has so many students that the top two percent still turns out to be many people.

Like the U.S., India has a lot of internal differences and debates on education, and both countries have pluralistic societies with great cultural diversity. Also, both countries are large democracies and hence are difficult to reform quickly. While that can be frustrating, it is likely to be better in the long run. And, change does happen. For example, people no longer question whether learning can be measured, which they certainly did eight to ten years ago. Now there is a consensus in India that low-stakes assessment at least doesn’t do harm, though there’s great resistance to high-stakes testing.

There are also substantial differences between the two countries. State differences vary much less in India, both in reality and in perception. India has no government high-stakes testing requirements. India places a very strong social value on education, especially math and science, and has no student discipline or motivation issues. And the culture of research in education is almost absent in India.

Rajagopalan said that it seems that in the U.S., a reasonably good education system is let down by societal, parental and student attitudes, while in India, student and parent aspirations are thwarted by an indifferent system.

Rajagopalan’s organization is based on the idea that if you can provide data showing very clearly what the current state of education is, what exactly students understand and don’t, and if you follow it up with support to teachers, this could lead to change. To that end, they’ve developed a

low-stakes test called Asset that is taken by over 400,000 students in private schools in 2010-2011.

Based on this test, EI created a study highlighting how much learning is rote even in the best schools. Using questions drawn from the TIMSS exam, the study showed that in the top schools of the country, if you measure for understanding, the students do pretty badly compared to international benchmarks. On the other hand, if you check for procedure, they do pretty well. The study was featured on the cover of *India Today*.

Challenging Perceptions about Learning in 'top' Indian Schools – Nov, 2006



TOP SCHOOLS IN INDIA SHOW AN ALARMING GAP IN STUDENT LEARNING

INDIAN STUDENTS FARE POORLY IN COMPARISON TO COUNTERPARTS

LEARNING IS ROTE-BASED AND DOES NOT FOCUS ON REAL KNOWLEDGE

EI's approach seems to have had a significant impact on private schools over the last eight to ten years. Certain state governments are now trying this at a much larger scale to improve public schools.

Even apparently straightforward questions, designed to check if a basic concept has been understood, can show gaps even among advanced students. For example, sometimes students believe that angles with larger arms are themselves larger. Because there is so much emphasis on rote learning, with students reciting memorized definitions, teachers don't always see that these gaps in understanding exist, making these kinds of problems especially valuable, even though they are basic.

Indian children do great on questions that are completely rote, but even on a rote question that's just a little out of the way, children will slip up.

After the test, EI explains the results to teachers, who are often shocked at what their students can't do. EI analyzes the data in many different ways to increase what teachers learn from the test.

For examples, these graphs (below) show the percentage of students in grades 4, 6, and 8 who get a particular question correct at different performance levels. The x-axis shows the total number of questions a student has gotten correct, and the y-axis shows the percentage of those students who got this particular question right. These graphs show that the weaker students (who are lower on the x-axis)

Rajagopalan recounted this story that Richard Feynman told about his observations of education in Brazil:

In regard to education in Brazil, I had a very interesting experience. I once attended a lecture which went like this: 'Two bodies... are considered equivalent... if equal torques... will produce... equal acceleration. Two bodies are considered equivalent if equal torques will produce equal acceleration.' The students were all sitting there taking dictation, and when the professor repeated it, they checked it to make sure they wrote it down all right.



Richard Feynman

After the lecture I talked to a student: 'You take all these notes - what do you do with them?'

'Oh, we study them,' he says. 'We'll have an exam.'

'What will the exam be like?'

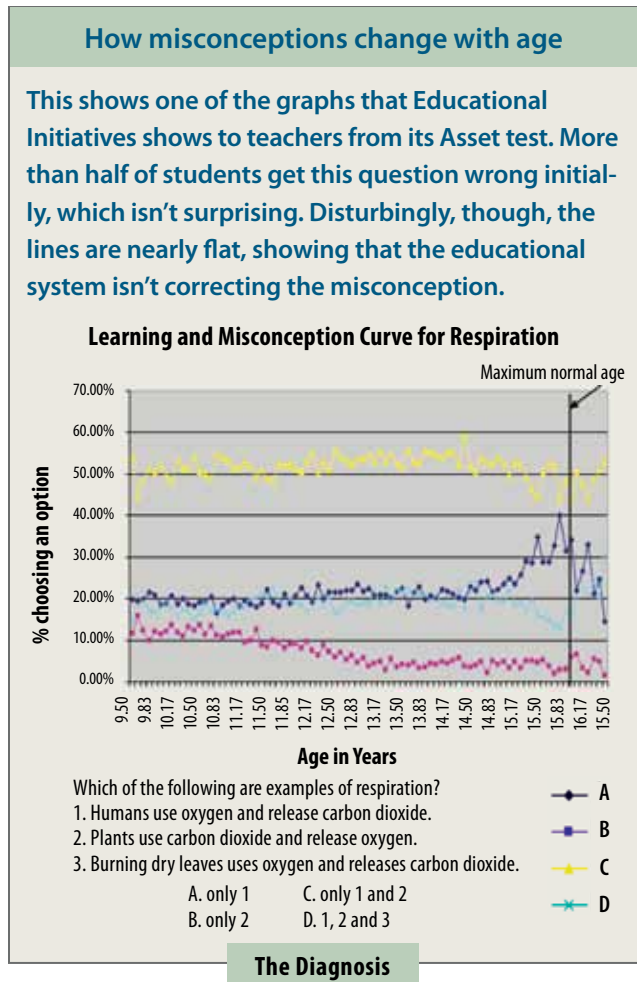
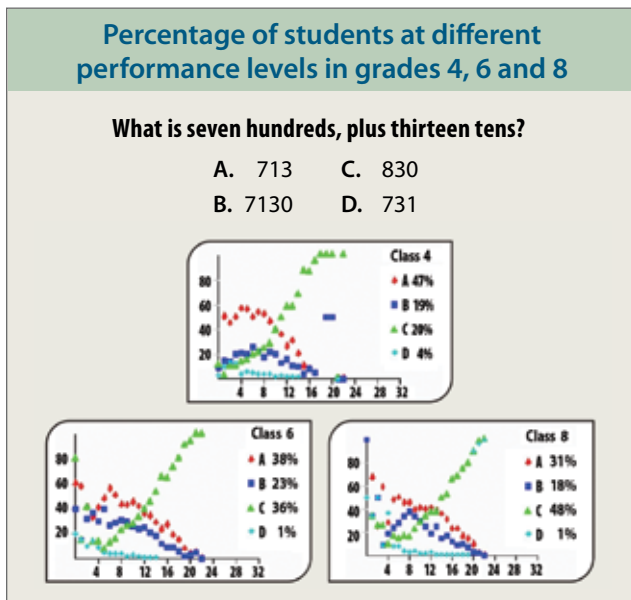
'Very easy. I can tell you now one of the questions: When are two bodies equivalent? And the answer is, Two bodies are considered equivalent if equal torques will produce equal acceleration.' So, you see, they could pass the examinations, and 'learn' all this stuff, and not know anything at all, except what they had memorized.

After a lot of investigation, I finally figured out that the students had memorized everything, but they didn't know what anything meant. When they heard 'when light passes through a medium,' they didn't know that it meant a material such as water. Everything was entirely memorized, yet nothing had been translated into meaningful words. So if I asked, 'When are two bodies equivalent?' I'm going into the computer with the right keywords. But if I say, 'Look at the water,' nothing happens - they don't have anything under 'Look at the water!'

Rajagopalan said that the situation in India is, unfortunately, quite similar to this, especially in the public schools but also in the private ones.

are typically drawn in by answer A (drawn in pink), and that doesn't change over the years. A student who doesn't get this in grade 4 probably never will. EI uses graphs like these as a basis for discussions with teachers and education planners, and it tremendously increases the teachers' understanding of their students and motivation for their work. This discussion is key to the power of EI's work.

Almost 90% of schools that try the Asset test continue to use it, even though they have to pay for it, and they have to convince parents that it's worth the cost. They have to see value in it, because there is no requirement to use it.



ASSET™ Respiration

Class 8

1 Why was the question asked?

Children learn about the basic life processes from grade 3. But they often see certain processes as specific to certain organisms, the photosynthesis as specific to plants or respiration as specific to animals. This question was asked to check if students really understood the difference between the two processes photosynthesis and respiration. It was asked across classes, 5 to 8, to see if there is any difference in performance.

Question

All living organisms need to respire to stay alive. Which of the following are examples of respiration?

1. Humans use oxygen and release carbon dioxide.
2. Plants use carbon dioxide and release oxygen.
3. Burning dry leaves uses oxygen and releases carbon dioxide.

Options

● only 1 ● only 2
● only 1 and 2 ● 1, 2, and 3

2 What did students answer?

Only about 19% of class 8 students selected the correct answer (B). We asked this question to students of class 5 to 8. The most surprising factor that the performance is almost the same across the classes. In fact, the percentage of students selecting the most incorrect answer (C) increases with the class.

It clearly indicates that fundamental processes like respiration and photosynthesis are not clearly understood by children.

3 Learnings

There seems to be a confusion between the two concepts of Photosynthesis and Respiration. At the first level students think that the process whereby plants use carbon dioxide and release oxygen can also be referred to as respiration. They might be a misconception based that needs to be checked. At the second level they think that at any time, plants can use only one gas. i.e. when the plants are using say carbon dioxide they cannot simultaneously use oxygen.

Some students who give the correct answer (B) show an incorrect understanding when probed, according to their plants photosynthesis during the day and respires during the night. There has to be a need to carry out photosynthesis and there is no time during the night, they switch over to respiration as it is also required for them to produce energy.

As whether they know that respiration is required for producing energy or not, the fact that both, respiration and photosynthesis, occur simultaneously in plants is just not clear. They need to think that only one of the processes can occur at a time (only one gas can enter or exit) and as to the meaning plants photosynthesis and as long as they require.

There are many students who seem to misunderstand the same effect relationship in this case. They say that carbon dioxide goes out to animals to use to produce food. They are not aware of the need to photosynthesis. They are not aware of the need to photosynthesis. They are not aware of the need to photosynthesis. They are not aware of the need to photosynthesis.

4 How do we handle this?

- Teachers need to understand that both these processes occur simultaneously.
- Let them understand the process of respiration without any regard to a source. i.e. just teach the process independent of animals or plants. Just make them clear that why the process is important and what happens in the process. Then lead them to which all organisms will need to respire, and from there lead them to the time at which respiration occurs.
- Many students have a misconception that plants or animals take in only one gas, carbon dioxide or oxygen at a time. It is important to correct this and teach that it's the mixture of gases that is taken in and out of which the important gas is selected (by flowing through lungs in case of animals and diffusion in case of plants).
- Another aspect that needs to be emphasized is that all parts of the plant respire. Leaves, stems, roots and flowers do! They can show this and get their understanding from the air through pores. The pores in the leaves are called stomata (singular: stoma). The pores in the branches of trees are called lenticels.

Build resources:

Website: <http://www.ck12.org/ck12:middle-school-biology:photosynthesis-and-respiration:the-correct-answer> & <http://www.ck12.org/ck12:middle-school-biology:photosynthesis-and-respiration:the-correct-answer>

Video: <http://www.ck12.org/ck12:middle-school-biology:photosynthesis-and-respiration:the-correct-answer>

Teacher Sheets

Post-assessment teacher support sheets show the depth of information given about problems such as Basic Shapes, Measurement, Perimeter, and Respiration (shown at left).

This teacher support is essential to the value of the testing.

AUSTRALIA

Helen Chick, a mathematics educator at the University of Melbourne, and Elizabeth Burns, the deputy principal and director of studies at Loreto, a girls' school in Melbourne, described how reasoning and sense-making are incorporated into the mathematics curriculum in Australia.

The structure of Australia's educational system is similar to the U.S., with primary school for kindergarten through grade 6 and secondary school for grades 7 through 12. Schooling is compulsory to age 16. Australia has seven states which developed their school systems and curricula independently, though a national curriculum was being developed at the time of the conference. There are large Catholic and independent school systems alongside the public one. In Victoria, one of the more populous Australian states, about a third of schools are Catholic or independent; such schools comprise at least 20 percent of schools in other states. These private schools receive a bit of federal funding, including money for new buildings through a stimulus effort after the global financial problems.

After many previous efforts, a national curriculum was developed in math, English, history, and science, in 2009 and 2010. At the time of the conference, the math curriculum was in draft form, and the feedback period had just closed. Part of the impetus for this curriculum was the move to testing.

States began testing around the turn of the millennium. Teachers have used testing for diagnostic purposes. However, there was a move to change that to national testing in around 2006. National testing began even prior to a national curriculum. Initially, the results were distributed to schools as an internal diagnostic document, but in 2010, results were published more widely. The analysis compares a school to the state as a whole, and also to schools of similar socio-economic standing. This testing has caused a huge amount of fuss, with arguments that it encourages teaching to the test or even cheating.

The Australian mathematics curriculum documents in 2010 identified reasoning and understanding (or, equivalently, sense-making) as two of four key proficiencies. The growth and nature of reasoning is described this way: "Students develop increasingly sophisticated capacity for logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying, and generalizing."

Similarly, understanding (or, equivalently, sense-making) is discussed this way: "Students build robust knowledge of adaptable and transferable mathematical concepts, make connections between related concepts and develop the confidence to use the familiar to develop new ideas, and the 'why' as well as the 'how' of mathematics." [Note from time of publication: By 2014, the curriculum documents had been revised, and these statements mention proof more explicitly.]

By comparison, the National Council of Teachers of Mathematics (NCTM) standards define them this way: "Instructional programs from prekindergarten through grade 12 should enable all students to:

- recognize reasoning and proof as fundamental aspects of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs;
- select and use various types of reasoning and methods of proof."

The Australian definitions in 2010 did not emphasize proof as much as the NCTM standards do. Chick and Burns reported that proof tends not to be emphasized in the classroom, which has been a widespread criticism. Various curricula have been used in Victoria, the state that Chick and Burns are from, and while the merits of the different curricula are debated, the reality is that changes and development of curricula may have fairly little impact on classroom practice as some teachers will continue to do whatever they were already comfortable doing.

Different states have taken very different approaches to the use of technology. Victoria has mandated the use of calculators for a long time, including calculators with Computer Algebra Systems, but New South Wales, the most populous state, has only limited use of non-graphing scientific calculators in state exams.

There is similar variation in textbooks. About half of primary school classrooms have a prescribed textbook, with the others using worksheets in class. Secondary schools nearly all use textbooks. Most Australian textbooks do attempt to explain mathematical rules in a logical, mathematical sense, modeling mathematical reasoning, but they're often not ideal mathematical proofs. Sometimes these limitations are acknowledged, for example, when something is explained by analogy.¹ However, it's not clear whether teachers even present the arguments that are presented in the textbooks.

This is a big issue in general. Elementary teacher education generally exposes future teachers to little in the way of reasoning and sense-making activities. Secondary teachers are required to have a degree in math or science, but still, their mathematical exposure and competence vary. There are a lot of mathematical and pedagogical skills required to choose good tasks, to ask good questions, to interpret students' responses, to ensure that good math is being done, and to know models and explanations that are mathematically valid and pedagogically appropriate.

¹ Stacey, K., & Vincent, J. (2009). Modes of reasoning in explanations in Australian eighth-grade mathematics textbooks. *Educational Studies in Mathematics*, 72, 271-288.

ASSESSMENT

The Common Core State Standards are just the beginning of the effort to coordinate education across the country. A key part of making them effective is to develop smart assessment tools. After all, if we don't assess reasoning and sense-making, we aren't really requiring it. But doing so is a challenge; it's far more straightforward to assess procedural knowledge. Several speakers described efforts to develop more sophisticated assessment tools.

RACE TO THE TOP ASSESSMENT CONSORTIA

Jason Zimba, a mathematics and physics professor at Bennington College in Vermont, was part of the writing team for the Common Core State Standards. He has now founded Student Achievement Partners, which helps teachers implement the CCSS in a way that will help their students achieve. He advises some of the consortia that are applying for money under the Race to the Top program, which will be described below.

The CCSS have these overarching mathematical practice standards:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

But connecting these practices to the mathematical content is not so obvious, and assessment lies in that intersection.

Some aspects of the practices seem easier to assess than others. For example, justifying conclusions, explaining reasoning to other students, and being precise seem somewhat less difficult.

Modeling with mathematics seems to be at a medium level of difficulty. You need sophisticated tasks, a tradition of content development that we don't really have in this country, and subtle rubrics, because models can be wrong. That would take some doing at scale.

Harder would be deviating from a known procedure to find a shortcut, because that's both sophisticated and opportunistic. Considering analogous problems is a difficult skill that arises only in the context of very difficult tasks. Persevering requires tests that stimulate failure. So those are harder to assess, particularly at scale.

They don't all need to be assessed separately. Some can be assessed within the content: reasoning abstractly and quantitatively, and making use of structure. Zimba argues that those can be embedded well in the content, and hopes that someday perhaps all these will be.

Considering the practices tends to pull assessment down to the classroom level and to bring teaching closer to assessing. They loosen you up and make you think about assessment in more abstract ways, such as questions in the classroom. For example, "Did you mean the Pythagorean Theorem or its converse?" "Martin, what did you think of Kendra's explanation?" "How many solutions were you expecting to find?" This kind of thing — asking questions, listening to answers, following up on them — is what Socrates would simply call teaching.

Assessing the practices only makes sense when you're evaluating growth. We need to compare each individual student to his own "past mathematician." Socrates didn't know anything about complex numbers, but he recognized the virtues of precision, reason, argument, and perseverance, which are the deepest values of education. In a conversation with Socrates, the standard against which you are judged is who you were when we first began to speak. Similarly, comparing one student to another, or one student to an abstract standard, becomes less feasible and less relevant as the goals become more profound. Zimba argues that individual growth may be best measured over multi-year time scales. Portfolios might hence be good for some. The practices might be easier to observe if students

are faced with less routine, more complex tasks with long chains of reasoning including skills mastered in previous years. That's difficult and daunting.

Note that it is better to assess the practices imperfectly than not to assess them at all.

Leaving the practices, consider the standards that call for procedural fluency. "Fluent" means fast and accurate. A standard that involves fluency involves time. It means more or less the same as fluency in language — you're not halting, or having to stop to correct yourself. When you're fluent, you flow. Assessing that would involve classroom observation or classwork like quizzes, or technology that involves time or rhythm. So it's not only the conceptual standards, the reasoning, or the practices that raise issues of assessment. Assessing any set of standards is challenging.

Zimba speculated that at the time of the conference, some long-overdue innovations in assessment might be about to occur, thanks to some investment from the federal government. The Race to the Top Fund Assessment program is awarding \$350 million to consortia of states to develop new assessment tools. In particular, it will award:

- One to two awards of \$160M each for development of a shared comprehensive assessment system in grades 3-8. A consortium of 15 or more states may apply.
- One award of \$30M for development of a high school assessment system. A consortium of 5 or more states may apply.
- Applications are due June 23, 2010.
- To be implemented across the consortium states by the 2014-2015 school year.

A middle school comprehensive assessment system must:

- Measure student knowledge and skills against a common set of college and career-ready standards in math and English language arts.
- Cover the full range of those standards, including standards against which student achievement has traditionally been difficult to measure.
- As appropriate, elicit complex student demonstrations or applications of knowledge and skills.
- Provide an accurate measure of student achievement across the full performance continuum, including for high and low achieving students.
- Provide an accurate measure of student growth over a full academic year or course.
- Include one or more summative assessment components.

- Be administered at least once during the academic year in grades 3 through 8 and at least once in high school.
- Produce student achievement data and student growth data that can be used to determine whether individual students are college- and career-ready or on track to being college- and career-ready.
- Assess everyone, including English learners and students with disabilities.
- Produce data, including student achievement data and student growth data, that can be used to inform.
 - Determinations of school effectiveness for purposes of accountability under title 1 of the Elementary and Secondary Education Act;
 - Determination of individual principal and teacher effectiveness for purposes of evaluation;
 - Determinations of principal and teacher professional development and support needs; and
 - Teaching, learning and program improvement.

In addition:

- Points are awarded based on the extent of collaboration and alignment between the states' elementary and secondary systems and public institutions of higher education.
- Consortia should work with the Department of Education to develop a strategy to make student-level data available on an ongoing basis for research, including for prospective linking, validity and program improvement studies (consistent with student privacy).
- Consortia should use technology to the maximum extent appropriate to develop, administer and score assessments and report assessment results.

The idea is to deepen assessment, to make it more useful, and to reflect a wider range of standards. Assessment needs to look more like good teaching, and instead of more assessment, we need better assessment.

While the practices will be assessed, Zimba isn't certain that the scores need to contribute to an annual growth score. Development might take more than a year, and accountability might then naturally rest at the school or district level. Given that the assessments will be used to evaluate individual teacher effectiveness, the consortia will have to sort out the issues that arise when those standards are better assessed by teachers in the classroom, informally. That might be the case for some content standards too, not just the practices. The good news is that the request for proposals include assessment of the practices.

PANEL DISCUSSION

Four people presented their varying experiences and expertise with assessment.

Henry Kranendonk

Henry Kranendonk is a retired curriculum specialist in the Milwaukee Public Schools and a board member of the National Assessment Governing Board (NAGB), which oversees the budget for the National Assessment of Educational Progress (NAEP) assessments and the writing of the national report card. The NAGB is looking at new technologies for assessment.

In particular, the NAGB is experimenting with a science assessment. Students are given a task, an investigation in science, and a shelf of resources and objects that they can work with. The first wave of data from this shows that the students are highly engaged. NAEP is also looking at new subject areas to assess. The newest test is on technology and engineering literacy, which will first be assessed in 2014. The test will be computerized.

Assessments of reasoning must tell us:

1. How students approach and think about a complex problem or task.
2. How students begin developing a solution to the task.
3. When students become flexible enough in their thinking to be able to do genuine problem solving.

An investigation needs to be carefully designed to be useful for assessment. In probability and statistics, for example, NAEP came up with these characteristics that are needed in an investigation of probability and statistics:

1. It must be meaningful and authentic.
2. The data can either be collected or is provided for analysis.
3. There must be variation in data that draws out questioning.
4. It must have data that can be organized.
5. A focus or research question needs to emerge.
6. There need to be opportunities for students to provide reflections.
7. By the end, either an answer to the research question emerges or other questions evolve.

Linda Gojak

Linda Gojak spent 28 years as an elementary math specialist and mathematics teacher in grades 5-8. For the last ten years, she has worked with teachers at the John Carroll University as director of the Center for Mathematics and Science Education, Teaching and Technology. She believes and shares with teachers with whom she works the mantra, “Everything you do in mathematics should make sense.”

We assess students for many reasons including to enhance student learning, guide instruction, inform parents, evaluate student achievement, and evaluate the mathematics program. The 1995 assessment standards of the National Council of Teachers of Mathematics define assessment as “the process of gathering evidence about a student’s knowledge of, ability to use, and disposition toward mathematics and making inferences from that evidence for a variety of purposes.”

The standards of mathematical practice should be woven in throughout the teaching process, to drive instruction and to drive assessment. They can’t be separated from the content standards, because effective teaching should include the standards for mathematical practice. The practice standards also must drive assessment, though it’s still a question as to what that should look like.

Assessment tasks need to be set in a context. Problems with mere “naked numbers” do not have meaning for students. Rich, contextual problems help students to make sense of a situation. Problems should also have multiple entry points, so that students who struggle can approach it on one level and gifted students can do so on another. Explanation and justification should be an expected part of any task, a requirement that eliminates most multiple choice questions.

More appropriate forms of assessment are rich tasks or investigations, journals, projects, student folios, interviews, and self and peer assessments. Still, thoughtful implementation is essential for these tasks to be worthwhile. For example, students might be asked to include everything in a journal, but they (or the teacher) may not be clear why they are doing this or what mathematics they are learning from keeping a journal.

In order to construct good assessments, teachers must understand the mathematical content deeply, select tasks that reflect this content, anticipate a variety of strategies and responses from students, ask appropriate questions in order to obtain information on student understanding, and know what to do next based on student responses.

If we assess what we value, then we must assess student learning for sense-making and reasoning.

Kimberly Seashore

Kimberly Seashore taught in the classroom and then for nine years at the Lawrence Hall of Science, and then became a graduate student at UC Berkeley with Alan Schoenfeld.

Seashore starts with three premises: that students are smart; that teachers are smart; and that people get smarter by doing things that make them think. Following those principles, it is essential not to steer away from things that teachers find challenging.

Seashore is currently involved in three projects. The first is the Algebra Teaching Study (ATS), which looks at how to capture all the rich stuff that happens in algebra classes with a classroom observation tool. The second is on diagnostic lesson development. The Shell Center develops excellent lessons, which Seashore is helping them adapt for American students. Third is the Strategic Education Research Project (SERP), which will be described further below.

Assessment figures into all of these projects, in improving student learning outcomes (SERP, ATS), developing teachers' understanding of core mathematics concepts and reasoning (SERP, Diagnostic Lesson Development), increasing teachers' capacity to use classroom data (SERP, Diagnostic Lesson Development), and identifying teaching practices that lead to robust student understanding (ATS).

SERP is a multiyear university-school district partnership, where the school district gets to direct the program. The district wants to improve middle school math, especially word problems. The teachers are very dedicated, in the middle of their careers, although some have not pursued every imaginable professional development opportunity. The teachers and researchers collaborate to understand student thinking and reasoning and to develop rich tasks and lessons. They then create tools that can be used by teachers across the district to improve student performance on word problems.

Rather than asking, "How do we teach students to solve word problems?" they asked, "What are they already doing when they try to solve word problems? What can we build on that is productive? How can we help students be more productive?"

Assessment isn't a tool, it's a process. It consists of activities by teachers and students, which provide information to be used as feedback to modify the teaching and learning activities. It requires all three of the following:

1. Tools to gather data about student thinking.
2. Analysis of that data.
3. Actions that can be taken to improve student learning based on the data analysis.

If you don't do all three, it's not assessment.

In SERP, the tools are student interviews, problem stems, diagnostic classroom lessons, and finally a commitment from the district to change their assessments.

At the beginning of the project, teachers didn't know why students couldn't do word problems. So they interviewed students who seemed to be representative of the class and who aren't getting it about a particular problem, like this one:

Interview problem

A five-pound box of sugar costs \$1.80 and contains 12 cups of sugar. Marella and Mark are making a batch of cookies. The recipe calls for 2 cups of sugar. Determine how much the sugar costs for the cookies.



Here's a sample interview:

Student: So a five-pound box of sugar costs \$1.80 and contains 12 cups of sugar.

Teacher: What are you thinking?

S: I'm thinking I should divide.

T: OK.

S: I'm thinking I should add.

T: OK.

S: I think that I should add 12 cups of sugar and \$1.80.

T: OK, why?

S: Because the recipe calls for sugar. I'm thinking I should add \$1.80 plus 12 cups. I don't really get the question, though.

T: You don't understand what you're supposed to be looking for?

S: No.

Nevertheless, the student calculates that $1.80 + 12 = 1.92$

T: But you're adding the cups of sugar to the dollar eighty?

S: I think you should add 12 to 2. Or maybe you should multiply it. That would give me 24.

T: What do you know?

S: I know that a five-pound box of sugar costs \$1.80 and contains 12 cups of sugar. So 12 divided by 2 and then I'll get... But I don't know why I would do divide 12 by 2.

T: Is there a picture you could draw that would help?

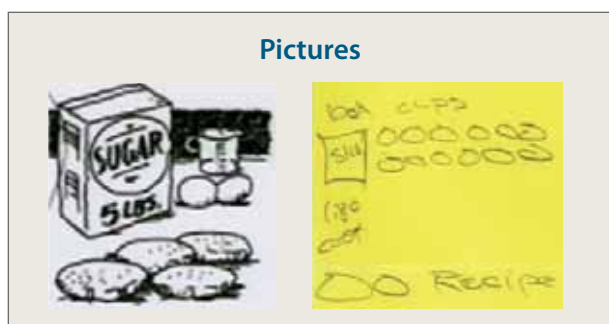
S: No, because I've already got one.

T: You mean the one they drew for you?

S: Yeah.

T: Is there a different picture you could draw that would help?

S: No. Maybe. OK, I guess I'll go for it.



The student was able to represent that there was a box, that there were twelve cups, and that she needed two of them. So that was pretty productive — much more than the picture she was given.

This student is not an outlier, though she's not doing well. With prompting, the student was able to get pretty close to a final answer.

The teachers observed that students were jumping into solving the problems before even understanding what the problem was asking them. They were focusing on getting an answer, independent of the question. They knew they had a bunch of numbers, and a bunch of operations, and so they tried them at random, looking to the teachers for cues when they got close to the answer. The teachers realized that in class, they were nodding when the students got to the right answer. They also realized that interesting pictures and useful diagrams are not the same, and the teachers were sometimes rewarding beautiful pictures that didn't contain mathematically useful information.

They developed two ideas from these observations. One is that they gave the students “problem stems.” For example, they'd tell the students, “The dragonfly, the fastest insect in the world, can fly 50 feet in 2 seconds. Make up a meaningful mathematics problem that uses this information.” That got the students thinking about the questions rather than the answers. The teachers also realized that they needed to focus on student diagrams.

The teachers asked for more “diagnostic problems” that will help teachers evaluate the gaps in student understanding, so that is what the project is focusing on now, creating diagnostic problem packets tied to the curriculum at each grade level. The district has then committed to including the packets in the course guide and to mirroring these problems on district benchmark tests.

Dan Teague

Dan Teague is a teacher at the North Carolina School of Science and Mathematics. He pointed out that the traditional rules of the game of teaching math are simple: we, the teachers, show students what to do and how to do it, we let them practice for a while, and then we give them a test to see how closely they can match what we did. The game is either won or lost on test day.

But this doesn't involve thinking. After all, thinking takes time. Thinking comes into play precisely when you cannot do something "without thinking." You can do something without thinking if you really know how to do it well. If your students can do something really well, then they have been very well prepared. Therefore, if both you and your students have done your jobs perfectly, they will proceed through your test without thinking. If you want your students to think on your test, then you will have to give them a question for which they have not been fully prepared. If they succeed, fine; in the more likely case that they do not, then they will rightfully complain about not being fully prepared. You and your student will have both failed to uphold your respective ends of the contract. Given how we mathematicians value thinking, it's a wonder we've gotten ourselves into this mess at all!

Therefore, to assess thinking — that is, reasoning and sense-making — we need totally different forms of assessment.

Here's a problem that Teague uses in a number of different classes to assess reasoning:

In 1981, two new varieties of a tiny biting insect called a midge were discovered by biologists W. L. Grogan and W.W. Wirth in the jungles of Brazil. They named one the Apf midge and the other the Af midge.

The biologists discovered that the Apf midge is a carrier of a debilitating disease, while the Af midge is quite harmless and a valuable pollinator. In an effort to distinguish the two varieties, the biologists took measurements on the midges they caught. The two measurements taken were of wing length and antennae length, both measured in centimeters. They only measured 15 midges, and this was their data:

Midge Data									
Af Midges									
Wing Length (cm)	1.72	1.64	1.74	1.70	1.82	1.82	1.90	1.82	2.08
Antenna Length (cm)	1.24	1.38	1.36	1.40	1.38	1.48	1.38	1.54	1.56
Apf Midges									
Wing Length (cm)	1.78	1.86	1.96	2.00	2.00	1.96			
Antenna Length (cm)	1.14	1.20	1.30	1.26	1.28	1.18			

This is a rich problem in which sense-making can happen.

The natural first step is to graph the data:

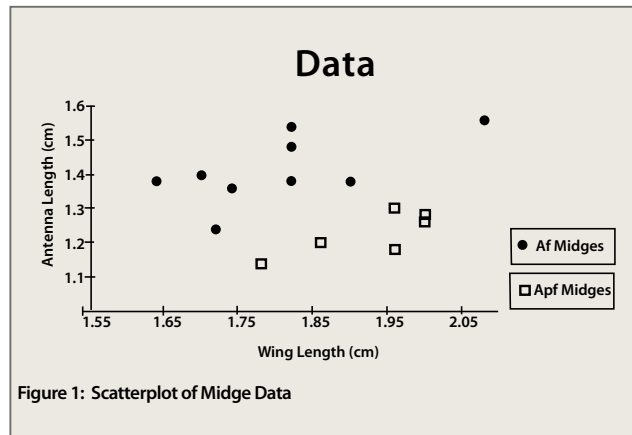
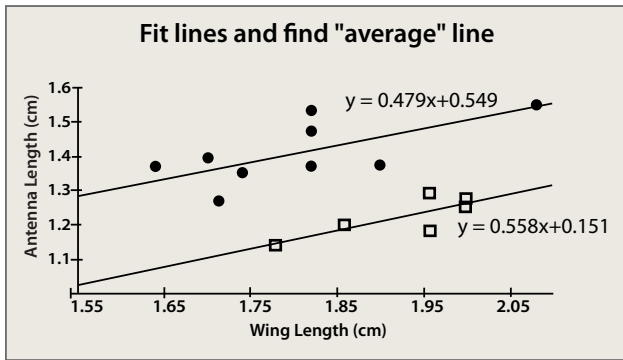


Figure 1: Scatterplot of Midge Data

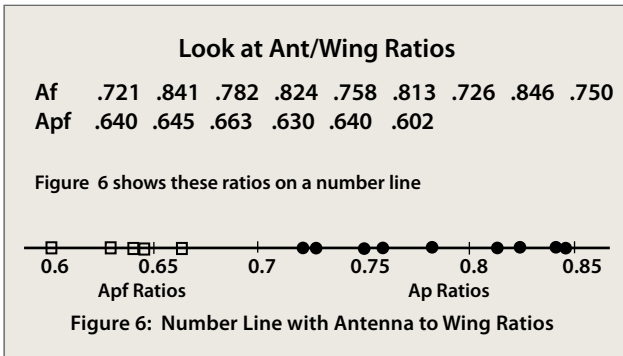
The harmless midges are in black, while the dangerous ones are in white. Students can easily find a line to divide the two groups, but how do you justify the line you picked? Of all the lines, why that one? That is the hard part.

If they've studied regression, the first thing they'll do is to fit a line to each type. Often, they'll take the average of the two slopes and average of the two intercepts and choose that as the dividing line, because that's the closest to something they've been taught to do. But it misclassifies one of the midges. They're good students because they did what they were taught to do, so it must be the right answer. So as a teacher the first thing you have to do is to break that down, to convince the students that there's more to it than that.

Another approach is to say that no Apf has been found above one line and no Af has been found below another, so maybe these two lines could form boundaries.



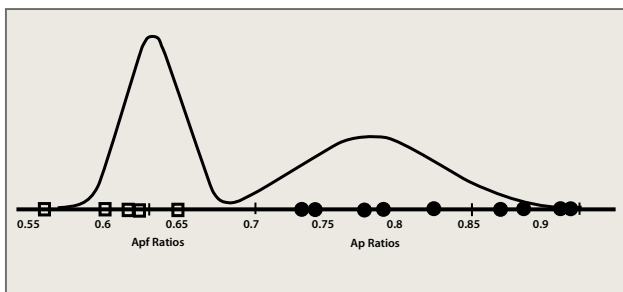
Or, since 9 of the 15 are up here and only 6 of the 15 are down here, it's an unequal split. Maybe that should affect where the line should go. Or, you could argue that it's better to be safe than sorry, so perhaps we should use the higher line. All these are using mathematical ideas in a reasonable, justifiable way to try to answer this question.



Another approach would be to look at antenna/wing ratios. Then you get a nice lean division.

That turns the two-dimensional problem into a one-dimensional problem. But you still have to find a point to draw the division and need to support that with some reason.

Or you could do a normal probability plot and show that each of those subsets is consistent with a random sample from a normal distribution.



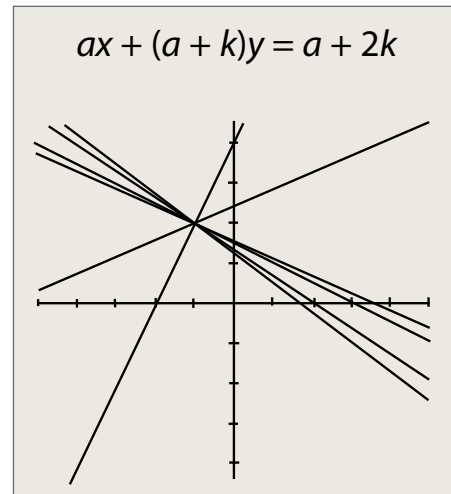
One could then approximate it with a random distribution with a mean of .637 and a standard deviation of .020. One could then decide to make a division that will allow for an error in 1% of the cases, or 5% of the cases. Or one could figure out where it's equally likely to come from either distribution.

There are lots of ways kids can use what they learn to solve this problem. Each is using reasoning and sense-making to come up with a solution. We usually have them write up a solution to submit as a final product. But a lot of the important reasoning isn't going to be in it. It's in the false starts that they decided to jettison. So to evaluate it, you have to watch their thought process as they develop it.

A problem like this draws on almost all the mathematical practices and allows one to evaluate them.

Teague gave a second example of a rich problem kids can tackle: Consider the family of linear functions $ax + by = c$ in which a , b and c are in arithmetic progression, i.e., $b = a + k$ and $c = a + 2k$. Examples of such functions would be $3x + 5y = 7$ or $6x + 11y = 16$. What characteristics does this family share?

Graph a bunch of them, and this is what you see:



This is a bit of a surprise: They all go through the point $(-1, 2)$. Why?

The standard "solution" is to solve the system of equations:

$$\begin{aligned} ax + (a+k)y &= a + 2k \\ bx + (b+n)y &= b + 2n \end{aligned}$$

You get the solution $x = -1, y = 2$. But this isn't satisfying, because although you've found the solution, you haven't made sense of the problem. Why does $(-1, 2)$ have to be a common point?

Examining the equation $ax + (a + k)y = a + 2k$ is revealing. Notice that there are two k 's on the right hand side. If the equation is going to hold for all values of a , then there must be two k 's on the left hand side too. The only way to get that is if $y = 2$. Setting $y = 2$ means that I have two a 's on the left hand side. Since there is only one on the right hand side, that can only work if $x = -1$.

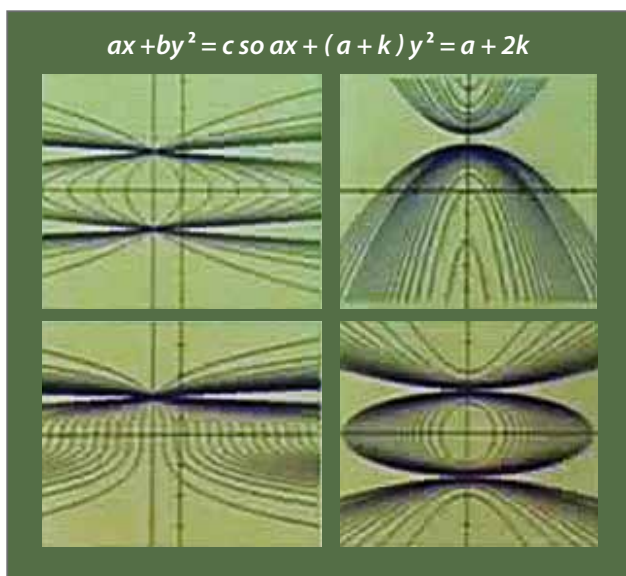
That is an explanation that makes sense of it.

Now, let's modify the problem. Can we find other interesting, related results?

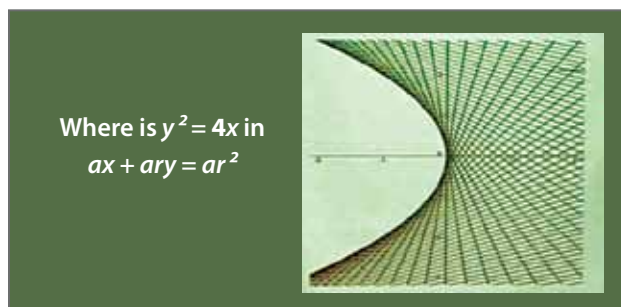
Kids tend to initially ask, "What about a geometric progression?" That's an interesting question, but there's another way to modify the problem as well: consider a non-linear function. So, what characteristics does the family of functions $ax + by^2 = c$, where a, b, c are in arithmetic progression?

These functions will all have this form:

$ax + (a + k)y^2 = a + 2k$. It turns out that the same reasoning holds: There's a $2k$ on the right hand side, so there must be a $2k$ on the LHS. So $y^2 = 2$, and $y = \pm\sqrt{2}$. And x still has to be -1 . So all the solutions must go through $(-1, \sqrt{2})$ and $(-1, -\sqrt{2})$. And indeed, if we graph it, we end up with pictures like these:



Now, let's modify it in the other direction, making a, b, c in geometric progression. If we graph them, it looks like this:



These lines are all tangent to a parabola, which turns out to be $y^2 = -4x$. Why?

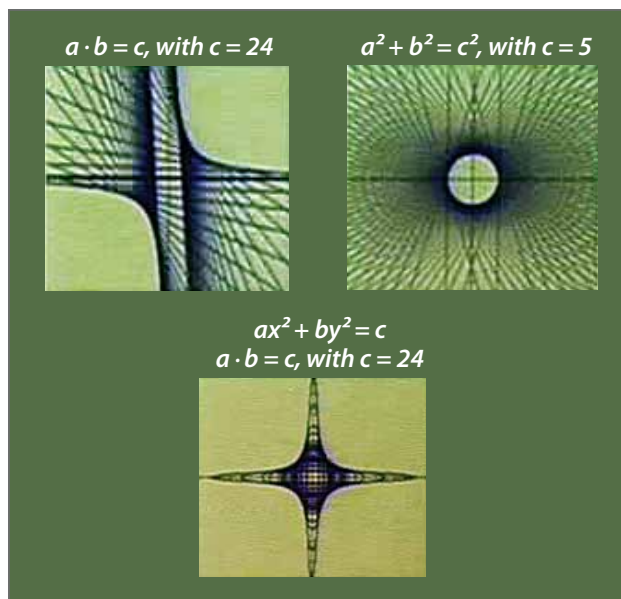
If you use the standard solution and solve the equations simultaneously, you get that $x = -rk$, $y = r + k$. But that's not very revealing.

The real question is, where is $y^2 = -4x$ in $ax + ary = ar^2$?

We can think of this as a quadratic equation in r , which is promising, since we're trying to explain something that's quadratic. If we use the quadratic equation to solve for r , we find that $r = (y + \sqrt{y^2 + 4x})/2$. For r to be real, we need to have $y^2 + \sqrt{4x} \geq 0$, so $y^2 < -4x$ is excluded. That's where the parabola is coming from.

Teague tells students to produce the most compelling image they can that can go with a nice mathematical argument telling him why it has to be that way, and the students do wonderful, beautiful, great things.

The same kind of arguments can be made about other relationships between a, b , and c . Here are other relationships students have found:



These are the kinds of problems we want to pose to students and say, “Show me your thinking.” You have to give them a problem that’s rich enough to think about.

Here’s another beautiful problem, one drawn from a Putnam exam:

Intermediate Value Theorem

Early in the season, Pat was hitting fewer than 80% of her free throws.

At the end of the season, she was hitting more than 80% of her free throws.

- Must there have been a time during the season at which she was hitting exactly 80% of her free throws? If so, explain why. If not, give a counter-example.
- If the answer to a) was yes, find all values of p which have this intermediate value property (you can't go from below p to above p without going through exactly p). If the answer to a) was no, are there any values of p for which this intermediate value property does hold? If not, explain why there can be no such values.

Calculus students will immediately point out that the intermediate value theorem doesn’t apply because this situation is discrete, not continuous. But of course, the theorem not applying doesn’t mean that the conclusion is necessarily false. In fact, it turns out that there’s something special about 80%. Eventually, students discover this and then ask if there are other special numbers. It’s a beautiful problem.

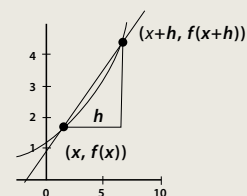
And one final favorite problem:

Classic definition of Derivative

It is traditional to define the derivative as:

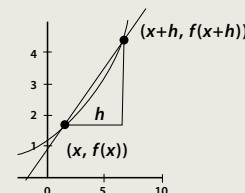
$$\lim_{x \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

The secant line geometry of this suggests that we pick a horizontal distance h , take whatever change in y we need to create the secant line and compute the slope. We then shrink h to zero and consider the limiting slope.



The Hernandez Derivative

Why not the other way around? Why not fix a vertical distance h , take whatever change in x we need to create a secant line and compute the slope. Then shrink h to zero and consider the limiting slope.



What happens if you do it this way?

A student asked Teague this. Does it work?

Teague said that when he tells other teachers about these kinds of problems, they typically say, “My kids can’t do that!” And he responds by saying, “Of course not! You haven’t taught them how.” They always do badly the first time they tackle a rich, challenging problem. After all, Teague points out, you played badly the first time you played tennis. No one can do something hard well from the beginning.

The poet Marianne Moore was once asked,

“What is poetry?” She said,

“Poetry is about imaginary gardens with real toads.”

Teague argued that this is the perfect description of mathematics. We make math up. But it has real toads: We send people to the moon and bring them back based on it. This describes the duality of math in a very interesting way. We do a lot with the toads, and we think we can draw students into the garden by following the toads.



TECHNOLOGY

Computers of various forms can now do essentially all of the calculations that students learn to perform in primary and secondary school. They cannot, however, reason or make sense of mathematics for students directly. Used with care, however, they can help students do that for themselves — naturally raising questions, inviting free investigations, deepening students’ understandings of algorithms, etc. Teachers and researchers discussed ways that technology can aid students’ sense-making and reasoning.

Nicholas Jackiw

Nicholas Jackiw, a senior scientist at KCP Technologies at the time of this presentation, designed The Geometer’s Sketchpad and, with it, invented the Dynamic Geometry paradigm that has become widespread in educational technology. Today, Sketchpad is one of the most widely-used educational technologies for school mathematics in the world. Many textbooks use the software. Academic research has found that Sketchpad use has positive impact on student achievement,² conceptual understanding,³ motivation and engagement.⁴ A large-scale assessment of technology trends in American education reports that mathematics teachers across the country find Sketchpad to be “the most valuable software for students.”⁵

The National Council for Teachers of Mathematics’ Standards explicitly recommends the use of Dynamic Geometry at several grade levels,⁶ and the Common Core State Standards call for its use particularly in high school geometric constructions and transformations (though it doesn’t call for it at all in elementary or middle school).

Jackiw argued that dynamic geometry should be a far more central tool to be used throughout the school curriculum, starting in kindergarten and continuing through college.

He began by describing what dynamic geometry is and demonstrating it, which is far more illuminating. While *The Geometer’s Sketchpad* is commercial software, a video showing what it can do is available at <https://www.keycurriculum.com/products/sketchpad>, and that website also

offers free demonstration versions of the software. And www.dynamicgeometry.com has many resources for The Geometer’s Sketchpad users, including classroom activities.

The central idea for the software is that you construct mathematical objects in a visual environment, and you can then manipulate and change them, while the object continues to obey all the definitions you gave it. So, for example, you can create a triangle by connecting three lines together, and then you can drag the vertices to different positions while the software preserves its triangularity. The object then transcends being a single example and becomes something much closer to the general case of all such things.

Jackiw demonstrated some of the ways this can be useful. For example, he drew the three altitudes of a triangle and then began transforming the shape of the triangle. It quickly became apparent that the altitudes continued to intersect one another at a single point no matter how he changed the triangle. Sometimes the orthocenter — i.e., the point of intersection of the altitudes — was inside the triangle and sometimes it was outside. That raised a natural question about how the orthocenter behaved as it passed from inside to outside. A bit of play rapidly suggested that when the orthocenter crossed the triangle, the triangle was always right and the orthocenter passed through the vertex at the right angle. While examining separate examples might convince you the orthocenter can exist inside, or outside, its triangle, only being able

² Battista, Michael T. Shape Makers: A Computer Environment That Engenders Students’ Construction of Geometric Ideas and Reasoning. *Computers in Schools*, 17(1/2), (2002): 105–120, and Hollebrands, Karen. “The Role of a Dynamic Software Program for Geometry in High School Students’ Understandings of Geometric Transformations.” *Proceedings of the 24th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Columbus, Ohio: ERIC (2007): 695-705.

³ Frekering, B. G. *Conjecturing and Proof-Writing in Dynamic Geometry*, PhD Thesis, Georgia State University (1994).

⁴ Sinclair, N. *Mathematics and Beauty: Aesthetic Approaches to Teaching Children*, New York: Teachers College Press (2006).

⁵ Becker, H. J.; Ravitz, J.; and Wong, Y. *Teaching, Learning and Computing National Survey*, University of California, Irvine: Center for Research on Information Technology and Organizations (1999).

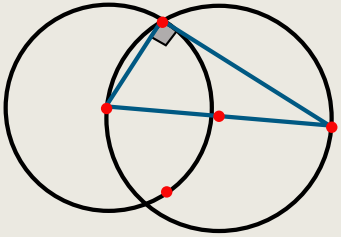
⁶ National Council of Teachers of Mathematics. *Principles and standards for school mathematics*, Reston, VA: National Council of Teachers of Mathematics (2000).

to witness and control this movement continuously—to *push* the center outside, or back into, its triangle—allows you readily to explore this boundary condition. Proving your result would be a separate step, but the software makes experimentation and questioning natural, and the knowledge thereby gained gives insight into the structure of a possible proof.

The first use of dynamic geometry that people tend to think of (particularly if they haven't used dynamic geometry much themselves) is for geometric constructions, and it indeed is quite useful for that. Jackiw demonstrated, for example, the process of figuring out how to construct a line tangent to a given circle through a given point using the software. Coming up with the construction requires remembering the fact that the angle contained in any semicircle is always right.

High School Geometry: Circles [G-C]

4. (+) Construct a target line from a point outside a given circle to the circle.

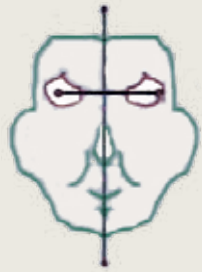


But Jackiw argues that there's not a lot of sense-making and reasoning in this process. Instead, one merely looks over a trove of known facts to see which will be helpful in the given situation, and then applies it.

Other ways of using dynamic geometry support much deeper types of reasoning. For example, students can start thinking about transformation as early as kindergarten, examining lines of symmetry and symmetric properties of shapes. Jackiw drew half a face and then used the software to reflect it, forming an object with mirror symmetry built into its definition. He could then change the location of the axis of symmetry and see the effect it had on the overall shape. He then connected corresponding points on the two sides of the face, and this led to a natural hypothesis that the axis of symmetry might perpendicularly bisect each of these lines.

Half-Head Reflection

1. Draw half a head on one side of the mirror.
2. Reflect it, then explore.



Jackiw created this image of a face by reflecting half a face. Connecting corresponding points suggests the idea that the lines are bisected perpendicularly by the axis of symmetry.

At the other end of the curriculum, dynamic geometry allows students to develop transformations not often studied in school. Any constructed dependency between two points can define a transformation of the plane, which is a much more powerful and abstract mathematical idea. One can, for example, define circle inversion: To invert through a circle with center O , take any point P to a point P' defined by the relationship $OP \cdot OP' = OR^2$, where OR is the radius of the circle. One can then use that construction to define a transformation of the whole plane, allowing the user to explore the global properties of that definition easily.

The Common Core State Standards say that we should turn to dynamic geometry as one of a number of technologies that we should know how and when to use, which seems to be based on the notion that appropriate technology is somehow layered on top of or independent of the mathematical content of the curriculum. Jackiw thinks that's a problematic notion, arguing instead that technologies always mediate our understanding and even definition of "content," and that changing technologies inevitably changes our perception of and relation to specific content. While he is pleased that at least they're admitting it into the stable of approved tools, his view is a more evangelical one.

First, he'd like to see dynamic geometry used as early as in kindergarten. A growing body of research demonstrates its usefulness at that level, and it's also useful at the college level. Countries that use Sketchpad system-wide amortize the investment of a student's time in learning the software over many years.

Second, he argues that dynamic geometry’s usefulness extends far beyond geometry to any situation in which geometric visualization is relevant. Algebra, calculus, number operations all need dynamic visualization. He argues that it’s like a word processor’s relationship to handwriting. In mathematics, Dynamic Geometry is potentially appropriate wherever a blackboard or piece of paper is appropriate.

Dynamic geometry supports several mathematical “habits of mind.”⁷ It is ideally suited to helping students “understand the value of example sets,” because it provides more examples of the phenomenon in question than you would ever otherwise see. Furthermore, those examples are ordered as you drag, providing additional insight. You’re making a trajectory through the space of all possible examples.

Another mathematical habit of mind that dynamic geometry naturally supports is “attention to extreme configurations, invariance, and generalization (mathematical and epistemological).” Extreme configurations pop out in dynamic geometry. In even the simplest case of playing within a triangle, everyone who does this ends up creating a collapsed triangle, for example. And invariance pops out obviously, as in the intersection of the altitudes of a triangle.

Dynamic geometry is also powerful because it draws on embodied cognition. The experience of dragging a triangle is very different from watching it. When doing it yourself, a feedback loop occurs between your hand and your eye, and this relationship is the core of all cognitive development.

Kurt Kreith

Kurt Kreith, a mathematician at the University of California at Davis, spoke about “Teachers in the Information Age.”

Norbert Wiener, who founded the field of cybernetics, wrote a book called *God and Golem, Inc.* (a golem is a kind of created servant). The question Wiener asked is whether these machines will be our servants or our masters. And that’s ultimately the challenge for teachers: to make computers the servants of students’ understanding.

When students use calculators mechanically — for example, entering bivariate data sets into a calculator and then pushing a button to spit out the correlation, when neither the students nor the teacher knows the definition of “correlation” much less the subtleties of the concept — they are acting as the calculator’s servants. And, unfortunately, such tasks are common.

There are three alternatives to this mechanical approach:

1. Emphasize *transparent* forms of technology. When a calculator does arithmetic, for example, the programming may be mysterious, but what it’s doing for you is not. Avoid having technology do things that students don’t understand.
2. Give the teachers greater control over the extent to which technology serves as a black box. For example, Kreith wondered if a dynamic geometry program could allow a teacher to set it to do synthetic geometry only, without measuring anything.
3. Create situations in which the student “teaches the machine” rather than visa-versa. Kreith has done this in a variety of areas, having students create spreadsheets that develop the procedures that underlie topics such as positional notation, long division, and the Euclidean algorithm. Many of these make use of Excel’s built-in MOD function to make accessible some of the mathematics that is central to the modern world of bar codes, parity checks, and credit card encryption.

⁷ Cuoco, A., Goldenberg, E.P., Mark, J. (1996). Habits of mind: An organizing principle for mathematics curricula. *Journal of Mathematical Behavior*, 15, 375-402.

Susan Addington

Susan Addington of California State University, San Bernardino, demonstrated some ways to solve algebra word problems without algebra, using spreadsheets. She demonstrated the method using this word problem: Mary is four years older than Rico. Ten years ago, Mary was three times as old as Rico. How old are they now?

To begin, the student simply guesses. Let's say Mary is 15. Then it's clear that Rico must be 11, since Mary is four years older. Instead of doing that calculation, though, you create a spreadsheet in Excel and tell the computer how to do the calculation by creating a formula. The next cell in the row would contain a formula to give Mary's age ten years ago (5), then a cell with Rico's age ten years ago (1), then three times Rico's age ten years ago (3). Then you have a cell subtracting that from Mary's age ten years ago (2). If the guess were correct, the final cell would be 0, but in fact it's 2.

So the next step is to copy the formulas so that you create a range of possible ages. Then you can simply scan down the final column to find the 0. That occurs when Mary is 16 and Rico is 12.

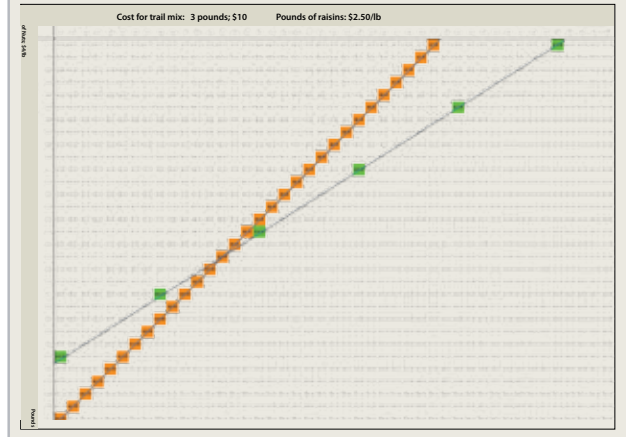
Solving an algebra word problem without algebra

Mary is 4 years older than Rico. Ten years ago Mary was 3 times as old as Rico was then. How old are they now?

Mary's age now	Rico's age now	Mary's age 10 years ago	Rico's age 10 years ago	3 x Rico's age 10 years ago
9	5	-1	-5	-15
10	6	0	-4	-12
11	7	1	-3	-9
12	8	2	-2	-6
13	9	3	-1	-3
14	10	4	0	0
15	11	5	1	3
16	12	6	2	6
17	13	7	3	9
18	14	8	4	12
19	15	9	5	15

This method, called a guess-and-check table, is well known to some algebra teachers and is included in the CPM curriculum. The check is an equation satisfied when the guess is correct, and the formula falls right out. This trains students to write equations for word problems.

Solving a problem of two linear equations with the intersection of two lines



The above is another problem in which a table can be useful: you want to make 3 pounds of trail mix from nuts and raisins. You have \$10 to spend. Nuts cost \$4 per pound. Raisins cost \$2.50 per pound. How much should you buy?

You can simply make a table of possible combos of raisins and nuts. You color a cell green when the combination costs \$10 and yellow when it weighs three pounds. The colored cells suggest lines; the intersection of the two lines is the solution.

While this is a time-consuming method to solve the problem, it effectively illustrates the idea that solving a system of two linear equations is the same as finding the intersection of two lines.

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Main Office 510-642-0143 • Fax 510-642-8609
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