

**CIME**

Critical Issues in Mathematics Education Series  
Teaching and Learning Algebra • Workshop 5 • 2008

# Transitions

Transitions: Organizing Principles for Algebra Curricula  
The Transition from Arithmetic to Algebra  
The Algebra Needed for College

Editor: Julie Rehmeyer



**MSRI**

Mathematical Sciences Research Institute

Published by the  
Mathematical Sciences Research Institute  
2014 • Berkeley, California

---

# Transitions

Organizing Principles for Algebra Curricula  
The Transition from Arithmetic to Algebra  
The Algebra Needed for College

Critical Issues in Mathematics Education Series  
Teaching and Learning Algebra  
Workshop 5 • 2008

By Julie Rehmeyer



---

Published by the Mathematical Sciences Research Institute  
2014 • Berkeley, California

# CIME

*Critical Issues in Mathematics Education*



## Teaching and Learning Algebra: 5th Workshop

### Organized by:

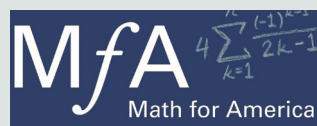
Al Cuoco, <i>Chair</i>	Center for Mathematics Education	James Fey	University of Maryland
Deborah Ball	University of Michigan	Megan Franke	University of California, Los Angeles
<i>Ex officio</i>		Roger Howe	Yale University
Hyman Bass	University of Michigan	Alan Schoenfeld	University of California, Berkeley
Herb Clemens	Ohio State University	Ed Silver	University of Michigan

### MSRI Educational Advisory Committee:

Michèle Artigue	Université Paris	David Eisenbud	MSRI
Deborah Loewenberg Ball	University of Michigan	Roger Howe	Yale University
<i>Chair</i>		Maria Klawe	Harvey Mudd College
Hélène Barcelo	MSRI	Jim Lewis	University of Nebraska, Lincoln
Hyman Bass	University of Michigan	Robert Megginson	University of Michigan
Sybilla Beckmann	University of Georgia	Robert Moses	The Algebra Project Inc.
Herb Clemens	Ohio State University	Alan Schoenfeld	University of California, Berkeley
Ricardo Cortez	Tulane University	Katherine Socha	Math for America
Ted Courant	Bentley School	Hung-Hsi Wu	University of California, Berkeley
Alissa Crans	MSRI		



MSRI receives major funding for the Critical Issues in Mathematics Education Workshop series from the National Science Foundation DMS-0441170, Math for America, and the Noyce Foundation.



The workshop speakers were chosen for their ability to articulate widely-held perspectives on mathematics education, but this choice is not meant as an endorsement of those perspectives. The content of this booklet is not intended to represent the views of the organizing committee, the Mathematical Sciences Research Institute, or the sponsors of the workshop.

# Contents

Acknowledgements .....	ii
Introduction .....	iv
<b>Question 1: Organizing principles for algebra curricula</b>	<b>5</b>
<b>Curriculum</b>	
1 The CME Project .....	6
2 The Interactive Mathematics Program .....	10
3 College Preparatory Mathematics .....	14
4 University of Chicago School Mathematics Project .....	16
<b>Proposal</b>	
Developing School Algebra Through a Focus on Functions and Applications .....	18
<b>Question 2: The transition from arithmetic to algebra</b>	<b>21</b>
<b>Presentations</b>	
“Why are Word Problems so Damn Hard?” .....	22
Alan Schoenfeld of the University of California, Berkeley	
What is Algebra, Really? .....	28
Mark Saul of The Center for Mathematical Talent, New York University	
How the Ideas and Language of Algebra K-5 Set the Stage for Algebra 8-12 .....	30
E. Paul Goldenberg of the Education Development Center, Inc.	
Supporting the Transition from Arithmetic to Algebra .....	31
Deborah Schifter of TERC Investigations, Susan Jo Russell of Education Development Center, and Virginia Bastable of Mount Holyoke	
Easing Kids from Arithmetic to Algebra with Strip Diagrams .....	34
Sybilla Beckmann of the University of Georgia	
<b>Question 3: The algebra needed for college</b>	<b>37</b>
<b>Presentations</b>	
A Master’s Degree Program to Improve the Teaching of Middle School Mathematics .....	38
Daniel Chazan of the University of Maryland	
The Precision and Rigor that is Essential for Teaching Algebra .....	40
Hung-Hsi Wu of the University of California, Berkeley	
The Uses of Higher Algebra for Teachers of School Algebra .....	43
Al Cuoco of the Center for Mathematics Education	
Teaching Algebra, Not Just Learning It .....	46
Deborah Ball of the University of Michigan	

# Introduction

In 2004, the Mathematical Sciences Research Institute (MSRI) launched a workshop series, *Critical Issues in Mathematics Education*, to provide opportunities for mathematicians to work with experts from other communities on the improvement of mathematics teaching and learning. In designing and hosting these conferences, MSRI seeks to encourage such cooperation and to lend support for interdisciplinary progress on critical issues in mathematics education.

The main goals of these workshops are to:

- Bring together people from different disciplines and from practice to investigate and work on fundamental problems of education.
- Engage mathematicians productively in problems of education.
- Contribute resources for tackling challenging problems in mathematics education.
- Shape a research and development agenda.

This booklet documents the fifth workshop in the series, *Teaching and Learning Algebra*. The workshop brought together mathematicians, mathematics educators, classroom teachers, and education researchers who are concerned with improving the teaching and learning of algebra across the grades.

For over two decades, the teaching and learning of algebra has been a focus of mathematics education at the precollege level. This workshop examined issues in algebra education at two critical points in the continuum from elementary school to undergraduate studies: at the transitions from arithmetic to algebra and from high school to university. In addition, workshop participants discussed various ways to structure an algebra curriculum across the entire K-12 curriculum. The workshop design was guided by three framing questions:

**1:** What are some organizing principles around which one can create a coherent pre-college algebra program? There are several curricular approaches to developing coherence in high school algebra, each based on a framework about the nature of algebra and the ways in which students will use algebra in their post-secondary work. We seek answers to this question that articulate the underlying frameworks used by curriculum developers, researchers, and teachers.

**2:** What is known about effective ways for students to make the transition from arithmetic to algebra? What does research say about this transition? What kinds of arithmetic experiences help preview and build the need for formal algebra? In what ways does high school and undergraduate mathematics depend on fundamental ideas developed in the transition from arithmetic to algebra? What are some effective pedagogical approaches that help students develop a robust understanding of algebra?

**3:** What algebraic understandings are essential for success in beginning collegiate mathematics? What kinds of problems should high school graduates be able to solve? What kinds of technical fluency will they find useful in college or in other post-secondary work? What algebraic habits of mind should students develop in high school? What are the implications of current and emerging technologies on these questions?

The workshop speakers were chosen for their ability to articulate widely-held perspectives on mathematics education, but this choice is not meant as an endorsement of those perspectives. The content of this booklet is not intended to represent the views of the organizing committee, the Mathematical Sciences Research Institute, or the sponsors of the workshop.



## Organizing principles for algebra curricula

What are some organizing principles around which one can create a coherent pre-college algebra program? There are several curricular approaches to developing coherence in high school algebra, each based on a framework about the nature of algebra and the ways in which students will use algebra in their post-secondary work. We seek answers to this question that articulate the underlying frameworks used by curriculum developers, researchers, and teachers.

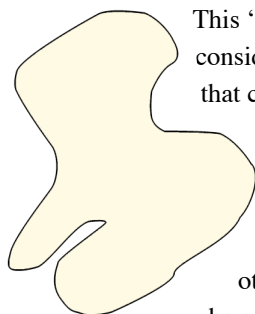
To address this question, speakers described the principles underlying four effective algebra curricula, plus one speaker proposed a rather different approach.



## Curriculum 1: The CME Project

The Center for Mathematics Education administers the CME Project, an NSF-funded coherent four-year curriculum, published by Pearson, that follows the traditional American course structure. Al Cuoco, the lead developer for the project, described its goals and approach.

The three central goals are for students to make connections among the various disciplines of mathematics; for them to learn to use general purpose tools rather than tricks that only work in isolated circumstances, like FOIL or keyword word problem solving techniques; and, most importantly, for students to learn a style of work that is indigenous to mathematics but is applicable to a wide variety of situations beyond mathematics.



This ‘style of work’ is exemplified by considering this question: Is there a line that cuts this object in half? For someone with appropriate mathematical training, the answer comes quickly: Imagine moving a line smoothly from one side of the blob to the other. At the beginning, the blob will be entirely on the right side of the line, and at the end, it will be entirely on the left side. By continuity, then, at some point half the blob will lie on one side and half will lie on the other. This kind of argument

This ‘style of work’ that is indigenous to mathematics is exemplified by this problem.

comes up in many contexts both within and outside of mathematics, and one of the goals of the CME Project is for students to truly master these broadly useful methods.

To this end, the CME Project has undertaken its “habits of mind” approach.

Here are five algebraic habits of mind that it highlights:

### 1. Seeking regularity in repeated calculations.

This allows students to move to greater abstraction through grounding in specific examples.

### 2. Chunking.

In other words, students learn to look for ways to change variables to hide complexity.

For example, the expression  $9x^2 - 6x + 1$  can be seen as  $(3x)^2 - 2(3x) + 1$ , and this can be seen as  $z^2 - 2z + 1$  where  $z=3x$ . This latter expression is far easier to factor.

### 3. Reasoning about and picturing calculations and operations first.

Before diving in, students learn to imagine how calculations will go without doing them. Sometimes, a complex calculation can be avoided entirely; sometimes it can be simplified; and if nothing else, this visualization helps students keep track of the big picture when they do calculate.

### 4. Purposefully transforming and interpreting expressions.

Different forms of expressions are useful or informative in different ways. Learning to recognize this helps students see the meaning in the expressions.

### 5. Seeking and modeling structural similarities.

Here are some examples illustrating these habits of mind and showing how they can be taught to students.

#### 1. Seeking regularity in repeated calculations

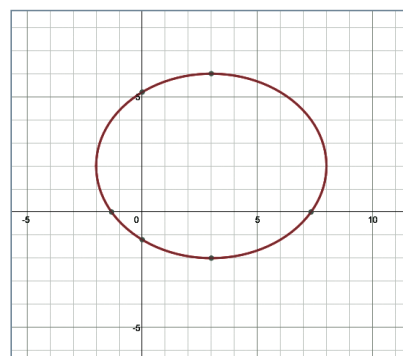
Consider this problem from a precalculus class that Cuoco had just taken over:

$$\text{Graph } 16x^2 - 96x + 25y^2 - 100y - 156 = 0.$$

The kids in Cuoco's class factored it, getting

$$\frac{(x-3)^2}{25} + \frac{(y-2)^2}{16} = 1,$$

and they then produced this picture:



Cuoco thought, “Boy! These kids really know how to do something.” Then he asked one kid after another, “Is the point  $(7.5, 3.75)$  on the

graph?” Not only didn’t they know, they had no way to tell, because for them, this equation was a code that allowed them to produce the picture. They had lost track of the fundamental fact that a point is on a curve if it satisfies the curve’s equation. This led to the idea that “equations are point testers,” which is strongly emphasized in the CME Project.

Students’ failure to grasp this is part of why linear equations can be difficult for them. They think that a linear equation is just something from which you read off the slope and  $y$ -intercept. So the CME Project has taken a different approach to slope, which seems to work well. Slope isn’t defined as an invariant of a line; instead, it’s defined as a function of two points. The curriculum then makes the assumption that points  $A$ ,  $B$ , and  $C$  are collinear if and only if  $m(A,B) = m(B,C)$ , where  $m$  is the slope. Later, the curriculum proves that using similar triangles in plane geometry, but in Algebra 1, the students accept it as an assumption. Finding the equation of a line between two points means finding a point-tester for that line.

So to find the line with the points  $A = (2, -1)$  and  $B = (6, 7)$ , students start by simply trying some points.

Is, for example, the point  $C = (3, 4)$  on the line?

Students can test that by calculating  $m(A, B)$  and  $m(B, C)$  and see if they are equal. In this case, the slope

$$m(A, B) = (7 - (-1)) / (6 - 2) = 8 / 4 = 2 \text{ and}$$

$$m(B, C) = (4 - 7) / (3 - 6) = -3 / -3 = 1, \text{ so } C \text{ isn't}$$

on the line. Students do this calculation with several points, until they notice a rhythm to the calculations.

They can then ask, what would you do to test to see if any point is on the line? They then repeat their calculations with an abstract point  $(x, y)$ :  $(x, y)$  is on the line if  $(y - 7) / (x - 6) = 2$ . This gives them a general point-tester, i.e., an equation. Reading the slope off the  $y = mx + b$  form of the equation comes later.

This habit of seeking regularity in repeated calculations is useful in other situations, like finding equations that model word problems, finding equations for curves, finding functions that agree with tables, establishing algebraic identities, and establishing proofs by mathematical induction.



Roger Howe

“The usual thumbnail history that you hear of algebra is that it was developed by Arabic or Islamic mathematicians in the middle ages. But in fact, algebra as we know it today was a product of the Renaissance. A central figure in this was François Viète. He introduced a revolution in algebraic thinking by essentially inventing the formula.

“Before Viète, problems were posed usually in oral form, and then some kind of a recipe or algorithm was demonstrated as an example of how to solve this sort of a problem. If you were told that the sum of two numbers was something and the difference of two numbers was something, then you would be given a recipe for finding the two numbers. But the idea of the numbers being quantities that you could work with and manipulate symbolically and get formulas for the solutions in terms of the symbols was lacking. This is what Viète brought to algebra.

“The tradition of oral algebra started at least in ancient Babylonian times. The earliest records are from 1800 BC, and it probably started earlier. It continued until the Renaissance. That’s a period of over 3,000 years.

“I take this as evidence that the idea of expressing relationships symbolically with variables is a difficult one. I think that we have to honor the difficulty of this idea with our students and give them a lot of opportunities to get used to the idea of variables. We have to let students know that these were really great inventions. The idea of writing  $x + 2$  or  $2x$  is a really great idea. It’s not, ‘Oh, of course, everybody knows this.’ It’s powerful and you can do a lot with it. If you didn’t think of it on your own, that’s OK.

“I think that every teacher should learn this history. I can’t think of a piece of mathematical history that’s more germane to the way we should teach than knowing about this sudden transition in algebra that happened at the end of the 16<sup>th</sup> century.

“That’s our first task: to help students to get used to the idea of using variables and to the symbolic notation that goes along with that.”

—Roger Howe, Yale University



### Silent Board Game

**Carol Cho** has been teaching for 40 years, and this is an assignment that she’s found especially helpful to students. It’s called the “silent board game.” She puts up a board like this one.

Silent Board Game					
a.	IN(1)	OUT(1)	b.	IN(1)	OUT(1)
		C		easy	
	L	N			light
		F		hot	cold
	Q			up	down
	W	Y		left	
			c.	IN(1)	OUT(1)
					⬠
				⬠	
				⬠	⬠
				⊙	

Rule:                      Rule:                      Rule:

Students are not allowed to talk. If a student thinks they can fill in one of the blanks, they raise their hand and come up to the board and write it in. If it’s incorrect, Cho will go to the board and erase it. Then another student will come up and write another answer, and if it’s correct, she smiles. Each correct answer gets them 2 points extra credit.

After a while, the students start really focusing: How come this student was right? How come mine was not right? Eventually, Cho stops the game and asks them what the rule is, and they might say, “The out is two letters after the in,” or “The out is the opposite of the in.” Often she’ll have to help them rephrase their formulation of the rule more precisely. Other boards involve an equation. Soon, they get pretty good at this. This game helps students learn how to go from a table to a rule.

Silent Board Game					
a.	IN(1)	OUT(1)	b.	IN(1)	OUT(1)
	8	17		100	51
	-2			4	
		9		6	4
	12	25		30	16
	10	21			31
			c.	IN(1)	OUT(1)
				4	16
				-1	1
					9
				12	
				-6	

Rule:                      Rule:                      Rule:

### 2. Chunking

Most teachers find that kids can handle problems like factoring  $x^2 + 14x + 48$ , which boils down to finding two numbers whose sum is 14 and whose product is 48. The problem comes with something like  $49x^2 + 35x + 6$ , a polynomial that isn’t monic. This is a particularly nice one because you can chunk it by setting  $z=7x$ ; then the expression becomes  $z^2 + 5z + 6$ . The curriculum calls it the hand method: Put a hand over the  $7x$  and it becomes monic.

So what about an expression like  $6x^2 + 31x + 35$ ? If it’s not so nice, make it nice! Multiply it by 6, and then divide the 6 out at the end:

$$\begin{aligned}
 &6(6x^2 + 31x + 35) \\
 &= (6x)^2 + 31(6x) + 210 \\
 &= z^2 + 31z + 210 \\
 &= (z+21)(z+10) \\
 &= (6x+21)(6x+10) \\
 &= 3(2x+7) \cdot 2(3x+5) \\
 &= 6(2x+7)(3x+5)
 \end{aligned}$$

Dividing out the 6 you “borrowed,” the factorization is  $(2x+7)(3x+5)$ . This saves a huge amount of time, and it’s another method that’s useful in many situations, like normalizing higher degree polynomials, deriving Cardano’s formula, solving trigonometric and exponential equations, completing the square, and analyzing affine transformations of graphs.

### 3. Picturing calculations

Cuoco would really like high school graduates to be able to do these problems in their heads:

1. Simplify  $(x-1)(x^4 + x^3 + x^2 + x + 1)$
2. Simplify  $(a+b)^2 - (a-b)^2$
3. Evaluate  $3(x-1)(x-3) + 5(x-1)(x-2) - 7(x-2)(x-3)$  at 1, 2, and 3.

Doing so requires students to be able to picture a calculation so that they can realize they don’t need to do it all out, and this skill is useful in many situations.

### 4. Transforming and interpreting expressions

Consider this problem:

Heron’s formula for the area of a triangle with height  $h$  and side-lengths  $a$ ,  $b$ , and  $c$  is:

$$4h(a,b,c) = \sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}$$

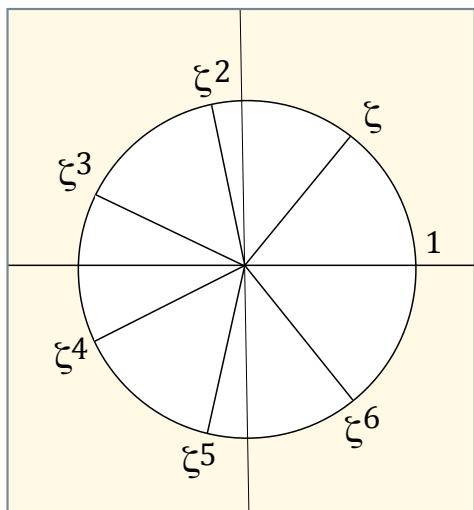
Give geometric and algebraic explanations for your answers:

- Why is  $h(a,b,c)=h(b,c,a)=h(c,a,b)$ ?
- When is  $h(a,b,c)=0$ ?
- Express  $h(3a,3b,3c)$  in terms of  $h(a,b,c)$ .

Students need to recognize the usefulness of the given form of Heron’s formula in order to do this problem effectively, and they need to be able to interpret its meaning.

### 5. Seeking and modeling structural similarities

Different types of mathematical objects sometimes have structural similarities which can be exploited to make a problem much simpler. This principle can be illustrated with this problem, which exploits the structural similarity between polynomials and complex numbers and would come at the end of precalculus. Here are the 7<sup>th</sup> roots of unity:



Zeta and its integer powers are roots of  $x^7-1=0$ . The non-real roots come in conjugate pairs. Since  $\zeta$  and  $\zeta^6$  are conjugates and reciprocals, their sum A is a real number. Same with  $\zeta^2$  and  $\zeta^5$  (summing to B) and  $\zeta^3$  and  $\zeta^4$  (summing to C).

Since they are real numbers, there should be a cubic equation with real coefficients for which these are roots. What are they?

To find it, we need to know  $A+B+C$ ,  $AB+BC+CA$  and  $ABC$ . We’ll start with  $A+B+C$ :

$$A + B + C = \zeta^6 + \zeta^5 + \zeta^4 + \zeta^3 + \zeta^2 + \zeta$$

But we know that  $\zeta^7-1=0$ , and we know in general that  $(x-1)(x^6+x^5+x^4+x^3+x^2+x+1)=x^7-1$ .

Substituting in  $\zeta$  for  $x$ , we know that  $\zeta^7-1=0$ , and since  $\zeta \neq 1$ , it must be that  $\zeta^6 + \zeta^5 + \zeta^4 + \zeta^3 + \zeta^2 + \zeta + 1 = 0$ , which means that  $A+B+C=-1$ .

The important thing here is that by modeling these complex numbers with polynomials, the problem becomes much simpler. Similar methods can be used to calculate  $AB+BC+CA$  and  $ABC$ , using a computer algebra system to calculate the polynomials and then simplifying it using the fact that  $\zeta^7-1=0$ . You find that  $AB+BC+CA=-2$  and  $ABC=1$ , so the cubic is  $x^3+x^2-2x-1=0$ . This previews really important material to come.

This habit of seeking and modeling structural similarities in algebraic systems is useful in other situations. E.g.,

- Matrices have structural similarities to linear transformations of the plane
- Arithmetic with integers have structural similarities to polynomials in  $i$  with an additional simplification rule
- Complex numbers have structural similarities to real numbers with  $i$  appended
- The matrix  $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$  can represent  $a+bi$
- The matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  can represent  $(ax+b)/(cx+d)$

An organization around algebraic habits of mind helps students see some coherence in algebra, provides students with general-purpose mathematical approaches, helps align school algebra with algebra as a scientific discipline, and helps students develop habits that are genuinely useful in the world.

There are equally useful habits indigenous to analysis and topology, for example, reasoning by continuity, looking at extreme cases, passing to the limit, extension by continuity, and using approximation. There’s no doubt that these habits are useful, particularly in scientific contexts. These ways of thinking form a solid basis for courses in geometry and “precalculus.”

For more information, go to [edc.org/cmeproject](http://edc.org/cmeproject).

## Curriculum 2: The Interactive Mathematics Program

Because the Interactive Mathematics Program (IMP) integrates algebra with geometry and statistics, Diane Resek, a co-director of the program, found it difficult to pull out the aspects of it that related only to algebra. Nonetheless, she described its goals and principles.

### Goals

**1. Motivating students to engage with the mathematics.** Typically students aren't engaged in math class — they're looking at the floor or the ceiling, unexcited, chewing gum, spacing out. The statistics show that half of algebra students flunk. So IMP aimed to change that, creating excited, engaged learners.

**2. Making students powerful problem solvers.** The ability to solve problems is becoming ever-more important, as this quote from the “Everybody Counts” report from the National Research Council in 1989 illustrates:

“From the accountant who explores the consequences of changes in tax law to the engineer who designs a new aircraft, the practitioner of mathematics in the computer age is more likely to solve equations by computer-generated graphs and calculations than by manual algebraic manipulations. Mathematics today involves far more than calculation; clarification of the problem, deduction of the consequences, formulation of alternatives, and development of appropriate tools are as much a part of the modern mathematician's craft as are solving equations or providing answers.”

This report came out just about the time that IMP was being developed, and the developers wanted students to do this kind of work in school.

**3. Preparing students for the future.** This includes enabling them to get into college, to do well in college, to make medical and financial decisions, and to prosper in jobs that don't yet exist. Students finishing the program need to be able to do well on college entrance tests and to be ready to take college courses taught in a standard way. Furthermore, people are changing jobs very quickly, including blue collar

and white collar workers alike. They have to learn new skills, to work with colleagues and to learn from books.

*To accomplish these goals, IMP embraced five core principles:*

### Principles

**1. Include key concepts and skills.**

The authors started with a list of concepts and skills they wanted students to learn. To compile this, they consulted NCTM standards, practices in other countries, their own classroom experience, and the advice of colleagues in other fields. They also took to heart the calculus reform maxim that less is more. A few examples of things they included:

- They emphasized proof and argumentation, starting in the first weeks. Students are, for example, given proofs with errors to analyze.
- They included problems such as using the distributive law to rewrite algebraic expressions.
- They asked students to explain things such as why division by zero is not well defined.

**2. Start with big problems.**

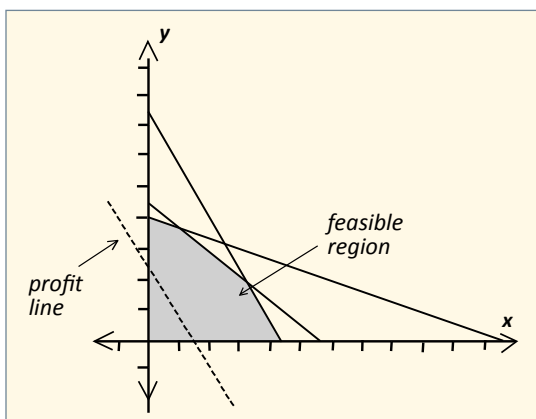
The curriculum is organized around big problems, rather than specific skills. There are five big problems each year for four years. Students work on each problem for one to two months. Inside that big problem, the skills were introduced in smaller problems, but with the focus remaining on the big problem. The primary reason for this was to help motivate the students, but it also helps the students develop problem-solving skills, since they have to refine and simplify the initial big problem to make it approachable.

Here's one example of a big problem, for a unit on solving two equations in two unknowns. The challenge in structuring this section of the course is that a single problem of this type does not qualify as a big problem — students can solve it very quickly by guess-and-check. The curriculum designers needed a situation where students would have to do lots

of these. They chose linear programming in two variables.

In particular, they gave students a problem about a small bakery shop that makes plain and iced cookies and is trying to decide how many dozens of each kind to make for the next day. The shop owners are limited by ingredients, oven space, and prep time, and of course, they want to maximize their profit. Students dive in and work on it bare-handed initially, to get invested in the problem. Each group tries to come up with the highest profit.

Then the students look at inequalities, how you operate with and graph them. Finally, they are able to interpret the problem in terms of this graph:



The x-axis represents dozens of plain cookies and the y-axis represents dozens of iced cookies, so each point gives a possible baking plan. If the point lies under one line, then there are enough ingredients to make that number of cookies; below another, there's enough oven space; and below a third, there's enough prep time. Therefore to satisfy all the requirements, it has to be in the shaded area.

Calculating profit requires a linear equation in  $x$  and  $y$ : if  $a$  is the profit on a dozen plain cookies and  $b$  is the profit on a dozen iced ones, then the total profit will be  $ax+by$ . So all the points that generate a profit equal to, for example, 100, lie on a straight line. The points generating a higher profit lie on a parallel line. Maximizing profit then boils down to figuring out which parallel line will be the last one that intersects

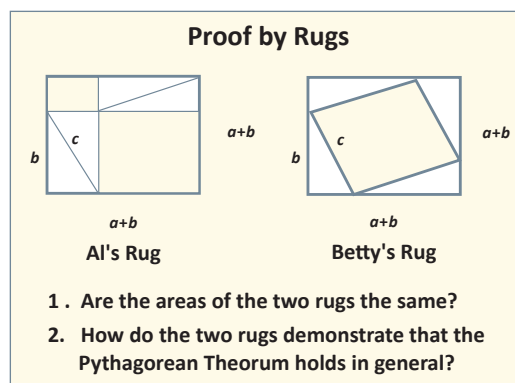
the shaded zone. This gets them to solve two equations in two unknowns.

Then they come up with an algorithm for the whole process. Different groups come up with different algorithms, but pretty much, they're variations on substitution.

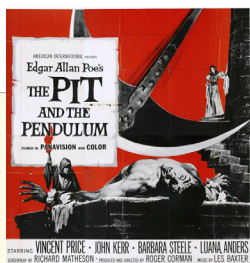
### 3. Actively involve the students

Keeping the students actively involved helps motivate students to engage with the mathematics. In the example above, the students actively engaged in coming up with their own algorithm rather than having an algorithm supplied to them.

When field testing, the authors found particular points where students weren't motivated. One such case was the Pythagorean theorem. We had them discover it for themselves, and the students were engaged with that, but then we wanted them to see a proof of it. The teachers went through a nice, clear proof, but we heard from teachers that the students were not engaged. This was one situation of several where we went back to the drawing board and put in something active.



This diagram shows the same area divided in two different ways, "Al's rug" and "Betty's rug." The students show that the two little squares that are white in Al's rug have the same area as the square in Betty's rug. The teacher then helps them to prove that Betty's rug is a square, which is a question that few students come to on their own. By that point, though, they're invested enough that they're willing to listen to the teacher.



## Edgar Allen Poe teaches math

Berkeley High School uses the Interactive Mathematics Program. Matt Bremer, a teacher there, gave this example of a “big problem” in IMP. Almost everything in their program is a “word problem,” and they choose five to six units of units of study in a year, each focused on a single big problem. How, for example, can a cookie store maximize their profits? When should a stunt man let go of a Ferris wheel to fall into a cart? Here’s one example in somewhat greater depth.

*The Pit and the Pendulum* is an Edgar Allen Poe story that tells a terrifying story of a prisoner during the Spanish Inquisition. In one part of the story, the prisoner is tied up at the bottom of a rat-filled 30-foot-deep pit, and above him, a pendulum armed with a blade is swinging, lowering itself toward him on every swing. Food has been left for him, so he rubs the food on the ropes restraining him, and the rats eat through them in just over a minute. On the twelfth swing of the pendulum, the blade is just about to slice his heart, but the rats finish the job and he manages to escape.

Students read the story and then ask how realistic this scenario is. Would a 30-foot pendulum swing a dozen times in just over a minute? To figure this out, students create their own pendulums of varying lengths and collect data on their periods. This then becomes a way for them to learn about regression models.

Here’s data some students collected:

Length in feet	Period in seconds
2	18.4
5	29.1
7	34.3
10	41.1
30	???

Students first tackle the question by looking directly at their data and trying to find patterns. They then use graphing calculators to find a curve that looks like this data, and then they tweak it to come up with something that fits the data reasonably well. They generally come up with a function that’s close to  $y = 13\sqrt{x}$  where  $y$  is the period and  $x$  is the length of the pendulum.

Then, with the help of a janitor who climbs to the top of the gym, the students construct a 30-foot pendulum to test their answer. This generates a sense of completion.

Active engagement was especially important to get students to remember definitions. An example is regression. The program gives them this situation: The star player of a basketball game has gotten injured, and the playoffs begin April 18. Should the coach pull him out or leave him in, hoping that he’ll be healed by then?

The coach has to make the decision on the basis of past data, which the students are given. They are then told that two other students have guessed that two different linear functions approximate the data. One of these functions is based on the sum of the linear distances and the other is based on the sum of the perpendicular distances. The students are then asked:

- Which student’s function seems to you to fit the data better, and why?
- Do you have a function that you think fits the data better than either of these? If so, what?
- Develop a mathematical procedure by which you might judge when one function fits data better than another.

At the end, the students are told that the definition chosen by mathematicians is the linear one, but by then, they’re invested enough that they don’t think it’s arbitrary and makes no sense.

#### 4. Introduce abstractions concretely

IMP introduces new ideas through stages over time. Graphing, for example, takes a number of days, starting pictorially and gradually adding numbers. With regression, the students are given some data and try to find best line using a straight edge or a piece of fettuccine. Then they try plotting the data intuitively with graphing calculators to see what line fits best. In the fourth year, they construct a procedure, and then, finally, they use the built-in facility on a calculator.

Abstract ideas are also often introduced with physical objects. For example, for graphing in three dimensions, strings are run from the front

to the back of the classroom for the  $x$  axis, side to side for the  $y$  axis, and from the ceiling to the floor for the  $z$  axis.

Metaphors are also useful. For exponential growth, for example, students discuss Alice in Wonderland. When she drank, she shrunk, and when she ate she grew. For each ounce of the drink her height would be halved, and for each ounce of cake, her height would be doubled.

**5 Use multiple representations**

Seeing different perspectives leads to deeper understanding, accommodates different learning styles, and develops ideas that can be applied to new problems. For example, students discover that  $2^0 = 1$  through a variety of representations:

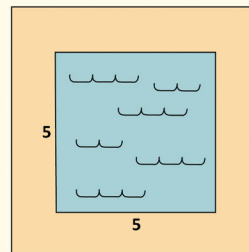
- Asking, “If Alice didn’t eat any cake, how big would she be and what number would her height be multiplied by?”
- Examining numerical patterns
- Graphically
- Deductively through the law of exponents

After this exploration, the curriculum presents the definition, hoping that at this point it won’t seem totally arbitrary. Finally, students are asked to write an explanation about exponents that are 0, negative or fractional. They then talk to an adult, explain it, and judge the person’s reaction.

All of this may seem unnecessarily elaborate, but it teaches the students that they can make sense of mathematics on their own, rather than it consisting of arbitrary rules. Some students figure that out without this kind of help, but not all. And it’s an equity issue: More people deserve to have this mathematical knowledge. Furthermore, people who could make valuable contributions to society are being excluded from math knowledge. Finally, evidence indicates that top students are not being harmed in this approach. Students from the top quartile in IMP did slightly better on standardized tests than those not in IMP. The difference wasn’t statistically significant, but at least the top students aren’t being harmed.



Elizabeth Phillips



1 ft  
1 ft  
border tile

Elizabeth Phillips presented this problem from the Connected Mathematics Project curriculum that shows how students can be helped to see that different algebraic expressions can be equivalent and can illuminate a problem in different ways.

The problem is this: A square pool is surrounded by one row of square border tiles. How many 1-foot square tiles,  $N$ , are needed for a border of a square pool with side length  $s$  feet? Find more than one way to represent this relationship. How can you convince your classmates that the expressions for the number of border tiles are equivalent?

Students come up with many different expressions, for example:

$$N = 4s + 4$$

$$N = 4(2+2) + 2s$$

$$N = 8 + 4(s-1)$$

$$N = 2(s+.5) + 2(s+1.5)$$

$$N = 4(s+1)$$

$$N = 4(s+2)-4$$

$$N = s+s+s+s+4$$

$$N = \frac{4(s+s+2)}{2}$$

$$N = (s+2)^2 - s^2$$

The last expression often creates confusion because it looks quadratic. The expression comes from viewing the tiles as the difference between the square formed by the outer edge of the tiles and the inner edge of them. Students use the distributive property in the problem to show that this expression is also symbolically equivalent to the others.



## Curriculum 3: College Preparatory Mathematics

CPM was created in 1999 by a non-profit organization, and much of the curriculum was written by teachers who then taught from the CPM books. It offers a curriculum for grades 6 through 12, all of which has been heavily tested in the classroom. One of the creators of the curriculum, Tom Sallee of the University of California, Davis described it.

The primary focus of CPM is to get more students to learn more mathematics in a way that causes them to retain their knowledge and to be able to transfer it into other academic subjects and out into the world.

The CPM developers have found that students' most common difficulties are more about learning than about math. One has to teach more than math: One has to teach how to learn, and specifically how to learn math. A lot of kids, especially from families that are underrepresented, don't learn good habits of thought at home. Organizing the curriculum to address this required changes at all levels: in teacher behavior, student behavior, classroom organization, and assessment.

To this end, the curriculum doesn't just have mathematical goals; it also has attitudinal goals. Students should come out of the program feeling confident they can figure out most problems on their own without being told how by the teacher, that they want to learn math, and that they want to understand what they learn.

Mathematically, each course is built around roughly six big ideas for the year. If students understand these big ideas deeply and integrate them, they'll be fine—but they really do need to understand them deeply. One of the keys to this deep understanding is the ability to move among different representations—written, tabular, graphical, symbolic—of the same concept. Students also need problem solving techniques, which CPM views as tools for both math and metacognition in general.

### Here are some examples of mathematical “big ideas”

- Multiple representations. There are many different ways of representing functions, including equations, graphs, tables, and contextual situations. Different representations are useful for different

things. Students need to be able to make connections among these representations and to move between them fluidly.

- Writing equations from word problems.
- Solving equations and systems of equations.
- Manipulating symbols.
- Proportionality. This is such a big idea that CPM spreads it over two years.

Ideally, students will come out of CPM having learned how to think. To that end, these fundamental approaches to learning are integrated into the curriculum. The first principle is that math is not a spectator sport: Work and engagement matter. Students need to learn to tackle things they don't already know how to do. Solving problems is the best way to learn new ideas, and to do that, students need to work with others and talk about math. This will help them to internalize these new ideas. CPM works to connect abstract concepts with concrete experiences. Kids tend to be able to think pretty clearly at a concrete level; the trick is to move them to the abstract level.

CPM recognizes that it takes a long time to learn a big idea, to internalize concepts, to learn algorithms, and to get mathematical habits of mind like generalizing or justifying your ideas.

To show how all this works in practice, consider how simultaneous equations are introduced. Before any rules are introduced for dealing with them, students are given this problem:

Some yodelers went on a gondola up a mountain for a party playing their xylophones. Two yodelers share one xylophone, so the number of yodelers on the gondola is twice the number of xylophones. The trip cost \$40 for the whole club, and each person cost \$2 and each xylophone cost \$1. How many yodelers and how many xylophones are on the gondola? Represent this problem with a system of equations. Solve it and explain how the solution relates to the number of yodelers. Represent it with a graph. Identify how the solution to this problem appears on the graph.

This is a challenging problem for students who haven't yet learned these concepts. CPM expects the students to jump in the deep end, though of course, it also provides support for both students and teachers, such as discussion points teachers can use to help the students. For teachers who are uncomfortable, there's further guidance. Teachers typically go immediately to the further guidance when they first begin CPM, but as they begin to understand that kids are smart, they give that up and give the students more time to solve it on their own.

Another example is how CPM teaches students to set up equations. CPM encourages them to guess and check, or guess and refine. This drives a lot of people crazy, thinking that the curriculum is trying to get students to guess the answer without understanding and without developing a general method. In fact, it is helping them to use their knowledge of special cases to generalize. Here's an example: Suppose a rectangle is 3 cm longer than it is wide and has a perimeter of 60 cm. What are its dimensions? The students start by guessing, and they're provided a blank version of this chart to fill out.

Setting up equations			
A rectangle is 3 cm longer than it is wide and has a perimeter of 54 cm. What are the dimensions? Write an equation that will allow you to solve this problem.			
Guess Side	Other side	Perimeter	=60?
10	13	46	low
15	18	66	high
14	17	60	high
x	x+3	2x+2(x+3)	=60

This student starts with 10 cm, calculates that the other side is 13 cm and that the perimeter then must be 46 cm, which is too low. So the student guesses higher: 15 cm. Then the other side is 18, and the perimeter is 66. That's too high, so the student tries 14. The other side is 17, the perimeter is 63, still high. So then the student does it with  $x$ : the other side is  $x+3$ , perimeter is  $2x+2(x+3)$ , and it equals 60. This process helps students to use what they know how to do with numbers to lead to the abstraction with  $x$ .

Notice also that this, like many problems in CPM, does double duty—in this case, reviewing the definition of perimeter while practicing setting up equations.

### Problems to help create mindful manipulators

These problems are designed to help students begin to notice things about expressions, rather than simply manipulating them without consideration. Do the equations have a solution? Explain how you know without solving them.

$$1. \frac{x+3}{2x+5} = 1$$

$$2. \frac{2x+3}{2x+5} = 1$$

$$3. \frac{x+3}{2x+6} = 1$$

Here are a few things one would hope students would notice about these. In the second one, the denominator is 2 bigger than the numerator, so it can't possibly equal 1! And in the third, the denominator is twice the numerator, so their quotient can't equal 1. It takes courage for a student to stop there and not solve it. Notice that they are perfectly allowed to solve the equation first—they just have to give an explanation that doesn't rely on that.

Without solving them, say whether the equations in 1-5 have a positive solution, a negative solution, the solution zero, or no solution. Give a reason for your answer.

- $7x = 5$
- $3x + 5 = 7$
- $3x + 7 = 5$
- $5 - 3x = 7$
- $3 - 5x = 7$

In the following problems, the solution to the equation depends on the constant  $a$ . Assuming  $a$  is positive, what is the effect of increasing  $a$  on the value of the solution? Does the solution increase, decrease, or remain unchanged? Give a reason for your answer that can be understood without solving the equation.

- $x - a = 0$
- $ax = 1$
- $ax = a$
- $x/a = 1$



**Elisabeth  
Phillips**

“My proposal is this: Presently, we have several K-12 algebraic curricula that embody one or more of the organizing principles discussed at this conference. We should take time to seriously study these curricula. What works for what students? Under what conditions? If we don’t, then we will have lost important information about what works and doesn’t work for students and teachers and we will end up repeating the same old patterns with very few students passing through the algebra gate.”

— Elisabeth Phillips, Michigan State University, coauthor of the *Connected Mathematics Project*

## Curriculum 4: UCSMP

The University of Chicago School Mathematics Project (UCSMP) was initiated in 1983 by professors in the departments of mathematics and education at the university. The secondary school curriculum developed by the project spends five years from algebra to calculus in order to make time for applications, statistics and discrete mathematics, plus a significant amount of algebra *before* the first formal algebra course. The third edition of the materials has recently been released. Zalman Usiskin of the University of Chicago described some history and organizing principles underlying algebra in this curriculum.

1. **The algorithmic approach:** The content is sequenced by skills following prescribed rules (algorithms), and in such a way that when you come upon a new skill, you are either putting together previously-learned skills or given a new rule.

During the 1950s, this was the only organizing principle for algebra, and the rules were presented as arbitrary facts to be learned. UCSMP goes far beyond this, but nevertheless, it retains the backbone of teaching students to perform algorithms reliably.

2. **The deductive approach:** Deduce the rules as theorems from the ordered field properties of the real (and later, complex) numbers, and in so doing, change the view of mathematics from a bunch of arbitrary rules to a logical and organized system.

Curricula in the new math era introduced the deductive approach and revolutionized math education. Again, UCSMP goes beyond this but has incorporated it as one of its guiding principles.

3. **Incorporating geometric transformations:** Transformations enable the notions of congruence, similarity and symmetry to apply to many different kinds of figures, not just the polygons and circles to which geometry is limited. Consequently, they provide a powerful set of ideas for dealing with graphs of functions and relations. The graph translation theorem is one powerful tool; it states that in a set of ordered pairs  $(x,y)$  described by a sentence in  $x$  and  $y$ , replacing  $x$  by  $x-h$  and  $y$  by  $y-k$  yields the same graph as applying the translation  $(x,y) \rightarrow (x+h, y+k)$  to the original relation. That along with a corresponding graph scale-change theorem allow students to understand the graphs of trigonometric functions, the similarity of all parabolas, for example, and the relationship between different conic sections. So in UCSMP, the geometry course must come between the two years of algebra study.
4. **Taking advantage of similarities of structure (isomorphism):** The word “isomorphism” is too unwieldy to use overtly with the

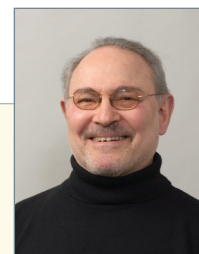
students, but correspondences can be pointed out. Properties of the additive structures of the real numbers correspond to properties of the multiplicative structure of the positive real numbers, for example,  $0a=0$  in the *additive* structure corresponds to  $a^0=1$  in the multiplicative structure. The isomorphism between composites of linear transformations and products of matrices has many payoffs, for example, in deriving the formulas for  $\sin(x+y)$  and  $\cos(x+y)$  from matrices for rotations.

- 5. The modeling approach:** The arithmetic operations are used to develop properties, formulas and applications for the corresponding algebraic expressions and functions. For example, by combining the use of subtraction to make comparisons and the use of division to calculate rates, one obtains the formula for slope. As another example, if a quantity grows with a constant growth factor  $b$  in every unit time interval, then it grows by a factor of  $b^n$  in an interval of length  $n$ , and since it remains the same in 0 time, this helps to justify that  $b^0=1$ .

This is a particularly important aspect of algebra if we want students to learn to use mathematics beyond arithmetic in their lives.

In the current edition of UCSMP, students are now using computer algebra systems. Usiskin argues that failing to do so is “morally wrong,” because technology can easily do many things that can only be considered “forced labor” on students. CASs are especially valuable for slower students, because it makes it easier for them to see the larger-scale patterns. Furthermore, it provides motivation for all students to learn to recognize equivalent forms of expressions, since machines don’t always output expressions in a desirable form.

In sum, there are many ways to organize the algebra we teach in schools. The key in each organization is to develop the sequence in a justifiable and understandable way and not be just a collection of isolated topics. Algebra plays many roles in mathematics and a single approach will not work for all needs.



Hyman Bass

“A functional definition of an algorithm is that it is something you can program a computer to perform. Given this, why should we teach algorithms? Let machines do the work, and liberate our thinking for higher order tasks. This logic seems quite compelling, and it is, often passionately, espoused by many educators, including some mathematicians.

“The tacit premise of this argument is that the primary reason for learning an algorithm is to be able to produce that outcome it enables, for example the result of a whole number division problem. I propose that the teaching of (numerical) algorithms supports two further important learning goals. One is that learning an algorithm—including why, not just how, it works—provides important insight into the nature of arithmetic operations and of our notational systems, notably place value. For example, the most direct way to understand the source of periodicity in decimal expansions of rational numbers is through examination of the long division algorithm.

“A second additional affordance of learning algorithms (including understanding of why they work) is to gain an appreciation of the very notion of algorithm. Ironically, algorithms are the enabling foundation of the very technology whose existence is offered as a reason not to teach algorithms. The prominence of the study of algorithms in computer science has bred important new branches of mathematics, such as complexity theory. Moreover, comparison of algorithms—for their efficiency, reliability, error-proneness, etc.—are interesting questions that are quite accessible and meaningful even in elementary mathematics.”

—Hyman Bass, University of Michigan

PROPOSAL

## Developing School Algebra Through a Focus on Functions and Applications

James Fey has a vision for an algebra curriculum that focuses on functions and applications and relies extensively on computer algebra systems to de-emphasize instruction in symbol manipulation. This came out of his experience teaching algebra. Like others he noted that the common conception of algebra is that it’s a dance of symbols. The word problems were commonly so staged that they were almost comical: Say “There are two trains” and everyone laughs. And he noted the usual results: students had a fragile mastery of limited technical skills; they learned special procedures for doing well-defined and inauthentic problems; and many developed a strong distaste for the subject.

Teaching calculus to students in the social sciences and management highlighted the consequences of this to him. The very notion of a variable was quite different in calculus from in algebra, since it represents a quantity that varies over time, rather than a fixed but unknown quantity. Equations in calculus show how quantities relate to one another, and expressions represent algorithms for calculating the value of a dependent variable from the values of an independent variable.

He’s come to think that algebra needs to be taught as a way of expressing relationships between variables. While its techniques are helpful for answering specific questions about variables (for example, finding the number that satisfies a particular condition), its greater power comes from its ability to express relationships between variables.

To give an example of this, consider this question: What average ticket price will maximize the operating profit of the Major League Lacrosse all-star game? Should the price be low, so that lots of people come and it seems like lacrosse is really growing? Or should it be higher, to get a lot of money out of a few people? What’s the break even price? Lots of variables are relevant to these questions: ticket price, tickets sold, income, expenses, profit... One could model the relationships with a bunch of functions, for example, if  $x$  is the ticket price:

Demand:  $n(x)=5000-65x$

Income:  $I(x)=5000x-65x^2$

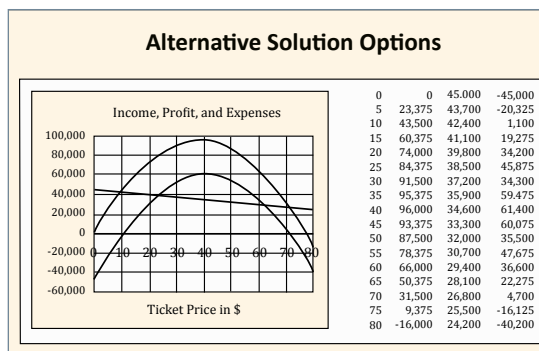
Expenses:  $E(n)=4n+25000$

$E(x)=45000-260x$

Profit:  $P(x)=-65x^2+5260x-45000$

Maximizing any of these functions, as a student might be asked to do in an applied calculus course, requires being able to do some algebra.

For almost all people, technology can give them satisfactory answers to questions that previously required lots of calculation. Most students would be happy, for example, getting an approximation for the optimum price by scanning the graph. A computer algebra system can also solve the problem exactly, too. CASs are pretty easy to use once a student learns their syntax. Given that, Fey wonders if it’s really justified to force a student who didn’t pass the algebra placement test initially to take a remedial course.



Furthermore, Fey argued that analyzing a graph produced by a CAS offers different and much richer information from what you get by analyzing profit, income, etc, by just doing symbol manipulation. If you solve a function for the break-even points, you’ll get two numbers from symbol manipulation. But the graph shows the whole scope of the relationship. At the peak, you can move a fair amount left and right without really changing how much money you make. Any price within a modest range is about the same as any other price.



Fey once allowed a student to use a graphing calculator throughout a calculus course, to see if access to a tool like that could allow a student to participate effectively and successfully when her weakness in symbol manipulation would otherwise have been quite a barrier. She was able to succeed. This led him to conclude that computers might allow us to rethink the priorities of an algebra course and particularly how much learning to manipulate algebraic symbols needs to be in the foreground of an algebra course.

That experience has changed his view about what algebra is really about. He came to conclude that the most important goals of algebraic reasoning are understanding and predicting patterns of change in variables, where variables represent things that change, not fixed but unknown quantities. Letters, symbolic expressions, and equations are invaluable tools for representing what we know or want to figure out about the relationship between variables. Students must learn to represent what they're thinking about in a symbolic way; technology can't do that for them. Algebraic procedures for manipulating symbolic expressions into equivalent forms are useful for developing insight into relationships between variables. But even here, calculating tools offer power alternative methods to gain insight and solve problems.

He has come to believe that algebra courses that focus on developing personal skill in algebraic symbol manipulation are a poor use of instructional time for all but a fairly small segment of the student population. In particular, such skill-oriented courses are an inappropriate requirement for getting a diploma, as does the growing requirement that students take a traditional Algebra II course. Traditional algebra courses don't even seem necessary for him for entry to college, even for majors that require a significant amount of quantitative reasoning.

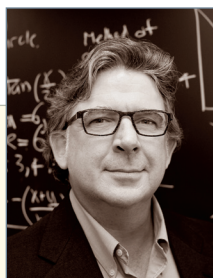
Fey argues that the essential dispositions, understanding and skills that ought to be at the heart of an algebraic experience are these:

1. **A disposition** to look for key quantitative variables in problem situations and for relationships among variables that reflect cause-and-effect, change-over-time, or pure number patterns. We want them to notice variables and relationships in their experiences and observe interesting number patterns.
2. **A repertoire** of significant and common patterns to look for: direct and inverse variation, linearity, exponential change, quadratic patterns, etc.
3. **The ability** to represent relationships between variables in words, graphs, data tables and plots, and symbolic expressions.
4. **The ability** to draw inferences from represented relationships by estimation from tables and graphs, by exact reasoning using symbolic manipulations, and by insightful interpretation of symbolic forms.
5. **The habit** of checking that the mathematics accurately describes the real world.

These goals suggest a presentation of algebra that begins differently from how it has traditionally. Rather than presenting algebra as a generalization of arithmetic, draw attention to the many interesting situations in science, business, engineering and technology where quantities change. The symbolic notations of algebra can be introduced naturally to precisely describe these observations of patterns. Then students can learn, with a very modest amount of personal symbolic reasoning proficiency, the array of computing tools to answer questions about them. Students who need to develop personal skills to do algebraic manipulations *without* technology can learn that when it appears essential, rather than doing so as part of a first step for all.

In some sense, this turns tradition on its head, by starting with conceptual understanding and problem-solving and ending with personal manipulation skills. This provides





**William  
McCallum**

“I would like to make a pitch for distinguishing between algebra and functions. I think of a function as a black box taking input to output. All the things we think about in algebra—such as expressions, equations, graphs and so on—I think of as the machinery inside the black box.

Different machinery can produce the same function:  $x$  goes to  $4x^2$  and  $p$  goes to  $(2p)^2$ , for example. Algebra can describe the machinery inside the black box.

“Functions are a useful vehicle for teaching algebra, but algebra is not *about* functions. You might use them as a way of teaching algebra though. Furthermore, although symbolic manipulation is an important part of the machinery, neither is algebra solely about symbolic manipulation.

“I propose: Algebra is about reasoning with numbers, operations and relationships using symbols. This includes working with expressions, equations, and functions.”

— William McCallum  
University of Arizona

students with intuitions about variables, expressions and equations that are an effective concrete grounding for later development of manipulative skills. First, they learn that these  $x$ 's and  $y$ 's and equations really *mean* something.

Fey acknowledges the following reasonable concerns about these ideas:

- 1. Is this algebra?** Fey is recommending putting factoring and solving and other forms of manipulation in the background and putting functions in the foreground. That sounds like analysis rather than algebra.
- 2. Does the function-oriented development serve well the variety of topics in which algebraic manipulation is useful?** Aren't there some topics in mathematics that aren't functions?
- 3. Don't users of CAS need some personal skill to understand how to utilize the tool?** Fey says that he has talked with several colleagues and asked them about what students can't do with CAS that they need to be able to do. They've said that there's a kind of flexibility and available to arrange things algebraically that a CAS just won't give you. Fey says that he hasn't heard the real killer example of that, but it's reasonable to imagine.
- 4. Is “just in time” skill development feasible pedagogically?** Some teachers say that students learn by doing it first, and that then it'll make sense to them and they'll see the structure later. That's an empirical question, and Fey says that 25 years ago, he had no evidence to support these proposals. But over that time, there's been a lot of development of pretty effective curriculum materials and experiments that show that kids can learn this way. So he argues that now it's not totally pie-in-the-sky, but an idea that has a lot of promise.

Fey points out that young people approach finding information differently from older people. They'll pull out their cellphone and call up a CAS to do a calculation for them. This technology environment for doing mathematics isn't going to go away; it's going to accelerate. So he argues that if we aim to provide the kind of mathematical understanding and skills that will be most useful and attractive to most students, this can make a strong claim for priority in school mathematics.



## The transition from arithmetic to algebra

What is known about effective ways for students to make the transition from arithmetic to algebra? What does research say about this transition? What kinds of arithmetic experiences help preview and build the need for formal algebra? In what ways does high school and undergraduate mathematics depend on fundamental ideas developed in the transition from arithmetic to algebra? What are some effective pedagogical approaches that help students develop a robust understanding of algebra?

Alan Schoenfeld

## “Why are word problems so damn hard?”

Prologue: *Why middle school math is important ...*

Schoenfeld has spent the past several years in middle school classrooms, so he’s had a good deal of opportunity to see the challenges that word problems offer middle school kids. But it’s not just an academic problem, he says. Being bad at the mathematics involved in such stories can have real world consequences, as described in this story.

A friend of his complained that his pain medicine was inadequate. His doctor asked how much morphine he was getting, and the friend said he was taking two pills every two hours. The doctor said, “We’ll double it to four pills every four hours.”

So, it’s not just middle school kids who have trouble with these kinds of issues. Getting the math wrong in cases like this can be painful, and possibly life threatening!

*...and why it’s harder than it looks.*

Typically, students are taught a simple model for how to solve word problems: Read the problem; make a model or diagram of the situation; write down equation(s); solve the equations; and check the answer. But the reality is that it’s not that simple, Schoenfeld argued. That model is akin to describing how you ride a bike this way: Put your foot on the pedal; swing your leg over; and start pedaling forward while you hold the handlebars. Those instructions make sense if you’ve been riding a bike for a long time—but for someone learning to do it for the first time, a lot of balance, motor coordination, and other skills that aren’t included in that account have to be acquired.

So in this talk, Schoenfeld aimed to unpack each of the steps described and add a few hidden steps, in order to suggest why solving word problems is as difficult as it is.

*Here are some of the challenges students face:*

### 1. Reading and language comprehension issues.

*Consider the problem:* “A five-pound box of sugar costs \$1.80 and contains 12 cups of sugar. Marella and Mark are making a batch of cookies. The recipe calls for 2 cups of sugar. Determine how much the sugar for the cookies cost.”

That looks really straightforward, but native English speakers are unlikely to notice that in one place the problem refers to a “batch,” and in another it refers to a “recipe.” For English language learners, that’s a decidedly non-trivial hurdle.

*Another problem:* “The upper Angel Falls, the highest waterfall on Earth, are 750 meters higher than Niagara Falls. If each of the falls were 7 meters lower, the upper Angel Falls would be 16 times as high as Niagara Falls. How high is each waterfall?”

This problem is a nightmare for a non-native speaker. There are upper falls, there are the words “higher” and “lower,” there are waterfalls, there are Niagara Falls.

*A third problem:* “The Java Joint wishes to mix organic Kenyan coffee beans that sell for \$7.25 per pound with organic Venezuelan beans that sell for \$8.50 per pound in order to form a 50 pound batch of Morning Blend that sells for \$8.00 per pound. How many pounds of each type of bean should be used to make the blend?”

The linguistic challenges here are huge, before you even begin to understand the mathematical structure. Other problems in the same book refer to “savings bonds,” “fungicide,” “red pigment,” and “processing a 24-exposure roll of film.” Imagine doing that in Swahili.

All of these problems were taken from middle school texts.

## 2. Mathematical disposition.

For many kids, math isn't about sense-making.

Here's an example that shows this starkly. A fairly typical student, working alone with his teacher, reads this task aloud: "A dragonfly, the world's fastest insect, can fly a distance of 50 feet in about 2 seconds. How long will it take for the dragonfly to fly 375 feet?"

Less than 1/5 second after reading the problem, the student says: "So, first I'll divide 375 with 50, and then— wait. Or, I will multiply ... 50, no wait, now what? This is dividing ... 5 times what can get 8?"

The teacher tries to slow the student down: "So you're thinking divide?"

The student says, "I'm not understanding. Do you look at 5 times the number first or is it the big number, this is 50 into it first?"

The teacher says, "What are the quantities we're looking at here? And what are you trying to find out?"

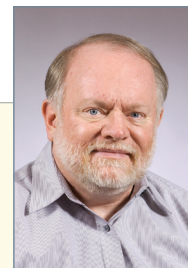
The student flounders: "Trying to find out how many seconds the dragonfly can fly in 375 feet ... wait ... How many seconds will it take it to fly 375 feet?"

The teacher encourages the student to draw a picture of the situation. The student draws a picture of a road, a town, a little dragonfly. The teacher focuses the student on the quantities, where they are in the picture, what they want to find out, and what they know. Then the teacher says, "So that looks great. What do you think we should do next?"

The student says, "I have an idea, maybe 50 times 375 divided by 2? ... That won't work."

The teacher says, "What are we trying to find out?"

Ed Silver



### Equality

Ed Silver of the University of Michigan pointed out that students' misunderstandings of the notion of equality can be a major stumbling block on the road to algebra, and there's been a fair amount of interesting research about this. Here are three different ways that kids can understand what the equals sign means:

1. Kids often first encounter the equals sign in the context of problems like  $8+7=_$ , and as a result, they can see the equals sign as being the signal to compile. That idea is reinforced by the fact that's exactly what the equals sign does on a calculator. But this understanding already starts breaking down if you give problems like  $8+7=_+3$ , or  $8+7+_ = 9+6$ . 15 is a very common answer to the first question for *sixth* graders, not just second graders. If a student's conception is rooted in this when they get to algebra, they're going to have big problems.
2. Another use of the equality sign is as punctuation, as if it were a comma:

$$12 \times 8 = 96 / 3 = 32 + 8 = 40 / 4 = 10$$

Compare:

$$12 \times 8 = 96$$

$$96 / 3 = 32$$

$$32 + 8 = 40$$

$$40 / 4 = 10$$

This is a crime against mathematics! But it's very common. In the U.S., we tend to permit abuse of the equals sign more than in some other places. Clearly, this is not a healthy use of equality.

3. Consider this problem:

On one balance beam, you have five A's on the left and three B's on the right. On another, you have one B on the left and an A plus two C's on the right. How many C balls would be needed to balance on A ball?

- a. 1
- b. 2
- c. 3
- d. 4
- e. 5

This is the notion of equality that's really helpful in algebra, but it's often least attended to in the developmental work in arithmetic.

This equality issue is obviously fundamental. A basic notion of algebra is that if a quantity can be expressed in more than one way, then these different expressions are, in general, equivalent, and students need to be facile with that to succeed.

**David Carraher**, a senior scientist at TERC, Technical Education Research Centers, described his work identifying opportunities to lay the groundwork for algebra in grades 3 through 5. He described two problems that show how children can begin to grasp the concept of variables in third and fourth grades.

Early in third grade, students are given two closed boxes. They are told that one contains John’s candies and the other contains Mary’s. Mary also has three more candies outside the box. The kids are told that both boxes contain the same number of candies, and they’re asked to show what they know about the candies John and Mary each have.

This is initially treated as an empirical problem, so the kids often start by shaking the box to try to guess how many candies it holds. Next, they’ll usually choose a particular number of candies—for example, they’ll say that John has 6 and then so Mary must have nine. A further step is made by a child who draws the boxes and refuses to commit herself to a particular number, thereby leaving it as an indeterminate value.



In a discussion, kids will make predictions about the number of candies each person has. Typically, some child will say something like, “Mary may have a total of 9 and John may have a total of 10, because we don’t know how many they have.” The children quickly realize that this doesn’t work. Eventually, they recognize that only the conjectures in which Mary’s amount is three more than John’s are acceptable.

They typically don’t use letters as variables, but they’ll use, for example, question marks. The teacher can say, “What if we call it  $N$ ? Is that reasonable?” Students are typically willing to buy into the idea that  $N$  can stand for the number of candies in the box. But sometimes they think  $N$  just means anything, with no constraints, and they’ll say, “How many candies does Mary have?  $N$ . John?  $N$ .”

By now, the student knows: “How many seconds will it take the dragonfly to fly 375 feet.” The student then gets enmeshed in a computation and sees it goes nowhere.

The teacher asks, “And we know what?”

Student: “It can fly 50 feet in 2 seconds.”

Teacher: “All right, what do you think? Well, if it could fly 50 feet ...”

Then a light bulb goes off for the student. “In 4 seconds it would be 100 feet, in 6 it would be 200, 8 would be 300, so 9 would be 350. There’s 25 missing, so  $1/2$  of it to get 375 so  $9\frac{1}{2}$  seconds to get 375 feet.”

Teacher: “That sounds pretty good...”

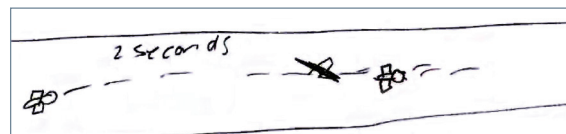
Although the student still didn’t have the details right, by the end, he was at least starting to reason constructively. This exchange shows that for many kids, math is not about sense-making. It’s not about taking a situation, figuring out how it all fits together, deriving relationships, symbolizing them and acting on them. It’s about doing what you’re told to do in classrooms, combining numbers to get an answer.

For example, take this problem, which Schoenfeld’s colleagues have given to hundreds of European kids:

“John wants to make wooden bookcases that are two feet wide. He has two five-foot long boards. How many two-foot long boards can he cut from them?” Seventy percent of kids say five.

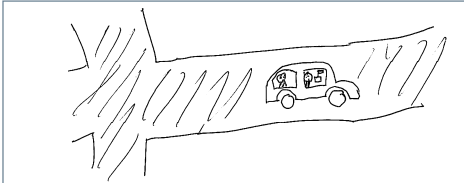
There are many other examples of this. There’s a famous problem, “There are 26 sheep and 10 goats on a ship. How old is the captain?” 76 out of 90 students said 36. After all,  $(26 - 10)$  gives 16 and a captain has to be older than that;  $(26 \times 10)$  is too old, and they don’t know how to divide the numbers, so it must be 36.

**3. In making a model or diagram step, students have to figure out what’s relevant and how to picture it.**



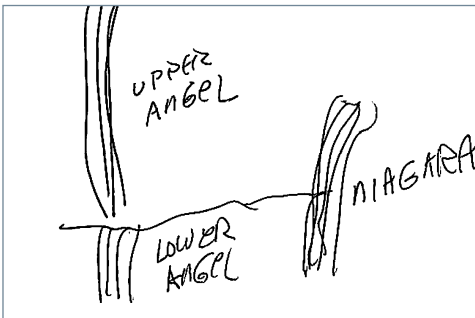
This is what the student drew in the dragonfly problem. It’s a scenic representation that doesn’t include the mathematically relevant information.

Or, here's another problem: "The local cab company charges \$1.25 for the first mile traveled and then \$0.35 for each additional mile. Natalie spent \$7.20 on a ride in a cab. How many miles did Natalie travel?"



If you look very closely, you'll see that there's a meter in the cab. That's better than many kids could do!



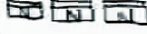
Consider this drawing a student made from the waterfalls problem:



All of us, if we drew that picture, would put the tops of the waterfalls in a straight line, because we know that it's the relative heights of the falls that matter. A student who starts with a picture like the above will have a much harder time mathematizing the problem.

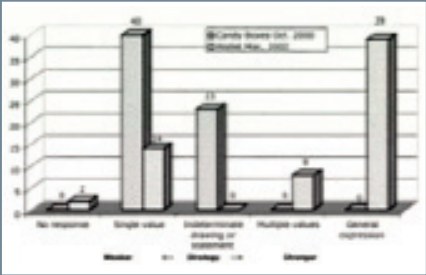
#### 4. Reading the math from the problem statement.

With experience, people learn an enormous amount about what to expect from a word problem. In the waterfalls problem, for example, a mathematically sophisticated adult will very quickly expect to solve the problem using two simultaneous equations in two variables. As soon as you recognize that, you approach the problem completely differently from someone solving it naively.

Mike	Robin
Mike has \$8 in his hand plus more money in his wallet.  \$8  $N + \$8 = \square$	Robin has \$23 money. Robin has 3 times as much money as Mike has in his wallet.  $N \times 3 = 2N$

**In fourth grade, 18 months later,** Carraher gives students a different problem: "Mike has eight dollars in his hand and the rest of his money in his wallet. Robin has all together three times as much money as Mike has in his wallet. Represent the amount of money Robin and Mike each have." This is a challenging word problem, and kids will dispute what this means. Does it mean that Robin has three times as much money as Mike? Kids will sometimes represent the situation with notation by drawing a wallet with an  $N$  on it, and say that  $N+8$  equals Mike's money. Then they'll draw three wallets for Robin. So the students are beginning to express algebraic reasoning, albeit without using algebraic notation.

Over a year and a half, these students show a clear shift in their thinking about variables. Forty children assigned a particular value to the candy boxes at the beginning. A year and a half later, fourteen still assigned a particular value in this analogous problem but there was a shift to using a symbolic representation either with letters or a combination of icons and letters. This suggests to Carraher that students can deal with statements about functions and relations between quantities at an early age, and that this is a focal point for early mathematics.

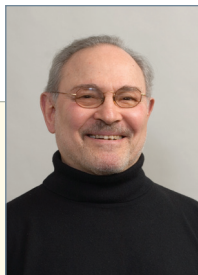


Further information on the early algebra studies by Carraher and his colleagues can be found here:  
<http://ase.tufts.edu/education/earlyalgebra/about.asp>

Carraher, D.W., Schliemann, A.D. & Schwartz, J. (2008). Early algebra is not the same as algebra early. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the Early Grades*. Mahwah, NJ, Erlbaum, pp. 235-272.

Schliemann A. D., Carraher D. W., Brizuela B. M. (2012). Algebra in elementary school. In L. Coulange & J.-P. Drouhard (Eds.) *Enseignement de l'algèbre élémentaire: Bilan et perspectives*. Special Issue of *Recherches en Didactique des Mathématiques*, pp. 109-124.





**Hyman Bass**

“In contrast with science, mathematical knowledge tends much more to be cumulative. New mathematics builds on, but does not discard, what came before. The mathematical literature is extraordinarily stable and reliable. In science, in contrast, new observations or discoveries can invalidate previous models, which then lose their scientific significance. The contrast is sharpest in theoretical physics. I. M. Singer once compared the theoretical physics literature to a blackboard that must be periodically erased.

“Given this, what saves math from sinking under the weight of its millennia of accumulated knowledge? It is a process that some (for example Bill Thurston) have described as compression. And algebra is the quintessential expression of this. For example, we create simple names and phrases that encapsulate, and cognitively rescale, very complex mathematical ideas. And we assimilate them fluently into the common language of mathematics. For example, two mathematicians may talk to one another fluently about a complex Lie group, but it might take two years of graduate instruction to explain to a non-expert what they are talking about. Yet the mathematicians talk about it as comfortably as would a child about a whole number.

“More concretely, consider the fact that the sum of two odd numbers is even. This statement, and its informal justification, are accessible to even young children. It typically rests on the notion of an odd number as one that can be ‘grouped by twos, with one left over,’ and, on this basis, a kind of generic proof can be offered. Once algebraic notation is available, the statement becomes,  $(2x+1) + (2y+1)$  is even for all integers  $x$  and  $y$ , and the very notation and basic rules of algebra render the proof almost mechanical.

“At the grand scale, compression is achieved by theories (group theory, number theory, Hilbert spaces, etc.) that synthesize and unify broad areas of mathematics.”

—Hyman Bass, University of Michigan

A researcher asked people what they expected of a math problem that started with the words, “a river steamer.” One subject said, “It’s going to be one of those river things with upstream, downstream, and still water. You are going to compare times upstream and downstream — or if the time is constant, it will be distance.” If you know that, you’ve got 70 percent of the problem done already.

After hearing five words of a triangle problem, one subject said, “This may be something about ‘How far is he from the goal’ using the Pythagorean theorem.”<sup>1</sup>

Knowing the tradition and the genre simplifies problems enormously. But beginning students aren’t at that point yet.

### **5. People’s conceptual models of the situations to be analyzed**

In the dragonfly problem, what makes the problem difficult for the student at the end (that is, once he finally understands the situation) is that he comes to the problem with an additive model rather than a multiplicative one. Mathematically sophisticated people approaching the problem see that distance per second is a convenient and powerful unit. Many students see this as an additive situation, with a non-standard yardstick: every 2 seconds the dragonfly advances 50 feet. The student’s perspective makes sense, it’s just not the most useful model for the problem. A teacher needs to recognize the model the student is using and help the student move to a more powerful mathematical perspective.

### **6. Meta-level knowledge**

Consider this problem: “Alan can mow the lawn in 40 minutes. David can mow the lawn in 50 minutes. How long does it take them to mow the lawn together? (Assuming two mowers, no crashes, etc.)” The key thing in understanding this kind of problem is that there are only certain things you can combine. In this case, it’s how much of the lawn each of them can get done in a certain amount of time, because those can be added. None of the other quantities can be added in this way. It’s not inherent in the equations; it’s inherent in one’s meta-knowledge of the situation.

### **7. Deriving the right equations**

There’s an art to picking the right variable, and it’s not easy to pick up. Consider this problem: “The length of a rectangle

<sup>1</sup> (Hinsley, Hayes and Simon, 1977, P. 97)

is five inches longer than twice the width of the rectangle. The perimeter of the rectangle is 112 inches. What are the dimensions (length and width) of the rectangle?”

With experience, one recognizes that it’s best to choose the width to be the independent variable, because then you can write the length as  $L=2W+5$ . But there’s no a priori reason to choose the width rather than the length, and if you choose  $L$  to be the independent variable, the problem becomes very much harder. Students will typically pick  $L$  because it’s the first variable that appears.

**8. Solving the equations**

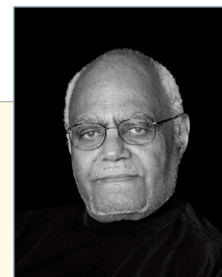
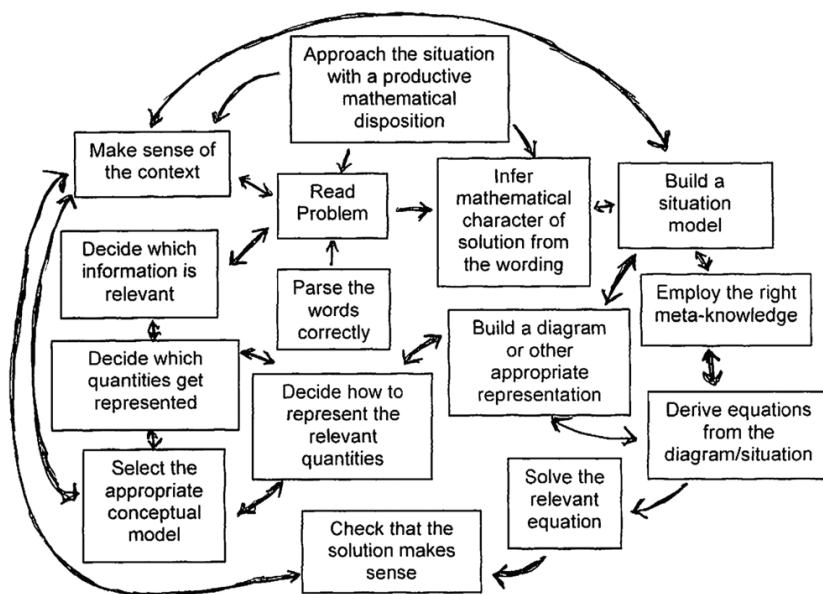
Solving the equations seems like it’s finally something straightforward and procedural — but even that isn’t quite true.

Choosing the strategy is non-trivial. For example, when do we substitute, when do we eliminate? That’s a kind of sophistication developed through time.

**9. Checking the answers**

Does it make sense? Can there really be 1-foot-long shelves in a 2-foot bookcase? Finishing up properly is every bit as much about sense-making as getting started, and students have to learn how to do that.

So the drawing at the bottom of the page gives a more realistic model of what problem-solving is like for our students. Every one of the boxes in the diagram requires complex pedagogical strategies and interventions, if you want there to be a good chance that students will get it! That’s why the job of middle-school teaching is so damn hard.



**Bob Moses**

“In order to change the country on big-ticket issues, we have to change the institutions and practices that have grown up as part of our history. In my experience, changing these requires a coalition of the top and the bottom. It’s crucially important for the people who are targeted by a damaging legacy to participate in the unraveling of it. The sharecroppers needed to know that they were party to the unraveling of their situation, and gaining their political access to the right to vote. It wasn’t something that was handed to them; they participated in earning it. The kids in the bottom quartile who are trapped in this system of education need to be part of that too.”

— Bob Moses, founder of the Algebra Project

Portrait featured in Mariana Cook’s *Justice: Faces of the Human Rights Revolution*. To purchase *Justice*, go to: <http://bit.ly/JusticeBook>  
Mariana Cook’s website: [www.marianacook.com](http://www.marianacook.com)

Mark Saul

## What is Algebra, Really?

Math teacher Mark Saul addressed the question of what algebra really is. He started by cautioning that he couldn't really answer that question, but he could offer a description of what algebra is for him.

First, he said, algebra is not letters in place of numbers. The problem  $5 + \text{what?} = 12$ , for example, isn't algebra, though perhaps it's pre-algebra. Graphing functions, which is usually taught in an "algebra" class, leads to analysis, not algebra. There are many ways to represent functions, and some of these representations are algebraic, but there is more to algebra than just the representation of functions and more to the representation of functions than just algebra.

*Assertion 1: The heart of algebra is NOT an understanding of the function concept.*

In looking at functions, and the role of algebraic variables in representing functions, students can come to understand something about algebra. This isn't unusual. For example, geometry is not characterized by the use of an axiomatic system, but in studying geometry, students can come to an understanding of something about the use of an axiomatic system. Fractions are not characterized by the expression of the probability of an event, but in computing probabilities, students can come to understand something about fractions. Baseball is not characterized by the speed at which a player runs, but in playing baseball, one can increase one's ability to run quickly.

Saul then presented three ways to think about algebra:

### 1. Algebra as "the general arithmetic," which was Newton's perspective on it.

Algebra gives a way of generalizing arithmetic patterns. For example,

$$25 = 5 \times 5 \text{ and } 24 = 6 \times 4;$$

$$49 = 7 \times 7 \text{ and } 48 = 8 \times 6;$$

etc., so

$$a^2 - 1 = (a + 1)(a - 1)$$

*Assertion 2: Students transitioning from arithmetic to algebra are learning to generalize their knowledge of the arithmetic of rational numbers.*

Key teaching questions for this:

"What is the next step in the pattern?"

"What is the 1000th step in the pattern?"

"What is the 1001st step in the pattern?"

### 2. Algebra as the study of binary operations.

Students come to shift their attention from the numbers they're operating on to the operation itself.

Take the problem "Solve  $2x + 5 = 13$ ," and contrast a solution through guessing and checking with an algebraic solution. For the algebraic solution, students have to understand that you can add the same thing to both sides of an equation and the equation still holds. That's moving to an understanding of the principles underlying the operations.

A hallmark of this level of work is that students begin to think about computations before performing them. Also, the "-tive laws" (commutative, associative, etc.) begin to have real meaning at this level. Students begin to see algebra as the study of "structures."

*Assertion 3: Students who are solving equations algebraically (and not arithmetically) are using the general properties of binary operations.*

Key teaching questions at this level:

"How are these equations the same?"

"What do you do next?" (before the student has actually done a computation)

"What do you want to do with the calculator?" (before the student has picked it up.)

### 3. As the study of the “arithmetic” of the field of rational expressions.

The mathematician I.M. Gelfand put it this way: “In arithmetic we can use letters to stand for numbers. In algebra, we use letters to stand for other letters.”

For example,

$$a^2 - b^2 = (a+b)(a-b)$$

If you let  $a=2x$  and  $b=1$ , then you get

$$4x^2 - 1 = (2x+1)(2x-1)$$

Or if you let  $a = \cos x$  and  $b = \sin x$ , you get  $\cos^2 x - \sin^2 x = (\cos x + \sin x)(\cos x - \sin x)$

On this level the form of algebraic expressions becomes important. Students can develop an intuition about which of several equivalent forms is most useful for a given situation. Algebraic expressions become objects of study, and not just their value at a given point.

Key teaching questions at this level:

“What plays the role of  $A$ ?”

“What plays the role of  $B$ ?”

The second transition that students make while studying algebra is to inductive reasoning: describing patterns, making conjectures, testing hypotheses, passing from specific cases to general rules. Empirical scientists do that all the time. Deductive reasoning involves examining assumptions, making definitions, “proving theorems,” passing from general statements to specific cases.

*Assertion 4: students making the transition from arithmetic to algebra are typically focused on learning and applying inductive reasoning, rather than deductive reasoning.*

What about the distributive law? Isn’t that an axiom, and hence involving deductive reasoning? Well, yes, but:

*Assertion 5: For us the distributive law is an axiom, but for kids first learning this stuff, it’s a way of computing.*

Applying the distributive law in a computation is, for us, an example of deductive reasoning. But for most students, most of the time, it is deductive reasoning *after* they’ve recognized deductions in other contexts.

Thus “justification of computation” is not a very effective step in learning about deduction. But, if this is done within a very conscious framework of, say, the field axioms, it can be a good example of a deductive system, with, say advanced students. (This is an empirical statement, on the basis of experience.)

So, Saul asks, how do we support students learning about the special nature of mathematical truth, which isn’t empirical? What are their typical intuitions about deductive logic? What are the steps in the development of this concept that we can anticipate them passing through?

*Assertion 6: Algebra traditionally in school math, including reform, is thought of in connection with inductive reasoning, and geometry with deductive.*

Saul ended his remarks with a list of questions he doesn’t know how to answer. How true is assertion 6? Are there places in algebra where we develop deductive reasoning? Are there places in geometry where we develop inductive reasoning?

How true “ought” assertion 6 to be? Is there a reason that algebra is conducive to inductive and geometry the opposite? Should we counteract that? How do we help students progress from inductive to deductive? And is “progress” the right word?

E. Paul Goldenberg

## How the ideas and language of algebra K-5 set the stage for Algebra 8-12

E. Paul Goldenberg of Education Development Center (EDC) described how algebra can serve both as a language and as a computational tool and argued that while most elementary school children can't use algebraic notation as a computational tool, they *can* use it as a language."

To us, expressions like  $(n-d)(n+d)$  can be manipulated to derive things we don't yet know or to prove things that we conjectured. But we can also use such notations, not manipulated, as a language to describe a process or computation or pattern, or to express what we do know, for example, that in all specific cases we've tried  $(n-d)(n+d) = n^2 - d^2$ .

Kids come with this unbelievable built-in apparatus. They are great abstracters. Take, for example, how they learn language. They abstract the meaning of a word like "dog" or "mommy" from chaotic data. First they may overuse the word dog for all kinds of animals, but then they refine it to apply only to certain kinds. Their drawings are similar abstractions, representing not what they see but what they know. Their phenomenal language-learning ability is why they can use some mathematical notation to express what they know before they can use it logically to derive what they don't know. They have some ability to quantify and to apply logic and are always building theories about the world. That's all stuff we can use in teaching them math. In learning math, they use the same tools.

Some algebraic ideas precede arithmetic. Part of understanding that eight things bunched together are "as many" as eight spread out seems built in and part may be developmental, but not until that idea is secure can a statement like  $3+5=8$  make sense. Children must believe that the number of objects *stays the same* when rearranged before it makes sense to say what "that number" *is!* Encoding this idea formally may come after arithmetic, but the algebraic idea, itself, comes first.

You certainly have to nourish kids to extend, apply and refine this built-in apparatus. They need experiments with breaking numbers up and rearranging parts for example, but they bring understanding that we shouldn't ignore. If you pull change out of your pocket and ask how much

money there is, with your fist closed, they should of course know that they don't know. But they never doubt that it's a fixed quantity. If you then show that money and ask, "Would you get the same number if you counted it in a different order?" You're almost making an assault on their prior logic.

Though children know that the number of objects doesn't change depending on how you count them, encoding that as  $a+b=b+a$  is a different matter. Like any language, algebraic language is a convention. You learn it from native speakers. In your native language, you get about half of your adult vocabulary and almost all of your grammar by the age of five, all inferred from use in context, not explicit definitions and lessons. When kids learn just by definition, you'll see them say, "Extinguish. That means put out." Then when they use it in a sentence they'll say, "Every night I extinguish the cat." Language learned from natural communication in context is absorbed quickly and with less distortion. "The same may be true of algebraic language," Goldenberg argued.

Linguistic strategies come up in math. For example, consider Michelle's strategy for subtracting 8 from 24. Well,  $24-4$  is easy, she says. Now 20 minus another 4. Well, I know  $10-4$  is 6, and 20 is  $10+10$ , so  $20-4$  is 16. So  $24-8$  is 16.

Breaking it up this way is algebraic. The knowledge that  $10-4=6$  is arithmetic. But the idea that  $24-4$  is easy is fundamentally *linguistic*, Goldenberg argued. He illustrated what he meant by this by asking the full name of a woman in the audience: Lisa Berger. What, he asked, if you take Lisa away from Lisa Berger? She responded, Berger. What if you take Berger away? She responded, Lisa. This, he said, is linguistically similar to taking "eight" away from "twenty eight."

This is linguistics, not math. Numbers didn't get born with names. We named them in order to make  $24-4$  easy.

Schooling should take advantage of the cognitive and linguistic strengths children bring to mathematics.



Deborah Schifter, Susan Jo Russell, and Virginia Bastable

## Supporting the Transition from Arithmetic to Algebra

Deborah Schifter, Susan Jo Russell, and Virginia Bastable have worked together since 1993, substantially on early algebra. Since 2001, they've had a series of grants to focus on early algebra. They authored *Developing Mathematical Ideas*, a professional development program that includes two modules on early algebra. They also developed *Investigations in Number, Data and Space*, a K-5 curriculum that includes early algebra components. Foundations of Algebra in the Elementary and Middle Grades is a project in which they are working with a group of teachers to see how the ideas of generalization and justification can develop in a classroom over the course of a year.<sup>2</sup> All three projects were funded by the NSF.

From their collaborations with elementary and middle grades teachers who found opportunities to address generalized arithmetic within their regular work on computation, Schifter, Russell, and Bastable identified four potentially important aspects of arithmetic experiences that underlie both arithmetic and algebra and, therefore, provide a bridge between the two. These are:

- Describing the behavior of the operations
- Generalizing and justifying
- Extending the number system
- Understanding notation

These aspects emerge naturally from work on computation already at the heart of elementary mathematics programs, and they can be highlighted and pursued by teachers who learn to recognize opportunities. Focusing on these aspects of arithmetic not only enables students to grow from arithmetic towards algebra, but also strengthens their understanding of computation.

Several classroom examples, documented with the help of collaborating teachers, were presented at the MSRI conference to illustrate these aspects.

**Example 1** illustrates the first two aspects: 1) describing the behavior of the operations, and 2) generalizing and justifying. It also previews the need for extending the number system.

The teacher, who worked with students struggling with grade-level mathematics, noticed they were making a common subtraction error:  $35 - 18 = 23$ . There are many possible interventions one might choose, but this teacher chose to have her students step back to consider the commutative property of addition. She asked them to articulate why, for example,  $17 + 9$  must be the same as  $9 + 17$ . After they demonstrated why these expressions are necessarily equivalent, she asked if subtraction works the same way. What happens with  $17 - 9$  versus  $9 - 17$ ?

Few of the students had experience with negative numbers, and most thought that  $9 - 17 = 0$ . One student came up with the following representation: she drew nine circles and then crossed them out, and drew 8 extra x's.



Her conclusion was that  $9 - 17 = 0$ , but other students looked at her representation and called the extra x's "invisible numbers." One student declared  $60 - 50 = 10$  and  $50 - 60 =$  "invisible 10." Throughout these explorations, the students—who had often felt unsuccessful and discouraged—were very engaged. The point of the exploration was not to introduce negative numbers; rather, the teacher wanted her students to think about the difference between addition and subtraction.

After these sessions, the teacher returned to the original problem,  $35 - 18$ . One student began to subtract in his old, incorrect way. Then he paused, said, "That won't work," and used another strategy to solve the problem correctly.

<sup>2</sup>Subsequent to this conference, the presenters completed a book for teachers of elementary and middle grades based on the work of this project. It includes extensive classroom cases and further elaborates the ideas of this presentation: Russell, Schifter, & Bastable 2011. *Connecting Arithmetic to Algebra: Strategies for Building Algebraic Thinking in the Elementary Grades* (Portsmouth, NH: Heinemann).





### Ed Silver on Teaching

Ed Silver of the University of Michigan argued that it's critical to consider teaching itself, not just curriculum. Good tasks and clever ideas are not sufficient.

Teachers will need help to use the materials well and to focus on the right stuff. This is particularly important with the transition to algebra, where the diversity in preparation, knowledge and skill of the teaching force is quite large, and they're being asked to do something that's not easy.

The task as presented in the textbook is not always the same as the task as set up by the teacher, which may be different from the task as enacted by teacher and students, and it's that, ultimately, that leads to student learning. For a variety of reasons, teachers may not present tasks the same way that the textbook does, from time issues to philosophy issues.

And teaching matters. The one thing the TIMSS videos found that distinguished high-performing countries from low-performing countries was how often and in what way teachers used high-demand tasks. In high-performing countries, high-demand tasks are used often and well.

But in the US, even if there are high-demand tasks in the textbooks, they tend to get watered down in the classroom. It happens frighteningly often. TIMSS could find almost no examples of high-demand tasks being used in high-demand ways in the US.

So the bottom line is that good tasks are important, but teachers also matter!

Some specific things that teachers need to know how to do:

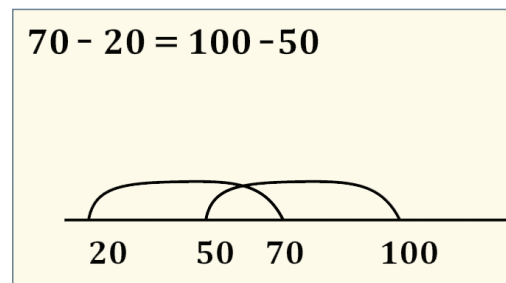
1. Resist the persistent urge to tell and to direct at all times, and instead to provide time and space for student thinking.
2. Know when and how to ask questions and to provide information to support student thinking rather than to replace it.
3. Help students accept the challenge of solving worthwhile problems, since we know that effort matters.
4. Sustain students' engagement at a high level.
5. Manage classroom interactions, such as orchestrating the treatment of multiple solution methods.

There is a growing body of evidence that teachers can be helped to do this better. They can learn how some of the things that they do with good intentions undermine student learning.

This intervention, based on a hypothesis about the source of the students' error, gave students access to ideas about the structure of arithmetic that will eventually help them in their transition to algebra. Their study of the behavior of subtraction led them to consider the meanings of the numbers in a subtraction expression, and how the relationship between those two numbers is not the same as the relationship between two addends.

**Example 2** illustrates the first three aspects of arithmetic experiences that lead to algebra (describing the behavior of the operations, generalizing and justifying, and extending the number system). It also highlights the work of a student who often needs additional challenge in math.

A fifth-grade class was discussing equivalent subtraction expressions, e.g.,  $70 - 20 = 100 - 50$ . A student, Alex, came to the board and drew a number line to show what was going on.



Alex explained, "You can see that the distance is the same. If you change one number, you change the other the same way. As long as both numbers change the same, you can make lots of new expressions." Other students played with his idea, sliding the 50-unit interval up and down the number line. One student observed that once the top end of the interval gets to 50, they can't move further to the left, because the bottom end of the interval is at 0.

Another student, Raul, said, "We could use the other numbers." The teacher asked, "What other numbers?" Raul responded, "The negative numbers

on the other side of zero.” Alex suggested using, 40 and  $-10$ , and showed the class how to write it:  $40 - -10 = 50$ .

At this point, several students started talking at once, pointing at the number line on the board. A student said, “No way, you can’t do that. How can you have a negative 10 and end up with 50?” Alex explained, “It’s like adding 10, because if you look on the number line you would have to jump 50 to get from negative 10 to 40. It’s the same as we did with 100 and 50 and 70 and 20.”

Although many students in the class still needed to think this through, the lesson was a first step toward making sense of subtracting negative numbers, using the number line to extend what they understood about subtracting positive numbers. Alex had a strong image for subtraction based on his work with the number line, and he was able to justify why the generalization extends to integers. His thinking gave the class an opportunity to consider new ideas about the operation of subtraction. At the same time, Alex was challenged to explain his conjectures and justify his generalization with a representation.

**In Example 3**, students engage with the first three aspects, as in Example 2, but also work to express their ideas using algebraic notation.

In a sixth-grade classroom, a student asked, “Is there a rule for predicting whether the sum is going to be negative or positive when you add a negative and a positive number?” Several students had ideas about this, including Nathan who said, “Let  $N$  be a negative number and  $P$  be a positive number. If  $N$  is bigger than  $P$ , then  $N+P$  equals something negative. If  $P$  is bigger than  $N$ , then  $N+P$  equals something positive.”

Nathan’s statement was recorded as: “Let  $N$  be a negative number and  $P$  be a positive number. If  $N > P$ , then  $N+P=N$ ; if  $P > N$ , then  $N+P=P$ .”

Both the written and spoken statements were problematic, but the discussion was not over. Rather, in

this classroom, such statements were treated as offerings to critique and edit. Nathan, himself, explained his statement could not be true because a negative number could never be bigger than a positive number.

Again, several students offered ideas until one student, Melinda, reminded the class of the idea of absolute value, which they had studied months earlier. By the end of the lesson, the students’ statement became, “The answer will have the same sign as whichever number has the larger absolute value.”

The next day, using the concept of absolute value, they worked on how to write the idea using symbols correctly.

These students were engaged in articulating a generalization about adding integers, both orally and with symbols. They worked together to clarify the general rule and to communicate their ideas with precision.

After describing additional examples, the presenters concluded by considering the question, “What does it look like when students *don’t* have experience with these aspects of arithmetic before they study algebra?” They offered examples from middle and high school showing students’ inability to apply key ideas of generalized arithmetic. For example, when sixth graders see an equation like  $100 - 50 = 70 - 20$ , they can only justify that the equation is true by solving each expression separately; it simply doesn’t occur to them that there is a way to reason about the equality of the two expressions or to see their relationship as an instance of a generalization.

Engaging with these aspects of early algebra in the elementary grades has the potential to:

- 1) strengthen students’ work on computation;
- 2) preview and build the need for algebra; and
- 3) support students who are struggling and students who need more challenge.

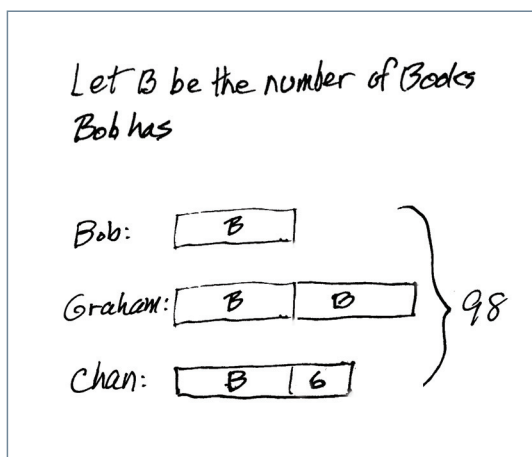
Sybilla Beckmann

## Easing Kids from Arithmetic to Algebra with Strip Diagrams

Sybilla Beckmann of the University of Georgia presented a session on a method that she has found especially helpful in easing kids from arithmetic to algebra. The method is widely used in Singapore, and Beckmann calls it “strip diagrams.”

Here’s an example of a problem that can effectively be solved using strip diagrams: Graham has twice as many books as Bob. Chan has six more books than Bob, and all together, Graham, Bob and Chan have 98 books. How many books does Bob have?

This information can be represented in a diagram like this:



Note that these diagrams don’t have to be drawn proportionally. Since you don’t know how many books Bob has, you don’t yet know how that number relates to 6, so you

### What professors need students to know about algebra

The standard error in statistics is given by  $\sigma/\sqrt{n}$ , where  $\sigma$  is the standard deviation and  $n$  is the sample size. Professors want students to recognize quickly that the standard error halves when  $n$  is quadrupled.

Many students don’t see this at all.

can’t draw them proportionally. As long as different things are drawn to be different lengths and the same things are drawn to be the same length, precision doesn’t matter. Beckmann has used this method quite a bit with prospective elementary teachers, and she’s found that they don’t usually bring that issue up.

Armed with this diagram, reasoning through it is fairly straightforward. The total number of books is 98, and the extra bits are 6, so if you take those off, you’re left with 92. The remaining four pieces are equal in size, so a single one will be one-fourth of that, or 23. Note that this didn’t require setting up any equations; one only had to draw the picture and reason about it.

But you can also use this as a transition to reasoning with equations. The standard way of doing it as an algebra problem is this:

$$\begin{aligned} B+2B+(B+6) &= 98 \\ 4B+6 &= 98 \\ 4B &= 92 \\ B &= 92/4 = 23 \end{aligned}$$

Each of these algebraic steps mimics the reasoning about the strip diagram above.

The critical issue with this kind of technique, though, is whether it can be applied to a broad enough set of problems that it becomes a genuine tool for thinking, rather than a special-purpose trick. Beckmann argues that strip diagrams do pass this hurdle, that they are an “extensible tool,” because they can be genuinely useful in helping students avoid common mistakes in arithmetic and then be carried through fairly complex problems in algebra. Essentially, they can be applied to any system of linear equations or any situation with additive and/or multiplicative relationships that are quantified between quantities.

Here’s an example from the 1999 TIMSS test for 8th grade. In Singapore (where strip diagrams are taught), 72% of students got this problem right; internationally, the average was 33%; and in the US, it was only 27%.

A club has 86 members, and there are 14 more girls than boys. How many boys and how many girls are members of the club? Show your work.

This can be represented in a strip diagram this way:

*Problems when quantities are related additively*

*1999 Times 8th grade to 1:*

*A club has 86 members, and there are 14 more girls than boys.*

*How many boys and how many girls are members of the club?*

*Show your work.*

$$\begin{array}{r} 86 \\ -14 \\ \hline 72 \\ \\ 2 \overline{) 72} \\ \underline{36} \text{ boys} \end{array}$$

Armed with this diagram, the reasoning becomes pretty straightforward.

They're also useful for much simpler problems. For example:

After Amanda got 14 more buttons, she had 52 buttons in all. How many buttons did Amanda have before she got more?

This is a second-grade word problem that's often tricky for kids, because "got more" makes them think they should add. Strip diagrams help with this, making it clear that you should subtract 14 from 52 rather than add.

A useful feature of strip diagrams for young children is that they're very simple. Often, if you just ask them to draw a problem, they'll draw every single button, or worse: If it's chickens, they'll put in every wattle and feather, and get distracted from the key features of the problem. Strip diagrams help to rein this in.

They are also helpful with multiplicative comparisons, which students find really confusing. Consider the phrase, "There are three times as many students as professors." Students will often write this down in the order the words come in and write  $3s=p$ . A strip diagram helps them keep this straight. Or consider this problem:

There are 10% more sea stars than crabs at the aquarium. If there are 84 sea stars and crabs all together, how many sea stars and crabs are there?

Students find this very difficult without strip diagrams.

Strip diagrams help with ratios as well. Consider this:

Blue paint and yellow paint are mixed in the ratio 3 to 5 to make green paint. How much blue paint do you need to make 96 gallons?

This language, the ratio 3 to 5, they have trouble with, but strip diagrams make it more concrete.

Strip diagrams are wonderfully suited to fraction problems like this one: Joey spent  $\frac{2}{3}$  of his money on a computer game that cost \$34. How much money did Joey have before he bought the game?

*How much money did Joe have before he bought the game?*

*2 parts  $\rightarrow$  \$34*

*1 part  $\rightarrow$   $\$34 \div 2 = \$17$*

*3 parts  $\rightarrow$   $3 \times 17 = \$51$*

$$\begin{array}{l} \frac{2}{3}x = 34 \\ x = 34 \cdot \frac{3}{2} \\ = 39 + \frac{3}{2} \end{array}$$

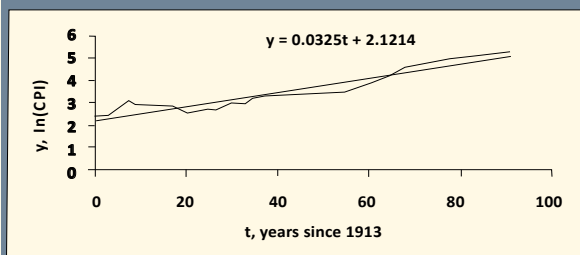
Two of these parts costs \$34, so one of them cost half of that, or \$17. All three, then, are  $3 \times 17 = 51$ , so \$51.

You can solve quite complex problems with these strip diagrams, which is why they were originally developed.

## What professors need students to know about algebra

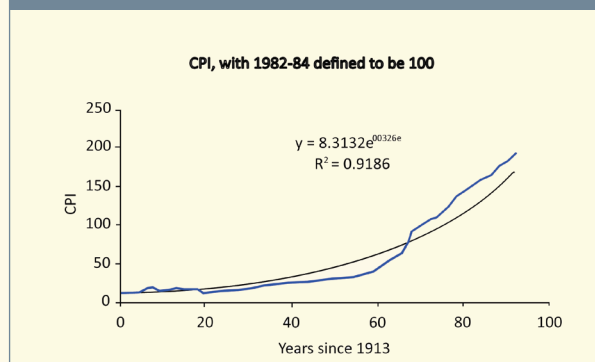
### An Economist's Use of Algebra

How fast has the CPI grown over last century?



An economics professor put up the graph above and commented that it showed that CPI has grown at an average rate of 3.25% a year. Students—including ones with strong quantitative backgrounds—were mystified by this because they weren't used to seeing graphs with a log function on one axis. Most would have understood had they been provided with the graph below.

### Consumer Price Index (CPI) Data (1913-2007)





## **The algebra needed for college**

What algebraic understandings are essential for success in beginning collegiate mathematics? What kinds of problems should high school graduates be able to solve? What kinds of technical fluency will they find useful in college or in other post-secondary work? What algebraic habits of mind should students develop in high school? What are the implications of current and emerging technologies on these questions?

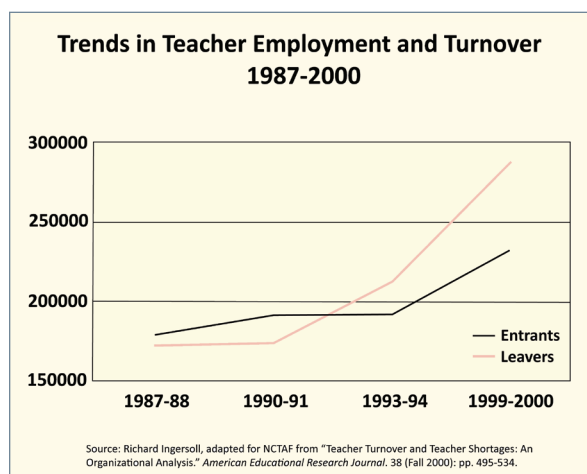


Daniel Chazan

## A Master's Degree Program to Improve the Teaching of Middle School Mathematics

The Montgomery County Public Schools came to the math department at the University of Maryland and said that they had a problem: They were concerned about their middle school math teachers. They wanted to have 80% of their students take and pass algebra by the end of 8th grade by 2010. Lots of students are now taking algebra in 6th and 7th grades, so they needed highly qualified teachers to teach those classes. And they were worried that their teacher corps wasn't up to the task.

This was particularly important because for some time now, more people have been leaving teaching than entering. When the baby boomers retire, that may get worse, and that may get quite dire within the next 5 to 10 years.



The trend is for more students to take more years of math in high school. This is good news overall, but it increases the need for good math teachers. Furthermore, students are being *required* to take more math courses, which changes the student body in more advanced classes and increases the demands on the teachers. And currently, there are concerns about the algebraic skills of future teachers who are *graduating* with major majors. Even at the end of college, their skills may be no stronger than when they were freshmen.

In most states there is no special middle school certification, so middle school teachers either have a high school or elementary school certification. With the recently increased demands on middle school math teachers, those with an

elementary certification may have an inadequate math background.

So the University of Maryland created a master's degree program for current middle-school math teachers with an elementary certification. Those teachers themselves felt that their mathematical knowledge was not sufficient to the task in front of them — though they were already teaching and quite committed. The hope is that investing in these teachers will keep them teaching for many years, justifying the investment.

These are quotes from teachers explaining why they decided to do the program:

"I think it was the kids, cause the kids were so... Like, they'd ask me questions, and I didn't know, but I'd want to find out for them. And they were willing to work and find out and come tell me. They'd come and say, 'Look at what I've found, Ms. \_\_\_\_.' I would look at it, and it was a little nine-year-old explaining stuff to me..."

"I remember thinking, 'You know what? I always wanted to go back and learn more math.' I wanted to take calculus, I really want to know how all that stuff works. And I thought, well, this might be a good place to start. And then I happened to interview for this other job, and now I'm a middle school math teacher."

"I'm really convinced at this point that if you don't have a deep understanding of the mathematics yourself, you're not gonna teach it very well. I thought before, 'Well, I have the textbooks. I can just follow that and the kids will be fine,' but it's really way more complicated than that. And that's why I joined the Masters program in the part, because I wanted to learn more math — just content, but now I can see where if you don't understand it all you can do is stand and deliver."

The school district approached the university in the fall of 2003, and the first cohort of fourteen teachers graduated in 2008. This is a drop in the bucket, because each large

district around us could easily provide another two cohorts tomorrow. The university doesn't have enough people to staff that, however.

The M. Ed. program takes three years and requires 30 credits. That consists of three math education courses, three math courses, three integrated math and math education courses and one educational inquiry course. The first calendar year is the algebra cycle, the second is data analysis, and the third is geometry. The teachers carry out an action research project in their classroom as well.

When the university sought state funding for the program, the state wanted to see more connection with the field, in schools. So the university created opportunities for the teachers to visit one another, to be videotaped, and to be observed by district or university personnel.

There are also "strands" that were added over time. The first was a short experience over five semesters to get the teachers conducting their own inquiry and doing their own mathematical research. Later, they added other strands on culturally relevant pedagogy, English language learners, and special education. They've targeted the program to work on schools that are in improvement.

Some teachers are from the Philippines, West Africa and India, which has made it challenging to get them accepted into the graduate school when their transcripts are from other countries.

Jim Fey and a number of graduate students initiated one of the combined pedagogy and content courses, a course focusing on the algebra curriculum.

The course objectives are to:

- gain enhanced understanding of the math of school algebra
- gain insight into the critical learning challenges that algebra students face
- gain understanding of various pedagogical models for teaching school algebra

- develop skill in applying knowledge about mathematics teaching and learning to lesson planning and classroom practice.

They organized the course around perplexing algebra questions that they had asked themselves as teachers. They used these problems to talk about students' conceptions and misconceptions and to look at different models and resources that are available.

The course that follows this course in the sequence is a typical math course that focused on algebra. The content is standard, but they try to present the material with a broader perspective and introduce a historical point of view.

The algebra inquiry strand task investigates the set of points that are equidistant from a line and a point not on that line. They first ask the teachers what the definition means, and the teachers then work on finding the set of points. They raise questions or conjectures, investigate them, and raise new ones based on their work. Many students wanted to prove that the set of points was a parabola. Some tried symbolic manipulations, but many tried using specific points. They discussed what constituted a proof, what different approaches could and could not offer, and whether proof established Truth.

Chazan offered some observations about the design of the program:

- Having the curriculum-related courses keeps the program job-related for the teachers.
- The mathematics is seen as challenging, but teachers feel a sense of accomplishment on completion.
- The inquiry experience adds an important dimension to how they think about themselves with respect to mathematics.
- There is more mathematics instruction in this program than in initial elementary certification and significantly more work on instruction, with people who are experienced teachers.

Hung-Hsi Wu

## The Precision and Rigor that is Essential for Teaching Algebra

Wu described this experience teaching professional development classes for teachers.

“Look!” he said, “There’s this wonderful theorem; if  $f(x)$  is a quadratic and  $r_1$  and  $r_2$  are the roots of  $f(x)=0$ , then for some constant  $c$ ,  $f(x)=c(x-r_1)(x-r_2)$  for all numbers  $x$ !”

Teachers: Silence.

Wu then went overboard explaining why it’s wonderful.

Teacher: What is there to prove?

This has happened to him three times.

The problem, he realized, is that the teachers confuse this with its converse. After all, the converse, that  $r_1$  and  $r_2$  are the roots of  $f(x)=0$  if  $f(x)=c(x-r_1)(x-r_2)$  for all numbers  $x$ , is trivial.

His conclusion: Because school algebra courses are not taught with the requisite precision and rigor, and because universities do not focus on eradicating the common misconceptions of preservice teachers, anecdotes of this type should be no surprise.

His belief, based on quite extensive experience of this nature, is that to produce teachers with the requisite content knowledge for teaching algebra, we must concentrate on teaching them the fundamentals of mathematics, that is, the proper use of symbols, precise definitions, precise reasoning, and coherent development of ideas. There is a particular urgency that teachers acquire this knowledge because they have to help students overcome common misconceptions in algebra.

The most basic task of learning algebra, in some sense, is learning how to use symbols fluently and correctly. This is a routine task if one goes about it the right way, but books do not always go about it the right way. The resulting confusion is immense.

For example, a basic etiquette in the use of symbols is to always say precisely what a symbol stands for. Consider: “ $st=ts$  for all real numbers  $s$  and  $t$ .” In this case, the symbols  $s$  and  $t$  stand for elements in an infinite set. Whenever a symbol stands for elements in a collection of more than one element, we informally refer to it as a variable. So  $s$  and  $t$  above are variables. The term

“variable” isn’t important, but it’s important for teachers to know that they are using a symbol for that purpose.

Sometimes symbols come in a slightly different form. Consider, “If  $a$ ,  $b$ , and  $c$  are fixed numbers, which number  $x$  would satisfy  $ax^2+bx+c=0$ ?” In this case, they stand for a fixed value throughout the discussion. We refer to these informally as constants. But again, the term is not important. What’s important is that you know whether the symbol you are using stands for a fixed number or an infinite collection of numbers.

The mathematicians in the audience may be astounded, Wu said, to learn that in school mathematics, “variable” has achieved the status of a mathematical concept crucial to the study of algebra. Here is a passage from a textbook that exemplifies this:

*A variable is a quantity that changes or varies. You record your data for the variables in a table. Another way to display your data is in a coordinate graph. A coordinate graph is a way to show the relationship between two variables.*

Sometimes the relationship between two variables can be described with a simple rule. Such rules are very helpful in making predictions for values that are not included in a table or graph of a set of data.

Another example:

*Variable is a letter or other symbol that can be replaced by any number (or other object) from some set. A sentence in algebra is a grammatically correct set of numbers, variables, or operations that contains a verb. Any sentence using the verb=(is equal to) is called an equation.*

*A sentence with a variable is called an open sentence. The sentence  $m=s/5$  is an open sentence with two variables. It is called “open” because its truth cannot be determined until the variables are replaced by values. A solution to an open sentence is a replacement for the variable that makes the statement true.*

In math, we strive for simplicity whether in teaching or in professional communications. These two examples (among

many) subvert this simplicity by elevating an informal piece of terminology to a fundamental concept and then formally defining it in abstruse language. This makes it very difficult for students to learn. We have to teach teachers how to circumvent these difficulties in the school classroom.

Moreover, the second example encourages the improper use of symbols by formalizing the concept of an “open sentence.” This appears to be quite common. In the education literature, one finds similar examples of asking students for interpretations of such “open expressions,” e.g.,  $\sqrt{6x-5}$ .

Again, we have to make sure that teachers know well enough not to engage in such counterproductive practices.

Definitions are generally conspicuous by their absence in school mathematics, but because algebra is the gateway course to higher math, this absence is no longer excusable. Absence here means it is never used in reasoning though it may be given. For example, the *graph of an equation*. However, if a definition isn’t used, then it would serve no purpose. For example, there is almost never any proof that the solution of a  $2 \times 2$  linear system is the point of intersection of the lines. Students learn by rote that such is the case.

What we must make our teachers and everyone else understand is that without precise definitions, there would be no mathematics, and that the role of definitions is to furnish a key piece of the foundation for the proofs of theorems. Students come to universities with little respect for definitions. The role of a definition in mathematics is generally not understood in the math education community. Both situations urgently need correction.

There are situations where school texts make it impossible to know whether something is a definition or a theorem, e.g.,  $(a^{1/n})^n = 1$  or  $a^{-n} = 1/a^n$ .

Consequently, many of our teachers cannot distinguish between a theorem and a definition.

Other basic definitions that are usually missing or ill-defined: the graph of a linear inequality, slope of a line, half-plane, the equivalence of expressions, polynomial form, exponential functions  $a^x$ , and constant rate.

Constant rate is especially fundamental.

### Problems to help create mindful manipulators

A person’s monthly income is \$I, her monthly rent is \$R, and her monthly food expense is \$F. In 1-4, say whether the expressions have the same value. If not, say which is larger, or that there is not enough information to decide. Briefly explain your reasoning in terms of income and expenses in each case.

1.  $I - R - F$  and  $I - (R + F)$
2.  $12(R + F)$  and  $12R + 12F$
3.  $I - R - F + 100$  and  $I - R - (F + 100)$
4.  $(R + F)/I$  and  $(I - R - F)/I$

The problem below is designed to illustrate that different forms of expressions can be useful in different ways.

The expression  $\frac{.6(t_1 + t_2 + t_3)}{3}$  is the contribution to a student’s final score from three test scores.

What is a different way of writing this?  
Which way should a student use in order to:

- calculate the total test contribution to their final grade
- calculate the effect of getting 10 more points on test 2

Here are some possible responses:

- i.  $\frac{.6(t_1 + t_2 + t_3)}{3}$
- ii.  $.2t_1 + .2t_2 + .2t_3$
- iii.  $t_1/5 + t_2/5 + t_3/5$

The first form is useful for thinking about the scores as the average of all three, with test scores contributing 60% of the grade, so it’s most useful for calculating the total test contribution to their final grade. The second or third forms are better for calculating the effect of getting ten more points on the second test.

## What professors need students to know about algebra

Consider the problem:

$$\text{Solve for } x: (x+5)(x-2)=8$$

When students see problems like this, the right hand side is usually 0, not 8. Many students don't understand that the 0 is crucial, and they solve a problem like this as though it equaled 0. For them, this kind of problem calls for a procedure that you just do, without further thought.

## A problem for preservice teachers

You are simplifying  $7-2(3-8x)$ . Which of the expressions is a correct next step?

	Correct	Not Correct
$5(3-8x)$		
$7-2(-5x)$		
$7-6-16x$		
$7-6+16x$		

For each expression, explain why you made the choice you did.

Here are some sample student answers:

- $7-2(-5x)$  isn't correct.  
Always do the parentheses first!
- $7-6+6x$  isn't correct.  
Where did the 16 come from?
- No answer.  
Can't use the distributive law because of the  $-$ .
- $7-6+16x$  is correct.  
You have to double multiply first.

What mathematical skills do we need to give teachers in our undergraduate courses in order to deal with these answers?

Another essential ingredient that's missing in the school curriculum is precise reasoning. Here the "precision" has to be appropriate to the grade level. Here are some examples of reasoning that's typically missing: why the graph of a linear equation is a line; why every line is the graph of a linear equation; why the graph of a linear inequality is a half-plane; why a quadratic function has a maximum or a minimum and why the maximum or minimum of a quadratic function is what it is; why study the exponential and log function; what underlies so-called proportional reasoning, etc.

Finally, why do we want teachers to have a coherent conception for the development of mathematical ideas? Here's an anecdote:

Q: If a student comes to you and asks why, if  $a \neq 1$  and  $a > 0$ ,  $a^t = a^s$  implies  $t = s$ , what would you tell her?

A:  $\log_a a^t = \log_a a^s$ , so  $t = s$ .

While the answer is 100% correct, mathematically, it is 90% certain that it is all wrong pedagogically. To answer this correctly, the teacher would need instant recall of the whole development that leads up to the definition of  $\log_a x$  and make an educated guess as to where the student's difficulty may lie, and then address that difficulty first. This is impossible without an understanding of that development in the first place. For the case at hand, it is unlikely that the student would understand the explanation using log because the definition of rational exponents precedes the introduction of the logarithm by quite a bit.

We want teachers to know, for instance:

1. The quadratic formula is not just "some formula," but the high point of a process that yields every desirable conclusion about quadratic functions or equations.
2. The subject of rational expressions is to polynomials as fractions are to whole numbers, and knowledge of fractions is a prerequisite for studying rational expressions.
3. The study of linear equations and straight lines depends on congruence and similarity.
4. The factor theorem (that  $f(r)=0$  implies  $(x-r)$  is a factor of  $f(x)$ ) is intimately related to the long division of whole numbers.
5. The precise definition of constant rate simplifies all discussions of rate problems in school mathematics.

This kind of knowledge facilitates teaching.

Al Cuoco

## The Uses of Higher Algebra for Teachers of School Algebra

Higher algebra comes up frequently in the high school curriculum, so it would be a very useful course for teachers to take in college.

Here are some examples of questions in high school algebra in which higher algebra comes up:

- Is  $\frac{x^2-3x}{x^2-9}$  the same as  $\frac{x}{x+3}$ ?
- When is  $x^n-1$  a factor of  $x^m-1$ ?
- Can a quadratic equation have more than two roots?
- At how many values do two polynomials have to agree before we can say they are equal?
- What does it even mean for two polynomials to be equal? As formal expressions, or as functions?
- Can a system of linear equations with integer coefficients have an irrational solution?
- Can a system of three linear equations in four unknowns have exactly three solutions?
- If a polynomial doesn't factor over the integers, can it factor over the rationals?

Here are examples from other courses:

- Why is arithmetic with complex numbers like arithmetic with polynomials?
- What does it mean for two functions to be equal?
- How can I find a polynomial that agrees with a table?
- Why do they use  $f^{-1}$  for inverse function?
- For polynomials, why does the  $h$  in the denominator of  $\frac{f(x+h)-f(x)}{h}$  always cancel out?
- Why can't you trisect a 60-degree angle with a straightedge and compass?

### Examples:

1. Here's a story from Deborah Schifter's book *Dialogue About Teaching*: The class is using calculators and estimation to get decimal approximations to  $\sqrt{5}$ . One student looks at how you do out long multiplication and realizes that none of these decimals would ever work, because if you square a finite

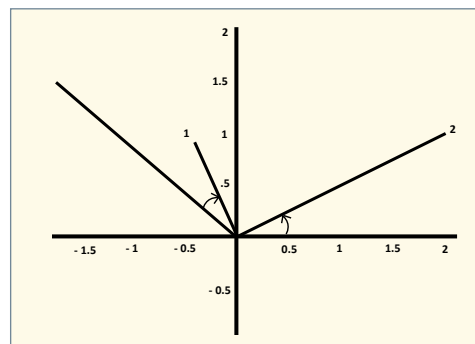
(non-integer) decimal, there'll be a digit to the right of the decimal point. So you can't ever get an integer. She deduces that  $\sqrt{5}$  can't be rational. What she said isn't correct, but there's something in there that we could learn from.

2. An example from a teacher, reported in a Reader Reflection by Walt Levisse in *The Mathematics Teacher*, March, 1997:

Nine-year-old David, experimenting with numbers, conjectures that, if the period for the decimal expansion of  $1/n$  is  $n-1$ , then  $n$  is prime.

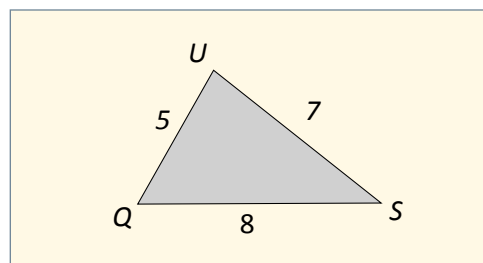
3. How can you help your students understand the "multiplication rule" for complex numbers? The usual way to do it is using the addition formulas for sine and cosine:

$$|zw|=|z||w| \text{ and } \text{Arg}(zw)=\text{Arg}(z)+\text{Arg}(w)$$



But what if the students don't know any trig? Some teachers worked out a way to do this using only high school algebra.

4. Algebra comes up when you try to generate "nice" problems, for example:



How big is angle Q? In this case, the cosine of angle Q turns out to be  $1/2$ .





Glenn Stevens

Glenn Stevens is the director of PRO-MYS (Program in Mathematics for Young Scientists), a six-week summer program at Boston University for motivated high school students. He described one unusual aspect of their approach:

“Definitions and axioms tend to come at the end, rather than at the beginning. We have a lot of rich mathematical experiences, and then it’s up to the kids and teachers to assemble and understand what those experiences mean. We give them a lot of help in doing that, but in the end, they must do it for themselves.

“We don’t think of axioms as foundational things. At the beginning of life, anything you experience and know to be true based on that experience is a perfectly legitimate axiom. But our goal is to *understand* the experiences we’re having. To do that, we look into that bag of statements that we know are true and look for relationships among them: This one is implied by that one; this one gets used all the time. The former get pulled out and called theorems, and the latter get flagged as especially important. So the axioms are distilled from an infinite list into a few simple statements that serve as central nodes of a compactly organized network. This distillation process helps us more fully understand the relationships between the different parts of our experience.

“Sometimes proofs are used for justification, and sometimes they’re used for exploration. This is a dichotomy that we stress very much during the program. After we’ve got our reduced inventory of seven or eight axioms for the integers, we go on to prove things like, there are no integers between 0 and 1. That’s silly—we already know there are no integers between 0 and 1—so why prove it? We’re testing our axiom system. We want to know that our axiom system is powerful enough to contain everything we know to be true.”

### Suggestions for teaching abstract algebra to teachers:

Instill a sense that algebraic objects are open to experiment. A computer algebra system allows you to do numerical experiments.

- Start with rings, field and polynomials — not groups
- Solve equations in these systems
- Develop in detail the structural similarities and differences between integers and the rational polynomials
- Construct the complex numbers as a quotient of real polynomials
- Construct more general splitting fields in the same way
- Tie Galois theory to:
  - The theory of equations
  - The roots of unity
  - The geometry for the regular  $n$ -gon
  - Cardano’s formula
  - The theory of straightedge and compass
- Stress some basics, like:
  - Extension by linearity
  - Change of basis
- Carefully develop vectoral methods in geometry, including:
  - The “extension program” from pairs of real numbers to  $n$ -tuple of real numbers.
  - The use of matrices to represent linear transformation
- Connect various uses of determinants:
  - As area and volume
  - As a test for linear dependence
  - As an algebraic tool (resultants, Cramer, matrix inverses...)
- Exploit matrix algebra as a formal “bookkeeping” tool
  - In adjacency and scheduling problems
  - As tools for solving recurrence equations
- Make connections among eigenvalues, geometry, and algebra

### *Some suggestions for number theory:*

Let general results evolve from numerical experiments

- Compare arithmetic in the integers, the integers mod  $n$ , the integers with  $i$ , and rational polynomials

- Examine Euclid’s algorithm in the integers, the integers with  $i$  appended, and rational polynomials
- Make localization and reduction general purpose tools
- Connect Pythagorean and Eisenstein triples to:
  - Norms from quadratic fields
  - Rational points on conics
- Develop the theory of repeating decimals
- Connect the Chinese remainder theorem to Lagrange interpolation
- Talk to Glenn Stevens, who knows more than anyone about how to get people engaged in elementary number theory

*Some suggestions for other things:*

Encourage reasoning about calculations and operations

- The theory of finite differences is useful in teaching
- Polynomial calculus is all algebra
  - The remainder when  $f(x)$  is divided by  $(x-a)^2$
  - The Taylor expansion about  $x=a$  is a polynomial identity
- The complex numbers can be used to derive the addition formulas for sine and cosine (rather than the other way around)
- Summatory polynomials are useful both in calculus and polynomials interpolation
- Function algebra gives an example of algebraic structure
- Statistics has deep connections to linear algebra
- Chebyshev polynomials bring coherence to trig addition formulas

In conclusion, algebra and algebraic reasoning around topics in the undergraduate mathematics curriculum can help prospective teachers enter the profession with a coherent view of secondary mathematics.

But this doesn’t come for free. Explicit connections to the daily work of high school teaching should be a part of every undergraduate course.

This doesn’t mean developing courses in high school mathematics from an “advanced” perspective. It means developing courses that develop the content and methods of undergraduate mathematics while taking seriously the profession-specific needs of high school teachers.

Cuoco has written an abstract book in collaboration with Joe Rotman that implements many of these suggestions, which can be found here: <http://www.maa.org/ebooks/textbooks/LMA.html>.



**Deborah  
Hughes  
Hallett**

Deborah Hughes Hallett has helped students at Harvard deal with the insufficiencies in their algebra background in order to succeed in college classes.

“Algebra turns out to be crucial for all sorts of parts of the college curriculum,” she observed. “Lack of knowledge of algebra keeps students from following the path they’d like to follow professionally. This can have a destructive effect on students’ ability to go into fields that are somewhat quantitative, but much less so than physics or math, like medicine, nursing, health management, business or economics.

“I work with mid-career students at the school of government who are mostly quite allergic to math. Now they want a master’s degree and they need to do some economics, so they need math. Even people who have advanced degrees in economics are going to struggle if their basic algebra isn’t OK. These are people with otherwise strong backgrounds, from strong schools, with good jobs.

“People’s big block isn’t their inability to do manipulations (though they usually can’t), but that the symbols don’t mean anything to them. They just look scary and blurry and weird, so they glaze over them. This is important because these people are cut off from the insights that algebra can provide.

“Going to college without being ‘symbolically literate’ is like going to college illiterate: Whole arenas of information are blocked off.”

Deborah Ball

## Teaching Algebra, Not Just Learning It

Although this was billed as a workshop on the teaching and learning of algebra, Deborah Ball argued that it ended up being more about the *learning* of algebra rather than the *teaching* of algebra.

It is not easy to focus attention on *teaching*. Several different reasons help to explain why this is:

- As English speakers, we have a limited and misleading language for talking about teaching. By contrast, Japanese has developed a detailed professional language for teaching. We lack words for the different segments of a classroom discussion or different types of problems used in instruction. For example, one could have terms to describe problems that launch the study of a topic versus the problems designed to provide practice. We don't even have a word for the fundamental transaction of teaching and learning. What goes on in classrooms is not just teaching, and not just learning. Ball tends to use the word "instruction" to approximate a word that comprises both, but that's not an ideal solution, she says, because we use "instruction" in other ways too. The result of this lack of language is that we lack the ability to distinguish things for which we don't have terms.

### What professors need students to know about algebra

Consider the expression:

$$n(n+1)(2n+1)/6$$

College professors would like students to see at a glance that it's cubic, and perhaps that the leading coefficient is  $1/3$ . Many students have to expand completely—a long procedure—to see that it's cubic. This is a serious disadvantage, because it makes life painful for them and presents plenty of opportunity to make mistakes.

- Some of the work is inside the teacher's head, so it is literally invisible.
- We lack frameworks for seeing teaching. We don't even see some things that are right in front of us as a result.
- We don't know the most useful scale to look. Some people like to talk about "teacher-directed" versus "student-centered" classrooms. But this is too gross a distinction to be useful. However, it is also not really useful to count how many times someone praises a student. The grain size in that case is too small. Teaching happens at different scales at different times.
- Disciplinary traditions and perspectives often focus closely on only part of the story. Sociologists and psychologists, for instance, bring their lenses and miss all kinds of things. Our interest in mathematics has similar limitations.
- Teaching isn't valued or understood.

Teaching is a thoughtful human construction designed to improve learning. Learning happens without teaching all the time, but we're interested in the improvement of students' learning. So teaching practice is both attentive and deliberative. Ball sometimes provocatively likes to say that lectures are the most extreme form of constructivism that she can think of, because, as the instructor talks, they leave it entirely to the students to construct the learning.

So what does it take to teach mathematics well? To begin, we need to understand that teaching is a form of mathematical work. For example, teachers:

- Use and analyze representations and examine equivalences among representations.
- Define terms and attend closely to language.
- Use and invent notation.
- Produce and analyze explanations.

- Generate simpler and more complex versions of a problem.
- Ask mathematical questions. For example: Why does this work? Does this work in all cases? Do we have all the solutions? How are these two representations related?
- Think of special cases to challenge a student's claim, or develop boundary cases.

These tasks are specific to the teaching of mathematics; they aren't generic to teaching as a whole. A history teacher, for example, doesn't have to do these things.

At the same, teaching is a very different form of mathematical work from doing mathematics, because teaching involves doing some things that are unnatural for the mathematically inclined. As much as teaching depends on mathematical instincts, habits of mind and practices, it also requires teachers to do things such as unpacking rather than packing mathematical ideas; listening to mathematically imprecise statements; refraining from automatically affirming correct statements; hearing what others say, not what you think they must mean; and sometimes even provoking errors.

Mathematicians tend to be oriented toward expressing mathematical ideas in as compressed a way as possible, but mathematics teachers need to manage the journey toward compression. They have to help students reach the end goal of competence with a compressed expression of mathematics, but at the same time they have to avoid "compression impatience." The desire for compression can make you rush and wish that something that will happen 12 years from now were happening now. You can't, for example, have the real number line in first grade. Teachers also must recognize opportune moments for compression, times when students will embrace it.

All of this is just a small part of how teaching is intricate work. As a community, we need to understand it better.

## What professors need students to know about algebra

**Consider the problem:** *When is  $\text{Lo}\sqrt{1-(v/c)^2}$  zero?*

College professors want students to see quickly that this will happen when  $v/c=1$ . This expression comes up in relativity, and the implication of this fact is that lengths shrink to zero at the speed of light. But many students don't see anything when they look at that expression.

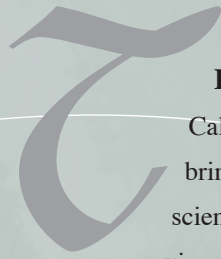
**Consider the problem:** *Simplify as much as possible*

$(5n-10)/(4-n^2)$ , assuming  $n \neq \pm 2$

Often students imagine that this equals 0 and so they solve for  $n$ . Another difficulty is that when they factor, they end up with an  $n-2$  at the top and a  $2-n$  at the bottom, and they don't recognize that these two expressions can cancel each other.

**Consider the expression:**  $\sigma/\sqrt{n}$ .

This is the formula for the standard error in statistics. Professors want students to recognize quickly that the standard error halves when  $n$  is quadrupled. Many students don't see this at all.



**THE MATHEMATICAL SCIENCES RESEARCH INSTITUTE (MSRI)**, located in Berkeley, California, fosters mathematical research by bringing together the foremost mathematical scientists from around the world in an environment that promotes creative and effective collaboration. MSRI's research extends through pure mathematics into computer science, statistics, and applications to other disciplines, including engineering, physics, biology, chemistry, medicine, and finance. Primarily supported by the U.S. National Science Foundation, the Institute is an independent nonprofit corporation that enjoys academic affiliation with nearly 100 leading universities as well as support from individuals, corporations, foundations, and other government and private organizations.

MSRI's major programs, its postdoctoral training program, and its workshops draw together the strongest mathematical scientists, with approximately 2,000 visitors over the course of a year. At any time, about eighty-five mathematicians are in residence for extended stays. Public outreach programs and VMath, the largest mathematical streaming video archive in the world, ensure that many others interact with MSRI throughout the year.

*For more information visit [www.msri.org](http://www.msri.org)*



Main Office 510-642-0143 • Fax 510-642-8609  
Mailing Address: 17 Gauss Way • Berkeley, CA 94720-5070  
[www.msri.org](http://www.msri.org)



