TRIVIALITY OF SYMPLECTIC $SU(2)$-ACTIONS ON HOMOLOGY

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Abstract. Lalonde and McDuff showed that the natural action of the rational homology of the group of Hamiltonian diffeomorphisms of a closed symplectic manifold (M,ω) on the rational homology groups $H_*(M,\mathbb{Q})$ is trivial. In this note, given a symplectic action of $SU(2)$, $\phi : SU(2)\times M \to M$, we will construct a symplectic fiber bundle $P_\phi \to \mathbb{C}P^2$ with fiber (M,ω) and use it to construct the chains, which bound the images of the homology cycles under the trace map given by the $SU(2)$-action. It turns out that the natural chains bounded by the $SU(2)$-orbits in M are punctured $\mathbb{C}P^2$'s, the counter parts of holomorphic discs bounding circles in case of Hamiltonian circle actions. We will also define some invariants of the action ϕ and do some explicit calculations.

1. Introduction

Let $\phi : G\times M \to M$ be a smooth action of a compact Lie group G on a smooth manifold M. The action induces a homomorphism on homology, called the trace homomorphism, $\partial_\phi : H_k(M,\mathbb{Q}) \to H_{k+d}(M,\mathbb{Q})$ defined as follows: If $\alpha \in H_k(M,\mathbb{Q})$ is a class represented by a cycle $a : A \to M$, then $\partial_\phi(\alpha)$ is the class in $H_{k+d}(M,\mathbb{Q})$ represented by the cycle $G\times A \to M$, $(g,x)\mapsto \phi(g,a(x))$, where d is the dimension of G. In general this homomorphism is not trivial (just consider product spaces $G\times M$). However, if M is a closed symplectic manifold and G is a compact Lie group acting on M in a Hamiltonian fashion, then McDuff and Lalonde showed that the homomorphism $\partial_\phi : H_k(M,\mathbb{Q}) \to H_{k+d}(M,\mathbb{Q})$ is trivial. Indeed, they have proved a much stronger result that the natural action of the homology of the group of Hamiltonian diffeomorphisms of the closed manifold (M,ω) on the homology groups $H_*(M,\mathbb{Q})$,

$$H_k(\mathrm{Ham}(M,\omega),\mathbb{Q})\times H_l(M,\mathbb{Q}) \to H_{k+l}(M,\mathbb{Q})$$

is trivial ([LM, LMP]).

Below is the main result of this note, which determines the chains bounded by the images of the trace homomorphism in the case of $G = SU(2)$.

Theorem 1.1. *Let $\phi : SU(2)\times M \to M$ be a symplectic action on a closed symplectic manifold (M,ω). Then there is a closed symplectic manifold (P_ϕ,ω_ϕ), which fibers over $\mathbb{C}P^2$ with fiber M such that,*

i) *the rational homology of the fiber M injects into the rational homology of P_ϕ,*
ii) *the symplectic form ω_ϕ restricts to ω at each fiber, and*

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iii) *if $\alpha \in H_k(M,\mathbb{Q})$ is a class represented by a cycle $a : A \to M$, then in the manifold P_ϕ, the cycle*

$$SU(2) \times A \to M \subseteq P_\phi$$

representing the class $\partial_\phi(\alpha)$, bounds a chain of the form ${\mathbb{C}P_0}^2 \times A \to P_\phi$, where ${\mathbb{C}P_0}^2 = \mathbb{C}P^2 - Int(D^4)$ is the punctured projective plane.

In particular, the induced homomorphism on homology $\partial_\phi : H_k(M,\mathbb{Q}) \to H_{k+3}(M,\mathbb{Q})$ is trivial.

Remark 1.2. Since $SU(2)$ is simply connected any symplectic $SU(2)$-action on a symplectic manifold is Hamiltonian ([MS]).

Example 1.3. Let $SU(2)$ act linearly on $\mathbb{C}P^2$ in the usual way (see the next section). Blowing up the isolated fixed point of the action we get an $SU(2)$-action on $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P}^2$. The action is Hamiltonian since it is algebraic. The orbit of a point with trivial stabilizer is a copy of $SU(2) = S^3$, which separates the two copies of the projective planes. So, the homology class represented by this orbit is trivial and it bounds a punctured $\mathbb{C}P^2$, not a 4-ball.

The next section is devoted to the proof of Theorem 1.1. In the third section, we will construct some invariants of the action $\phi : SU(2) \times M \to M$ and compute them in some cases. Finally, we will mention some applications of these results to the study of the topology of real algebraic varieties.

2. Proof of Theorem 1.1

Let (M,ω) be a closed $2n$-dimensional symplectic manifold and

$$\phi : SU(2) \times M \to M$$

a symplectic action. The proof of Theorem 1.1 consists of three parts. In the first part, we will construct a smooth symplectic fiber bundle $\pi_{S^4} : P^0_\phi \to S^4$ with fiber (M,ω) using the action $\phi : SU(2) \times M \to M$ as the clutching function. Moreover, the fibre bundle, both the total space and the base, will have an $SU(2)$ and an S^1-action both preserving a closed two form ω_{S^4} on P^0_ϕ, which restricts to ω on each fiber. Moreover, the projection map will be equivariant with respect to both actions.

In the second part, using a natural $SU(2)$ and S^1-equivariant degree one map $\mathbb{C}P^2 \to S^4$, where the $SU(2)$ and the S^1-action on the complex projective space are obtained from the natural actions of these groups on $\mathbb{C}^2$, we will pull back the bundle over the sphere to a bundle over $\mathbb{C}P^2$, which we will denote $\pi : P_\phi \to \mathbb{C}P^2$. Using the pull back of the two form ω_{S^4} on P^0_ϕ and the Fubuni-Study form on $\mathbb{C}P^2$ we will construct a symplectic form ω_ϕ on P_ϕ, which restricts to ω in each fiber. Moreover, the $SU(2)$ and the S^1-actions on P^0_ϕ will induce Hamiltonian actions on P_ϕ.

In the last part, we will consider a symplectic reduction of the total space of the bundle $\pi : P_\phi \to \mathbb{C}P^2$ using the S^1-action. Finally, Kirwan's Surjectivity Theorem on symplectic quotients ([K]) together with a topological observation will finish the proof.

2.1. **Symplectic M-bundles over S^4 with structure group $SU(2)$.** Given any smooth action $\phi : SU(2) \times M \to M$ we define the smooth manifold P^0_ϕ as the identification space

$$D^4_+ \times M \cup D^4_- \times M/(g,x) \sim (g, \phi(g,x)), \text{ for } (g,x) \in \partial D^4_+ \times M$$

where we identify $\partial D^4_\pm$ with $S^3 = SU(2)$. Note that we have a fiber bundle $P^0_\phi \to S^4$ with fiber M, induced by the projection maps $D^4_\pm \times M \to D^4_\pm$.

Another description for this bundle, which is more suitable to define the $SU(2)$-action on, is as follows: Let $\mathbb{H}$ denote the quaternion line and $U(\mathbb{H}^2)$ the set of unit length vectors in $\mathbb{H}^2$. Also identify $SU(2)$ with the set of unit quaternions $\mathbb{H}$. Let $L \to S^4$ denote the quaternion line bundle, whose unit disc bundle is the $SU(2)$-bundle $U(\mathbb{H}^2) \to S^4$, $(v_1, v_2) \mapsto [v_1 : v_2]$, for $(v_1, v_2) \in U(\mathbb{H}^2)$. Note that the latter map is nothing but the orbit map of the $SU(2)$-action on $U(\mathbb{H}^2)$ given by $(v_1, v_2) \mapsto (v_1 g, v_2 g)$, for $g \in SU(2)$ and $(v_1, v_2) \in U(\mathbb{H}^2)$.

Similarly, on $U(\mathbb{H}^2) \times M$ we have an $SU(2)$-action defined by

$$(g, (v_1, v_2), x) \mapsto ((v_1 g, v_2 g), \phi(g,x))$$

for $g \in SU(2)$, $x \in M$ and $(v_1, v_2) \in U(\mathbb{H}^2)$. The projection map $U(\mathbb{H}^2) \times M \to U(\mathbb{H}^2)$ is equivariant and taking quotients by the respective actions on $U(\mathbb{H}^2) \times M$ and $U(\mathbb{H}^2)$ we recover the fiber bundle $P^0_\phi \to S^4$.

There is a second $SU(2)$-action on $U(\mathbb{H}^2) \times M$, which commutes with the first one:

$$(h, (v_1, v_2), x) \mapsto ((h^{-1} v_1, v_2), x)$$

for $h \in SU(2)$, $x \in M$ and $(v_1, v_2) \in U(\mathbb{H}^2)$. Clearly, this induces an action on S^4 given as

$$(h, [v_1 : v_2]) \mapsto [h^{-1} v_1 : v_2]$$

which makes the projection map equivariant. Since the two actions commute the second action induces actions on both the total space and the base of the fiber bundle $P^0_\phi \to S^4$, which makes the projection map equivariant. Note that the action on S^4 is free outside the poles, namely $[0:1]$ and $[1:0]$, the only fixed points of the action.

There is also a left circle action on this space: Identify $\mathbb{H}$ with $\mathbb{C}^2$, on which $SU(2)$ acts by matrix multiplication. Also regard S^1 as the set of matrices $\{e^{i\theta} I_2 \mid e^{i\theta} \in S^1\}$, where I_2 is the 2×2 identity matrix. Now let S^1 act on $U(\mathbb{H}^2) \times M$ by

$$(e^{i\theta}, (v_1, v_2), x) \mapsto ((e^{-i\theta} I_2 v_1, v_2), x)$$

for $e^{i\theta} \in S^1$, $x \in M$ and $(v_1, v_2) \in U(\mathbb{H}^2)$. Since $e^{-i\theta}\, I_2$ commutes with all elements in $SU(2)$ we get an S^1-action on both the total space and the base of the fiber bundle $P^0_\phi \to S^4$, which makes the projection map equivariant. The circle action on S^4 commutes with the $SU(2)$-action described in the above paragraph and it is a free action outside the poles.

The Wang sequence for cohomology associated to the M-bundle $P^0_\phi \to S^4$ yields the isomorphism

$$0 = H^{-2}(M, \mathbb{Q}) \to H^2(P^0_\phi, \mathbb{Q}) \overset{rest.}{\to} H^2(M, \mathbb{Q}) \overset{\partial^*_\phi}{\to} H^{-1}(M, \mathbb{Q}) = 0$$

given by the restriction map, where the last map is the dual of the trace homomorphism $\partial_\phi : H_k(M, \mathbb{Q}) \to H_{k+3}(M, \mathbb{Q})$ ($k = -1$ in this case). In particular, there is unique cohomology class on P^0_ϕ, which restricts to the cohomology class $[\omega]$ on each fiber. Indeed, we will construct a two form representing this cohomology class, which we will use to get a symplectic form on the M-fiber bundle over $\mathbb{C}P^2$.

Lemma 2.1. *There is an $SU(2)$ and S^1-invariant closed two form ω_{S^4} on P^0_ϕ, which restricts to ω on each fiber and such that $[\omega_{S^4}]^{n+1} = 0$ in cohomology, where the $SU(2)$ and S^1-actions are the ones described in the above paragraphs. Moreover, the cohomology class of ω_{S^4} is uniquely determined by these conditions.*

Before we prove this lemma we need some preliminaries. Let $f : S^3 \times M \to S^3 \times M$ be the smooth map given by $f(g, x) = (g, \phi(g, x))$, $(g, x) \in S^3 \times M$. Consider the following diagram, where π_i, $i = 1, 2$, are the projections onto the second factors.

$$\begin{array}{ccc} M & & M \\ \pi_1 \uparrow & & \uparrow \pi_2 \\ S^3 \times M & \xrightarrow{f} & S^3 \times M \end{array}$$

Using the decomposition $T_*(S^3 \times M) = T_*S^3 \times T_*M$ we will write any tangent vector X on $S^3 \times M$ as $X = (X_S, X_M)$. Note that the differential of f has the form

$$f_* = \begin{pmatrix} Id & 0 \\ \frac{\partial \phi}{\partial g} & \frac{\partial \phi}{\partial x} \end{pmatrix}.$$

Hence $f_*((X_S, 0)) = (X_S, X^\sharp_S)$ and $f_*((0, X_M)) = (0, \phi_*(X_M))$, where $X^\sharp_S$ is the vector field on M generated by the vector X_S via the action.

The $SU(2) = S^3$-action on M is Hamiltonian means that there is a smooth map $\mu : M \to su(2)^*$ such that for any vector $X_S \in T_*S^3$ we have

$$i_{X^\sharp_S}\omega = d(\mu(X_S)).$$

Since $SU(2) = S^3$ is parallelizable choosing a global frame de_1, de_2, de_3 for the cotangent bundle for S^3 we can regard μ as a one form on $S^3 \times M$, namely

$$\mu(g, x) = A(x)\ de_1 + B(x)\ de_2 + C(x)\ de_3$$

for $(g, x) \in S^3 \times M$. One can easily check that $d(\mu(X_S)) = -i_{X_S} d\mu$. Now we can state the next lemma.

Lemma 2.2. *$\pi_1^*(\omega) - (\pi_2 \circ f)^*(\omega)$ is an exact two form on $S^3 \times M$, which vanishes on T_*M identically.*

Proof. Let $X = (X_S, X_M)$ and $Y = (Y_S, Y_M)$ be tangent vectors at any point of $S^3 \times M$. Then

$$\begin{aligned} I &= (\pi_1^*(\omega) - (\pi_2 \circ f)^*(\omega))\ ((X_S, X_M), (Y_S, Y_M)) \\ &= \omega(X_M, Y_M) - \omega(X^\sharp_S + \phi_*(X_M), Y^\sharp_S + \phi_*(Y_M)) \\ &= \omega(X_M, Y_M) - \omega(\phi_*(X_M), \phi_*(Y_M)) \\ &\qquad - \omega(X^\sharp_S, Y^\sharp_S + \phi_*(Y_M)) - \omega(\phi_*(X_M), Y^\sharp_S) \\ &= -\omega(X^\sharp_S, Y^\sharp_S + \phi_*(Y_M)) - \omega(\phi_*(X_M), Y^\sharp_S), \end{aligned}$$

because $\omega(\phi_*(X_M),\phi_*(Y_M))=\phi^*(\omega)(X_M,Y_M)=\omega(X_M,Y_M)$. Note that this calculation already shows that $\pi_1^*(\omega)-(\pi_2\circ f)^*(\omega)$ is identically zero on T_*M.

Now using $i_{X_S^\sharp}\omega=d(\mu(X_S))$ we can write
$I=-d(\mu(X_S))(Y_S^\sharp+\phi_*(Y_M))+d(\mu(Y_S))(\phi_*(X_M))$. Since

$$d(\mu(X_S))=-i_{X_S}d\mu$$

we get
$I=d\mu(X_S,Y_S^\sharp+\phi_*(Y_M))-d\mu(Y_S,\phi_*(X_M))$
$\qquad =d\mu(X_S,Y_S^\sharp)+d\mu(X_S,\phi_*(Y_M))+d\mu(\phi_*(X_M),Y_S)$.

On the other hand, similar calculations yield
$d\mu(f_*(X),f_*(Y))=d\mu(X_S,Y_S^\sharp)+d\mu(X_S^\sharp,Y_S)$
$\qquad +d\mu(X_S,\phi_*(Y_M))+d\mu(\phi_*(X_M),Y_S)$.

Comparing the two equations we deduce that
$I=d\mu(f_*(X),f_*(Y))-d\mu(X_S^\sharp,Y_S)$. For the last term we can write
$d\mu(X_S^\sharp,Y_S)=-d\mu(Y_S,X_S^\sharp)=-(i_{Y_S}d\mu)(X_S^\sharp)=d(\mu(Y_S))(X_S^\sharp)$
$\qquad =(i_{Y_S^\sharp}\omega)(X_S^\sharp)=\omega(Y_S^\sharp,X_S^\sharp)$. So, we have obtained

$$I=d\mu(f_*(X),f_*(Y))+\omega(X_S^\sharp,Y_S^\sharp).$$

Writing $\omega(X_S^\sharp,Y_S^\sharp)=I-(f^*(d\mu))(X,Y)$ we see that the map

$$(X,Y)\mapsto\omega(X_S^\sharp,Y_S^\sharp)$$

is a closed two form on $S^3\times M$. Moreover, it vanishes if X_S or Y_S is zero and hence it is identically zero on the T_*M component of the tangent space. So, the de Rham cohomology class represented by this closed two form evaluates zero on any two dimensional homology class of the product $S^3\times M$ provided that the homology class is represented by a cycle lying in some $\{pt\}\times M$. However, since S^3 has no first and second homology, by the Künneth formula, this de Rham class must be trivial. Hence, there is a one form u on $S^3\times M$ such that $I=(f^*(d\mu))(X,Y)+du(X,Y)$. This finishes the proof of the lemma. □

Proof of Lemma 2.1. By the isomorphism obtained from the Wang sequence, the cohomology class of ω_{S^4} is uniquely determined by that of ω (see the paragraph above Lemma 2.1).

We will regard the total space P^0_ϕ as the identification space

$$\mathbb{R}^4_+\times M\ \cup\ \mathbb{R}^4_-\times M/(t,g,x)\sim F(t,g,x)\doteq(t^{-1},g,\phi(g,x)),$$

for $(t,g,x)\in(\mathbb{R}^4_+-\{0\})\times M$, where we identify $\mathbb{R}^4-\{0\}$ with $(0,\infty)\times S^3$ in the obvious way. Let π_1 and π_2 denote the projections onto the M factors of the products $\mathbb{R}^4_+\times M$ and $\mathbb{R}^4_-\times M$, respectively. Also, we will denote the projection of $(0,\infty)\times S^3\times M$ onto the $S^3\times M$ component by π_{SM}.

Let $\omega_i=\pi_i^*(\omega),\ \ i=1,2$. Then by Lemma 2.2

$$\omega_1-F^*(\omega_2)=\pi_{SM}^*(dv_1)$$

for some one form v_1 on $S^3 \times M$. Let $v_2 = (f^{-1})^*(v_1)$, where f is as in Lemma 2.2. So, $\omega_1 = F^*(\omega_2 + dv_2)$ on $(\mathbb{R}^4_+ - \{0\}) \times M$. Let $\widetilde{\omega}_2 = \omega_2 + d(\rho(t)v_2)$, where ρ is a smooth function on $\mathbb{R}$, which vanishes on $(-\infty, 0.5]$ and equals one on $[0.75, \infty)$.

Now we have

$$F^*(\widetilde{\omega}_2) = F^*(\omega_2) + d(F^*(\rho(t)v_2)) = \omega_1 - dv_1 + d(\rho(1/t)v_1).$$

So, letting $\widetilde{\omega}_1 = \omega_1 + d((\rho(1/t) - 1)v_1)$ we obtain $F^*(\widetilde{\omega}_2) = \widetilde{\omega}_1$. By the choice of the function ρ the forms $\widetilde{\omega}_i$ are defined on all of $\mathbb{R}^4 \times M$, and indeed, they are equal to ω_i on $D_{1/2} \times M$, where $D_{1/2}$ denotes the disc of radius $1/2$ in $\mathbb{R}^4$. Moreover, since both dv_i and dt vanishes on T_*M, each $\widetilde{\omega}_i$ restricts to ω on each fiber $\{pt\} \times M$. Hence, together they define a global closed two form on P^0_ϕ, say ω_{S^4}, which restricts to ω in each fiber.

To see that $[\omega_{S^4}]^{n+1}$ is zero just open up $\widetilde{\omega}_i^{n+1}$ and note that $\omega_i^{n+1} = 0$.

The form ω_{S^4} may not be $SU(2)$-invariant. However, we can average it over the $SU(2)$-orbits to get an $SU(2)$-invariant form with the desired properties. Namely, let dH denote the Haar measure on $SU(2)$ with total volume one and define the average of ω_{S^4} as the form

$$(X, Y) \mapsto \int_{SU(2)} (\psi^*(h,p)\ (\omega_{S^4}))\ (X, Y)\ dH$$

where $\psi : SU(2) \times P^0_\phi \to P^0_\phi$ is the action map and the integration is over $h \in SU(2)$ for fixed vectors $X, Y \in T_p(P^0_\phi)$. Since on $U(\mathbb{H}^2) \times M$ the action is given by $(h, ((v_1, v_2), x)) \mapsto ((h^{-1}v_1, v_2), x)$ and the restriction of ω_{S^4} to each fiber, which is a copy of M, is just ω, so will be the restriction of the average of ω_{S^4}. Once, the form is $SU(2)$-invariant then we can average it to make also S^1-invariant in the same way. Since the two actions commute, averaging over the S^1-orbits will not spoil the $SU(2)-$invariance of the form. Also averaging commutes with exterior derivative and therefore the averaged two form will be still closed. $\square$

2.2. **Symplectic M-bundles over $\mathbb{C}P^2$ with structure group $SU(2)$.** By a fiber bundle $\pi : P_\phi \to \mathbb{C}P^2$, with fiber M and structure group $SU(2)$ we mean a group homomorphism $\phi : SU(2) \to \mathrm{Symp}(M, \omega)$, where the latter is the group of symplectomorphisms of the symplectic manifold (M, ω) and a principal $SU(2)$-bundle $P \to \mathbb{C}P^2$ such that P_ϕ is obtained from P via the representation ϕ in the usual way. The classifying space for $SU(2)$-bundles is $\mathbb{H}P^\infty$, whose 7th skeleton is $\mathbb{H}P^1 = S^4$ and therefore any principal $SU(2)$-bundle over a closed 4-manifold N is obtained form the universal bundle $L \to S^4$ by pulling it over N by a map $\xi : N \to S^4$. Since $e(L) = c_2(L) \in H^4(S^4, \mathbb{Z})$ is a generator we have $e(\xi^*(L)) = c_2(\xi^*(L)) = \deg(\xi)$.

Let $\xi : \mathbb{C}P^2 \to S^4$ be the map given by the formula

$$\xi([z_0 : z_1 : z_2]) = (\frac{2\bar{z}_0 z_1}{||z||^2}, \frac{2\bar{z}_0 z_2}{||z||^2}, \frac{|z_1|^2 + |z_2|^2 - |z_0|^2}{||z||^2})$$

where $||z||^2 = |z_0|^2 + |z_1|^2 + |z_2|^2$, for $[z_0 : z_1 : z_2] \in \mathbb{C}P^2$. Here we consider the 4-sphere as

$$S^4 = \{(w_1, w_2, t) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R} \mid |w_1|^2 + |w_2|^2 + t^2 = 1\}.$$

Note that $\xi([0:z_1:z_2]) = (0,0,1)$, the North pole and $\xi([0:0:1]) = (0,0,-1)$, the South pole. So, the map ξ maps the complex line $z_0 = 0$ to the North pole and is a diffeomorphism onto its image outside the line $z_0 = 0$. In particular its degree is one.

Consider the linear $SU(2)$-action on $\mathbb{C}P^2$ given as

$$[z_0 : z_1 : z_2] \mapsto [z_0 : a'z_1 + b'z_2 : c'z_1 + d'z_2]$$

where $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SU(2)$ is the inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$. Writing

$$w = (w_1, w_2) = (z_1/z_0, z_2/z_0)$$

we get

$$\xi([z_0 : z_1 : z_2]) = (\frac{2w_1}{1+||w||^2}, \frac{2w_2}{1+||w||^2}, 1 - \frac{2}{1+||w||^2}).$$

Hence, ξ becomes equivariant if we endow S^4 with the $SU(2)$-action given by

$$(w_1, w_2, t) \mapsto (a'w_1 + b'w_2, c'w_1 + d'w_2, t).$$

However, the latter is just the action of Lemma 2.1. Similarly, ξ is S^1-equivariant where the S^1-action $\mathbb{C}P^2$ is given by

$$[z_0 : z_1 : z_2] \mapsto [z_0 : e^{-i\theta} z_1 : e^{-i\theta} z_2].$$

Let $P_\phi = \xi^*(P^0_\phi)$, the pull back of the M-bundle $P^0_\phi \to S^4$ via the $SU(2)$ and the S^1-equivariant map $\xi : \mathbb{C}P^2 \to S^4$. Since ξ is an equivariant map the bundle $\pi : P_\phi \to \mathbb{C}P^2$ gets both $SU(2)$ and S^1-actions, for which the projection map π is equivariant. Moreover, the pull back cohomology class $\xi^*(\omega_{S^4})$ is invariant with respect to both actions and restricts to ω on each fiber. On the other hand, the cohomology class $[\xi^*(\omega_{S^4})]^{n+1} = \xi^*([\omega_{S^4}]^{n+1}) = 0$.

Let ω_{FS} denote the Fubuni-Study symplectic form on $\mathbb{C}P^2$. The form $\pi^*(\omega_{FS})$ is invariant under the $SU(2)$ and the S^1-action on the complex projective plane and is identically zero when restricted to each fiber $\{pt\} \times M$. Hence for any positive large enough constant $\kappa \gg 0$ the 2-form $\omega_\phi = \xi^*(\omega_{S^4}) + \kappa\, \pi^*(\omega_{FS})$ is a symplectic form on P_ϕ. Moreover, both actions on P_ϕ are Hamiltonian. This is obvious for $SU(2)$ since it is simply connected. For the S^1-action one can argue as follows: Since averaging commutes with exterior derivative locally we have $\omega_\phi = \pi_i^*(\omega) + dv + \kappa\, \pi^*(\omega_{FS})$ for some equivariant one form v on P_ϕ. Let χ be a vector field generated by the S^1-action. Since the form is invariant the Lie derivative of dv along χ will be zero. Now by the Cartan formula we get $i_{\chi^\sharp} dv = -d(i_{\chi^\sharp} v)$ and hence the S^1-action on P_ϕ is also Hamiltonian.

Remark 2.3. 1) By multiplying the last coordinate of the map $\xi : \mathbb{C}P^2 \to S^4$ by -1, if necessary, we can arrange so that the pull back $SU(2)$-bundle $\xi^*(L) \to \mathbb{C}P^2$ has $c_2 = -1$. Since $SU(2)$-bundles are determined by c_2 we see that $\xi^*(L)$ is smoothly isomorphic to $O(1) \oplus O(-1)$. Therefore, the construction of P_ϕ could be made just over $\mathbb{C}P^2$ using the bundle $O(1) \oplus O(-1)$ with appropriate $SU(2)$ and S^1-actions.

2) We will orient the bundles P^0_ϕ and P_ϕ as follows: The manifold P_ϕ is oriented with the orientation coming from the symplectic form ω_ϕ. Since P_ϕ is the pull back of P_ϕ^0 via the map $\xi : \mathbb{C}P^2 \to S^4$, whose degree is chosen as above, the orientation on P_ϕ induces one on P_ϕ^0.

2.3. **Hamiltonians and symplectic reduction.** Let $\mu : P_\phi \to \mathbb{R}$ be a Hamiltonian for the S^1-action on P_ϕ. Recall that the S^1-equivariant map $\xi : \mathbb{C}P^2 \to S^4$ maps the line $z_0 = 0$ to the North pole and sends the point $[0 : 0 : 1]$ to the South pole of the sphere. So, over some small S^1-invariant disjoint tubular neighborhoods U and V of the line $z_0 = 0$ and the point $[0 : 0 : 1]$, respectively, the bundle $\pi : P_\phi \to \mathbb{C}P^2$ is isomorphic to the product bundles $U \times M \to U$ and $V \times M \to V$, where the S^1-action on the M-factor is trivial. Moreover, by the construction, ω_ϕ when restricted to $\pi^{-1}(U)$ and $\pi^{-1}(V)$, is just $(\omega_\phi)_| = \omega + \kappa\ \pi^*(\omega_{FS})$. Since the action on the M-factor is trivial we see that the moment map restricted to $\pi^{-1}(U)$ and $\pi^{-1}(V)$ is just a multiple of the moment map $\mu_0 : \mathbb{C}P^2 \to \mathbb{R}$ of the S^1-action on $\mathbb{C}P^2$ plus a constant, which depends only on the open set U or V; i.e., $\mu(p) = \kappa\ \pi(\mu_0(p)) + C(p)$, for all $p \in \pi^{-1}(U) \cup \pi^{-1}(V)$, where C is a locally function on the union $\pi^{-1}(U) \cup \pi^{-1}(V)$.

We are ready now to prove the main theorem.

Proof of Theorem 1.1. Replacing the Hamiltonians by adding constants if necessary we can assume that 0 is a regular value for μ and hence for μ_0 such that $\mu_0^{-1}(0) = S^3$ lies in U. Note that this S^3 divides $\mathbb{C}P^2$ into two pieces. By multiplying all the symplectic forms with -1 if necessary we can assume that $\mathbb{C}P^2_0 = \mu_0^{-1}((-\infty, 0])$ is a closed tubular neighborhood of the line $z_0 = 0$ and $D^4 = \mu_0^{-1}([0, \infty))$ is a closed 4-ball with common boundary $S^3 = \mu^{-1}(0)$. The M-fiber bundle over these pieces are just products and

$$P_\phi = \mathbb{C}P^2_0 \times M \cup D^4 \times M/(g, x) \sim (g, \phi(g, x)), \text{ for } (g, x) \in \partial(\mathbb{C}P^2_0) \times M.$$

Let $\alpha \in H_k(M, \mathbb{Q})$ be a class represented by a cycle $a : A \to M$. We need to show that the class $\partial_\phi(\alpha)$, represented by the cycle $S^3 \times A \to M$, $(g, x) \mapsto \phi(g, a(x))$, is trivial in $H_{k+3}(M, \mathbb{Q})$. We can clearly view $S^3 \times A$ as a subset of the boundary of $\mathbb{C}P^2_0 \times M$. The identification in the above decomposition of P_ϕ maps $S^3 \times A$ into the other piece by the map $(g, x) \mapsto \phi(g, a(x))$. On the other hand, the radial contraction of D^4 to its center $\{0\}$ induces a radial contraction of $D^4 \times M$ to $\{0\} \times M$. Moreover, the composition of the identification map with the contraction will map $S^3 \times A$ into M exactly via the map $(g, x) \mapsto \phi(g, a(x))$. Since $S^3 \times A = \partial(\mathbb{C}P^2_0 \times A)$ the class $\partial_\phi(\alpha)$ is trivial in $H_{k+3}(P_\phi, \mathbb{Q})$.

Now consider the symplectic quotient $\mu^{-1}(0)/S^1$, which is equal to the product $S^3/S^1 \times M = S^2 \times M$, because S^1 acts trivially on M by the construction of the S^1-action. By the Kirwan's Surjectivity Theorem ([K]) the map, induced by the inclusion $\mu^{-1}(0) \subseteq P_\phi$, $\mathcal{K} : H^i(P_\phi, \mathbb{Q}) \to H^i(S^2 \times M, \mathbb{Q})$ is onto, for all i. So the restriction map $H^i(P_\phi, \mathbb{Q}) \to H^i(M, \mathbb{Q})$ is surjective. Hence the map in homology induced by the inclusion of a fiber $H_i(M, \mathbb{Q}) \to H_i(P_\phi, \mathbb{Q})$ is injective. Finally, by the above paragraph $\partial_\phi(\alpha)$ is trivial in $H_{k+3}(M, \mathbb{Q})$. $\square$

3. Some invariants of the SU(2)-action

In this section we will study the sections of the bundles P^0_ϕ and P_ϕ, define some invariants of the $SU(2)$-action on (M, ω), make some computations and mention some applications to the study of the topology of real algebraic varieties.

The orientations on the manifolds P^0_ϕ and P_ϕ, which we will need when we consider integrals over them, are the ones described in Remark 2.3.

3.1. Sections of P^0_ϕ.

The lemma below describes the $SU(2)$-equivariant (with respect to the $SU(2)$-action on P^0_ϕ and on S^4 described in Subsection 2.1) sections of the bundle $P^0_\phi \to S^4$ up to homotopy.

Lemma 3.1. *There is an $SU(2)$-equivariant section $s : S^4 \to P^0_\phi$ if and only if the $SU(2)$-action on M has a fixed point. Moreover, if $s_i : S^4 \to P^0_\phi$, $i = 1, 2$, are two such sections then the difference of $(s_2)_*([S^4]) - (s_2)_*([S^4])$ as a homology class is in the image of the map $\pi_4(M) \to \pi_4(P^0_\phi)$, induced by the inclusion of a fiber.*

Proof. Let $l(t) : [-1, 1] \to S^4$ be a one to one geodesic arc from the point $(0, 0, -1)$ to the point $(0, 0, 1)$. If $s : S^4 \to P^0_\phi$ is an equivariant section then s is determined completely by its values $s(l(t))$, $t \in [-1, 1]$. On the other hand, the points $(0, 0, \pm 1)$ are the fixed points of the action on the sphere and hence the points $s(0, 0, \pm 1)$ are in the fixed point set of action on M. Moreover, since the action on S^4 is free outside the poles, any section s defined on the arc $l(t)$ with $s(0, 0, \pm 1) \in M$ fixed points of the $SU(2)$-action, extends uniquely to a section. Indeed, the section $s : S^4 \to P^0_\phi$ is just the trace of the section $s(l(t))$, $t \in [-1, 1]$, under the $SU(2)$-action on P^0_ϕ.

The second statement follows the long exact sequence corresponding to the fibration $M \to P^0_\phi \to S^4$,

$$\cdots \to \pi_4(M) \to \pi_4(P^0_\phi) \to \pi_4(S^4) = \mathbb{Z} \to \cdots .$$

$\square$

Theorem 1.1 implies the following result.

Proposition 3.2. *If the $SU(2)$ action on M is symplectic then the homology class $s_*([S^4])$ of an equivariant section $s : S^4 \to P^0_\phi$ is determined only by the connected components of the fixed point set containing the fixed points $s(0, 0, -1)$ and $s(0, 0, 1)$.*

Proof. Assume the set up in the proof of the Lemma 3.1. If s_1 and s_2 are two such sections with $s_1(0, 0, -1) = s_2(0, 0, -1)$ and $s_1(0, 0, 1) = s_2(0, 0, 1)$ then the difference homology class can be identified with the trace of a loop in M based at one of these two fixed points. However, by Theorem 1.1 the latter is trivial. Now assume that $s_1(0, 0, -1)$, $s_2(0, 0, -1) \in F_0$ and $s_1(0, 0, 1)$, $s_2(0, 0, 1) \in F_1$, for some connected components F_0 and F_1 of the fixed point set. Join the fixed points in F_0 and F_1 by arcs contained completely in the fixed point sets. The trace of a path that lies in the fixed point set is just the path itself and hence these arcs do not contribute to the difference of the homology classes. This finishes the proof. $\square$

We will call a cohomology class $u \in H^4(M, \mathbb{Q})$ *monotone* if it vanishes on spherical classes, i.e., on the image of $\pi_4(M) \to H_4(M, \mathbb{Q})$.

We know that the cohomology class $[\omega_{S^4}] \in H^2(P_\phi^0, \mathbb{Q})$, which restricts to $[\omega]$ on each fiber with $\omega_{S^4}^{n+1} = 0$. Following [LMP, S], we denote the Chern classes of the vertical tangent bundle

$$T_*^{vert} = \ker(\pi_* : T_* P_\phi^0 \to T_* S^4)$$

by c_i^ϕ. These classes are clearly invariants of the action ϕ and hence, so is any integral of the form

$$I^0(k, k_1, \cdots, k_n) = \int_{P_\phi^0} \omega_{S^4}^k \ (c_1^\phi)^{k_1} \cdots (c_n^\phi)^{k_n}$$

where k, k_i are non negative integers with $2k + 2k_1 + \cdots + 2nk_n = 2n + 4 = \dim(P_\phi^0)$.

Let $x \in M$ be any fixed point of the $SU(2)$-action. Then the function $v \mapsto (v, x)$, $v \in S^4$, defines a section, say $s_x : S^4 \to P_\phi^0$. Note that the pull back bundle over S^4 of the vertical bundle via the section s_x is nothing but the associated complex vector bundle of the principal $SU(2)$-bundle $U(\mathbb{H}^2) \to S^4$ (see Subsection 2.1) corresponding to the representation of $SU(2)$ on tangent space $T_x M$. We have then the following result about the representations of $SU(2)$ on tangent spaces of the fixed points, which follows easily from Lemma 3.1.

Corollary 3.3. *Let (M, ω) and ϕ be as above. Assume that $c_2(M)$ is a monotone class. Then for any two fixed points x_1 and x_2 of the action on M, the vector bundles over S^4, corresponding to the $SU(2)$-representations at the tangent spaces T_{x_i}, $i = 1, 2$, have the same second Chern class.*

Remark 3.4. Lemma 3.1 and the above corollary are valid indeed for any smooth action on M since we do not make use of the symplectic form at all.

Example 3.5. 1) If the action $\phi : SU(2) \times M \to M$ is trivial then the integrals

$$I^0(k, k_1, \cdots, k_n) = \int_{P_\phi^0} \omega_{S^4}^k \ (c_1^\phi)^{k_1} \cdots (c_n^\phi)^{k_n}$$

are all zero, because in this case $P_\phi^0 = S^4 \times M$ and all the forms in the integral are trivial on $T_* S^4$.

2) Consider the standard action of $SU(2)$ on $(\mathbb{C}P^2, \omega)$, where $\omega = \omega_{FS}$, and consider

$$I^0(2, 0, 1) = \int_{P_\phi^0} \omega_{S^4}^2 \ c_2^\phi.$$

Note that $c_2(\mathbb{C}P^2)$ is a monotone class because $\pi_4(\mathbb{C}P^2) = 0$. It follows from the Wang sequence of the fibration that $c_2^\phi = \lambda[\omega_{S^4}]^2 + \pi_{S^4}^*(v)$ for some $v \in H^4(S^4, \mathbb{Q})$ and real number λ. Let $x_0 = [1 : 0 : 0]$, the only fixed point of the action, and denote the corresponding section of $P_\phi^0 \to S^4$ by s_{x_0}. Then

$$c_2^\phi([s_{x_0}]) = \lambda[\omega_{S^4}]^2([s_{x_0}]) + \pi_{S^4}^*(v)([s_{x_0}]) = \pi_{S^4}^*(v)([s_{x_0}]) = v([S^4]),$$

where the first equality follows from the fact $s_{x_0}^*([\omega_{S^4}]) = 0$ and the second one from that $\pi_{S^4} \circ s_{x_0} = id_{S^4}$. It follows from the representation of $SU(2)$ on the tangent space

$T_{x_0}\mathbb{C}P^2$ that $c_2^\phi([s_{x_0}]) = c_2(L) = -1$, where $L \to S^4$ is the canonical $SU(2)$-bundle (see Section 2.1).

Now, $I^0(2,0,1) = \int_{P^0_\phi} \omega^2_{S^4}\ c_2^\phi = \int_{P^0_\phi} \omega^2_{S^4}\ \pi^*_{S^4}(v)$
$= \int_{P^0_\phi - (\text{one fiber})} \omega^2_{S^4}\ \pi^*_{S^4}(v) = \int_{\mathbb{R}^4\times\mathbb{C}P^2} \omega^2_{S^4}\ \pi^*_{S^4}(v)$
$= (\int_{\mathbb{R}^4} v)\ (\int_{\mathbb{C}P^2} \omega^2_{S^4}) = v([S^4])\ \omega^2([\mathbb{C}P^2])$
$= -c_1^2([\mathbb{C}P^2]) = -9.$

Note that we can define analogous integrals over P_ϕ: There is a unique cohomology class $u_\phi = \xi^*([\omega_{S^4}]) \in H^2(P_\phi, \mathbb{Q})$, which restricts to $[\omega]$ on each fiber with $u_\phi^{n+1} = 0$. Similarly, we define

$$I(k, k_1, \cdots, k_n) = \int_{P_\phi} u_\phi^k\ (c_1^\phi)^{k_1} \cdots (c_n^\phi)^{k_n}$$

where k, k_i are non negative integers with $2k + 2k_1 + \cdots + 2nk_n = 2n + 4 = \dim(P_\phi)$. Note that the two invariants are indeed equal, where the first one may be more suitable for computations. However, if the action has no fixed points then the bundle over S^4 may have no section, as the next example shows.

Example 3.6. The bundle corresponding to the standard action of $SU(2)$ on $\mathbb{C}P^1$, $P^0_\phi \to S^4$, has no section. Otherwise, by deforming the section to a constant, say $[1:0]$, over D^4 we would arrive at the contradiction that the map $SU(2) \to \mathbb{C}P^1$ given by

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto [\bar{a} : \bar{b}] = \phi(g, [1:0])$$

is homotopically trivial, which is nothing but basically the Hopf map (see Section 2.2).

On the other hand, as we will see in the next section that the bundle $\pi : P_\phi \to \mathbb{C}P^2$ has always a section.

3.2. **Sections of P_ϕ.** Since the $SU(2)$ action on $\mathbb{C}P^2$ has a fixed point, an equivariant section exists if and only if the $SU(2)$-action on M has a fixed point. Note also that, any equivariant section of $P^0_\phi \to S^4$ pulls back to an equivariant section of $P_\phi \to \mathbb{C}P^2$. Indeed, these pull back sections are those equivariant sections $s : \mathbb{C}P^2 \to P_\phi$ such that $s(z_0 = 0)$ and $s([1:0:0])$ are two fixed points of the action on M (since the map $\xi : \mathbb{C}P^2 \to S^4$ maps the line $z_0 = 0$ to a pole of S^4 the bundle P_ϕ restricted to the line $z_0 = 0$ is trivial). In particular, for this class of equivariant sections of P_ϕ the analogues of Lemma 3.1, Proposition 3.2 and Corollary 3.3 will hold.

Another source of equivariant sections of the bundle is the set of points of M whose stabilizers is a circle. Namely, let $H = \mathrm{Stab}_{SU(2)}([0:1:1])$ and consider the path $r(t) = [1-t : t : t]$, $t \in [0,1]$, in $\mathbb{C}P^2$. Let x_0, x_1 be points in M with $\mathrm{Stab}_{SU(2)}(x_0) = SU(2)$ (x_0 is a fixed point) and $H \leqslant \mathrm{Stab}_{SU(2)}(x_1)$. Choose a section of the bundle over the path $r(t)$ with $s([1:0:0]) = x_0$ and $s([0:1:1]) = x_1$. Then this extends uniquely to an equivariant section $s : \mathbb{C}P^2 \to P_\phi$. Moreover, any equivariant section is of this form and the analogues of Lemma 3.1 and Corollary 3.3 will hold in this case also.

Unlike the bundle $P^0_\phi \to S^4$ the bundle over $\mathbb{C}P^2$ has always a section.

Lemma 3.7. *Let p_0 be any point in the fiber $\pi^{-1}([1:0:0]) = M$. Then the bundle $\pi : P_\phi \to \mathbb{C}P^2$ has a section $s : \mathbb{C}P^2 \to P_\phi$ with $s([1:0:0]) = p_0$.*

Proof. Recall the decomposition of P_ϕ from the proof of the Theorem 1.1

$$P_\phi = \mathbb{C}P^2_0 \times M \cup D^4 \times M/(g,x) \sim (g, \phi(g,x)), \text{ for } (g,x) \in \partial(\mathbb{C}P^2_0) \times M.$$

We define a section $s : \mathbb{C}P^2 \to P_\phi$ as follows: For $v \in D^4$ let $s(v) = (v, p_0)$. Over $\partial(D^4)$ the section looks like $v \mapsto (v, p_0)$ and over $\partial(\mathbb{C}P^2_0)$ it is given by the formula $v \mapsto (v, \phi(v^{-1}, p_0))$.

Since the action of the maximal torus $H = \mathrm{Stab}_{SU(2)}([0:1:1])$ on M is also Hamiltonian it has a fixed point, say $p_1 \in M$. Hence, $\mathrm{Stab}_{SU(2)}(p_1)$ contains H. Let $\sigma : [0,1] \to M$ be a path from p_0 to p_1. The $SU(2)$-orbit of this path,

$$SU(2) \times [0,1] \to M, \ (g,t) \mapsto \phi(g, \sigma(t))$$

gives a map $\beta : \mathbb{C}P^2_0 \to M$, whose restriction to the boundary $\partial(\mathbb{C}P^2_0)$ is the orbit of p_0. Now we can extend the section over $\mathbb{C}P^2_0$ as $v \mapsto (v, \beta(v))$. □

Remark 3.8. i) For the last part of the above proof we could use also the chain constructed in Theorem 1.1 bounded by the cycle $\partial_\phi(p_0)$.

ii) Note that if the point $p_0 \in M$ is not a fixed point of the $SU(2)$-action then the section of the above lemma is not equivariant. In particular, if $M = \mathbb{C}P^1$ with the $SU(2)$-action as in Example 3.6, then neither bundles have an equivariant section.

We believe that J-holomorphic sections of $P_\phi \to \mathbb{C}P^2$ deserve some attention also.

3.3. **Algebraic actions on real algebraic varieties.** The analogous constructions in case of Hamiltonian circle actions on closed symplectic manifolds can be done (cf. see also [LMP, S]). This result has the following consequence. Let $M = X_{\mathbb{C}}$ be a nonsingular projective complexification of a nonsingular real algebraic variety X and assume that we have a linear (hence Hamiltonian) S^1-action on $M = X_{\mathbb{C}}$. Assume that the action leaves the real part invariant and the action on X is free. Then the S^1-analogue of Theorem 1.1 implies that we can map $D^2 \times X$ into P_ϕ via an equivariant map, which extends the action $\phi : S^1 \times X \to X$. Moreover, the map descends to a map

$$D^2 \times_{S^1} X \to P_\phi$$

which shows that X bounds, in the manifold P_Φ, the mapping cylinder of the quotient map $p : X \to X/S^1 = B$. In particular, this implies that all the homology classes in X represented by cycles of the form $p^{-1}(A)$, where A is a cycle in B, are homologous to zero in the complexification $X_{\mathbb{C}}$, a fact proven in [O1] previously.

Lastly, we will mention another application along the same lines. The result mentioned in the introduction that the natural action of the homology of the group of Hamiltonian diffeomorphisms of a closed symplectic manifold (M, ω) on the homology groups $H_*(M, \mathbb{Q})$,

$$H_k(\mathrm{Ham}(M,\omega), \mathbb{Q}) \times H_l(M, \mathbb{Q}) \to H_{k+l}(M, \mathbb{Q})$$

is trivial ([LM, LMP]) has an immediate consequence in the study of topology of real algebraic varieties: Let X be a nonsingular compact real algebraic variety with a nonsingular projective complexification $i : X \to X_{\mathbb{C}}$. Clearly $X_{\mathbb{C}}$ carries a Kähler and hence a symplectic structure such that X is a Lagrangian submanifold. Define $KH_i(X, \mathbb{Q})$ as the kernel of the homomorphism $i_* : H_i(X, \mathbb{Q}) \to H_i(X_{\mathbb{C}}, \mathbb{Q})$ and $ImH^i(X, \mathbb{R})$ as the image of the homomorphism $i^* : H^i(X_{\mathbb{C}}, \mathbb{Q}) \to H^i(X, \mathbb{Q})$. In [O1, O2] it is shown that both $KH_i(X, \mathbb{Q})$ and $ImH^i(X, \mathbb{Q})$ are independent of the projective complexification $i : X \to X_{\mathbb{C}}$ and thus (entire rational) isomorphism invariants of X. We know also that the natural linear action of a unitary group on a complex projective variety is Hamiltonian. We have then the following corollary.

Corollary 3.9. *Let X and $X_{\mathbb{C}}$ be as above and G be a compact Lie group acting unitarily on $X_{\mathbb{C}}$, leaving the real part X invariant. Then the image of the trace map*

$$H_k(G, \mathbb{Q}) \times H_l(X, \mathbb{Q}) \to H_{k+l}(X, \mathbb{Q})$$

lies in $KH_{k+l}(X, \mathbb{Q})$.

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