Curvature-free upper bounds for the smallest area of a minimal surface.

Alexander Nabutovsky and Regina Rotman

May 11, 2004

Abstract

In this paper we will present two upper estimates for the smallest area of a possibly singular minimal surface in a closed Riemannian manifold M^n with a trivial first homology group. The first upper bound will be in terms of the diameter of M^n, the second estimate will be in terms of the filling radius of a manifold, leading also to the estimate in terms of the volume of M^n. After that we will establish similar upper bounds for the smallest volume of a stationary k-dimensional integral varifold in a closed Riemannian manifold M^n with $H_1(M^n) = ... = H_{k-1}(M^n) = \{0\}$, $(k > 2)$. The above results are the first results of such nature.

1 Main results

In this paper we prove an effective version of results of J. Pitts ([P] , section 4) establishing the existence of stationary integral varifolds of all dimensions $\leq i$ corresponding to a homology class $h \in H_i(M^n)$ for closed Riemannian manifold M^n. Namely, we are able to give a priori upper bounds for the smallest volume of a stationary integral varifold of dimension k in M^n in terms of the diameter or M^n or in terms of the volume of M^n. However, we are able to do this only in the situation when all homology groups of M^n of dimensions $< k$ vanish. Moreover, we will need the homological filling functions in these dimensions for our estimates. (These homological filling functions will be defined below.)

For readers who are not familiar with Geometric Measure Theory note that stationary integral k-dimensional varifolds are (singular) minimal submanifolds. The set of singular points has k-dimensional Hausdorff measure zero. The set of regular points consists of countably many k-dimensional submanifolds of M^n with positive integral multiplicities. The k-dimensional Hausdorff measure of the regular set where each point is counted with its multiplicity is called the mass of the varifold and must be finite. The stationarity of a varifold v has the following formal meaning: Let X be a smooth vector field on M^n. Consider the corresponding 1-parametric flow of diffeomorphisms Φ_t. Apply these diffeomorphisms to v and consider the mass of the resulting varifold as a function of t. By definition, v is stationary, if $t = 0$ is a critical point of this function.

We would like also to note that there exists an oriented analog of integral varifolds, namely, integral cycles. Since one can integrate k-forms over integral k-cycles, they can be regarded as a subset of the dual space to the space of k-forms and often are considered with a toplogy of the dual space. The dual space norm of k cycles is called their mass. F. Almgren proved ([A]) that the ith homotopy group of the space of integral k-cycles is isomorphic with the $(i + k)$th homology group of the ambient manifold. This fact plays a crucial role in the geometric measure theory. Yet integral cycles have the following technical disadvantage in comparison with integral varifolds: the mass is not a continuous function on the space of cycles. Indeed, consider a translate of a cycle z in R^n by a small vector. Consider the sum of this translate and $-z$ (that is, z taken with the opposite orientation). If $\varepsilon \neq 0$ then the mass of the cycle is equal to twice the mass of z. Hoewer, when ε become equal to zero the cycles cancel each other and the mass is equal to zero. If one would forget about the orientations and consider the corresponding varifolds, then this phenomenon cannot take place: the mass is a continuous functionional on spaces of varifolds.

Following an earlier work of F. Almgren J. Pitts developed a version of Morse theory for spaces of cycles. He described how one can start from a non-trivial homology class of the manifold and to assign to it non-trivial stationary integral varifolds in each dimension lower than the dimension of the homology class. He used the Almgren isomorphism between homology groups of the manifold and homotopy groups of the space of cycles here. Note that one gets here merely stationary varifolds instead of minimal cycles precisely because of the mentioned above technical problem with the possible vanishing of the mass of cycles in the limit.

In this paper we provide a quantitative version of Pitts's results. We combine his technique with our technique from [NR1] to derive upper bounds for the volume of the stationary integral cycles. Our results could, thus, be considred as a multidimensional generalization of results of [NR1]. Yet, they are different from the results of [NR1] in the following three aspects: 1. Here we need to assume that first $(k-1)$ homology groups of the Riemannian manifold vanish, whenever we did not need any assumptions about the ambient Riemannian manifold in [NR1]; 2. In [NR1] stationary 1-dimensional varifolds had only finitely many pieces (and the number of pieces was controlled in terms of the dimension of the manifold); and 3. Our present estimates involve not only diameter or volume but also defined below homological filling functions. However, we believe that the last limitation is unavoidable and the minimal volume of, say, a minimal hypersurface in a three-dimensional Riemannian manifold diffeomorphic to S^3 cannot be majorized only in terms of the diameter of the manifold. Our present results can be also compared with the results of [NR2]. There we considered only the case when the ambient manifold is diffeomorphic to S^3. Moreover, our estimates required a two-sided bound for the sectional curvature of the manifold. However there we got an upper bound for the area of a minimal surface diffeomorphic to S^2, whereas here we provide an upper bound for the area of a minimal surface of an unknown topological type.

In order to state our results we need the notion of homological filling functions. In the definition below we will be considering only singular chains with smooth simplices. The volume of a singular simplex will be defined as the volume of the standard simplex endowed with the pullback measure; the volume of a singular chain $\Sigma_i a_i \sigma_i$, where $a_i \in R$ and σ_i are singular simplices, is defined as $\Sigma |a_i|\ vol(\sigma_i)$.

Definition 1.1 (Homological filling function.) *Let M^n be a closed Riemannian manifold of dimension n. Let $\gamma(t)$ denote a closed piecewise differentiable curve. Then the homological filling function $FH = FH_1 : \mathbf{R}_+ \longrightarrow \mathbf{R}_+$ will be defined as follows: $FH(x) = \max_{\{\gamma(t) | length(\gamma(t)) \leq x\}} \min_{\{\Sigma_2 | \partial \Sigma_2 = \gamma(t)\}} Area(\Sigma_2)$, where Σ_2 is a singular 2-chain, and $Area(\Sigma_2)$ denotes its area.*

Definition 1.2 (Homological filling function in dimension $k > 1$.) *Let M^n be a closed Riemannian manifold of dimension n with the trivial k-th homology group. Then the homological filling function of order k, $FH_k : \mathbf{R}_+ \longrightarrow \mathbf{R}_+$ will be defined as follows: $FH_k(x) := \max_{\{\Sigma_k | m(\Sigma_k) \leq x\}} \min_{\{\Sigma_{k+1} | \partial \Sigma_{k+1} = \Sigma_k\}} m(\Sigma_{k+1})$, where Σ_k, Σ_{k+1} are singular chains of dimension $k, k+1$ respectively.*

Firther, recall that M. Gromov ([Gr]) defined the filling radius of a closed Riemannian manifold X^n embedded in a metric space Y as the minimal radius of a neighborhood of X^n in Y such that X^n bounds in this neighborhood. The filling radius of an abstract closed Riemannian manifold X^n is the filling radius of its Kuratowski embedding into $L^\infty(X)$ (Recall that the Kuratowski embedding assigns to each $x \in X^n$ the distance function to x. See a more formal and detailed definition of the filling radius below in the next section).

In this paper we will prove the following theorems:

Theorem 1.3 *Let M^n be a closed Riemannian manifold of dimension n with the trivial first homology group. Then the smallest area $A(M^n)$ of a possibly singular minimal surface in M^n satisfies the following inequalities:*

$$(1)A(M^n) \leq \frac{(n+1)!}{2} FH(2d(M^n));$$

$$(2)A(M^n) \leq \frac{(n+2)!}{6} FH(6FillRad(M^n));$$

$$(3)A(M^n) \leq \frac{(n+1)!}{6} FH(6(n+1)n^n\sqrt{(n+1)!}vol(M^n)^{\frac{1}{n}}).$$

Here $d(M^n)$ denotes the diameter, $FillRad(M^n)$ denotes the filling radius, and $vol(M^n)$ denotes the volume of M^n. If $n = 3$, then there exists a non-singular embedded minimal surface such that its area satifies the inequalities (1)-(3).

Formally speaking, "possibly singular minimal surface" means here "2-dimensional stationary integral varifold". More generally:

Theorem 1.4 *Let M^n be a closed Riemannian manifold of dimension n with $H_1(M^n) = H_2(M^n) = ... = H_{k-1}(M^n) = \{0\}$. Then for each $k \geq 2$ there exists a non-trivial stationary integral varifold of dimension k, such that its mass V_k is bounded from above by*

$$(1)V_k \leq \frac{(n+1)!}{k!} FH_k(k(FH_{k-1}(...(3FH_2(2d))...))).$$

$$(2)V_k \leq \frac{(n+2)!}{(k+1)!} FH_k((k+1)FH_{k-1}(...(4FH_2(6FillRadM^n))...)).$$

$$(3)V_k \leq \frac{(n+2)!}{(k+1)!} FH_k((k+1)FH_{k-1}(...(4FH_2(6(n+1)n^n\sqrt{(n+1)!}vol(M^n)^{\frac{1}{n}})...))).$$

If $k = n-1$ and $n \leq 7$ then we can ensure that a non-trivial stationary integral varifold satisfying the inequalities (1)-(3) is a smooth embedded hypersurface; if $k = n-1$ and $n = 8$, then we can ensure that it has only isolated singularities, if $k = n-1$ and $n \geq 9$ we can ensure that the Hausdorff dimension of its singular set does not exceed $k-7$.

Remarks.

1. The third inequalities in Theorems 1.3, 1.4 follow from the second inequalities and the above mentioned upper bound for the filling radius in terms of volume proven by M. Gromov, [Gr].

2. Theorems 1.3, 1.4 provide effective versions of the results by J. Pitts [P], Theorems 4.10, 4.11. Therefore our proofs provide the existence not merely stationary integral varifolds with volume ounded as stated in Theorems 1.3, 1.4 but stationary integral varifolds that are almost minimizing in a small annular neighborhood of every point precisely as in Theorems 4.10, 4.11 in [P]. (See 3.1(2) of [P] for the definition of almost minimizing varifolds.) In particular, our stationary varifolds are stable in a neighborhood of every point but finitely many, and the cardinality of the set of points, where stability is not guaranteed does not exceed some $N(n)$ depending only on the dimension of M^n.

3. In view of the last assertion of Theorem 1.3 one can ask for an upper bound of the smallest area of am embedded non-singular minimal hypersurface *diffeomorphic to* S^2 in a three-dimensional Riemannian manifold diffeomorphic to S^3. The existence of such a surface was proven by F. Smith and L. Simon ([S], see also [CD]). In [NR2] we obtained explicit upper bounds for the area of such a surface. These estimates are given in terms of an upper bound for the diameter of M^3, a positive lower bound for its volume and a two-sided bound for the sectional curvature. The methods of [NR2] have very little in common with the methods of the present paper.

2 Ideas of the proofs

This paper extends our earlier paper, (see [NR1]) in which we have found two curvature-free upper bounds for the minimal length of a strongly stationary 1-cycle. Strongly stationary 1-cycle can be considered as homological equivalents of closed geodesics and also as especially nice 1-dimensional equivalents of minimal surfaces with singularities. The proofs in the present paper generalize the proofs in [NR1]. Also they heavily use the results of J.

Pitts ([P]), and can be regarded as a quantitative version of his work. First we will present an informal explanation of the proof of the first estimate of Theorem 1.3.

For the sake of simplicity of the explanation assume that M is diffeomorphic to a round 3-sphere S^3. We are going to show that there exists a minimal imbedded surface in S^3 of area satisfying the upper bounds of Theorem 1.3

Let $f : S^3 \longrightarrow M$ be any diffeomorphism. Assume S^3 was triangulated into simplices of diameter smaller than δ. Let the standard 4-disc D^4 be triangulated as a cone over S^3. A k-simplex $[v_{i_0}, ..., v_{i_k}]$ of D^4 will be denoted σ_i^k, (or sometimes $\sigma^k_{i_0,...,i_k}$), where $k \in \{1, 2, 3, 4\}$.

The proof will be by contradiction. Suppose $A(M) > 12FH(2d+\delta)+5\delta$. We will show that in that case there exists a singular chain, such that $f_*([S^3])$ bounds. Here $[S^3]$ denotes the fundamental class of S^3. This trick is based on the obstruction technique first used in the paper [Gr].

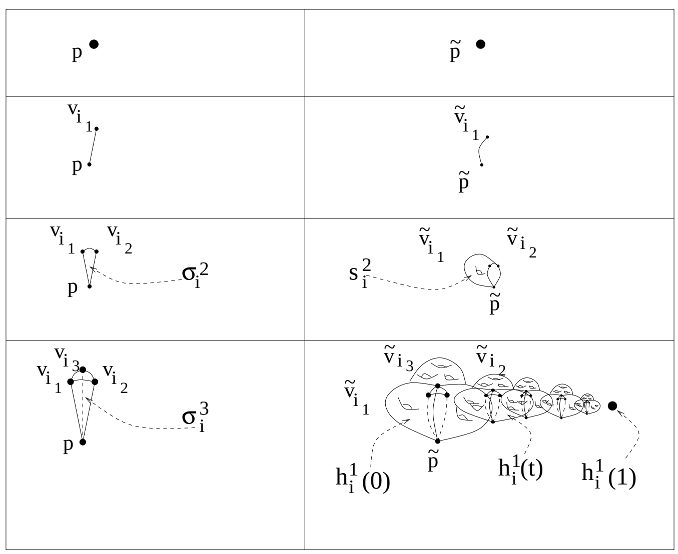

Figure 1: Table 1.

We will begin by extending the map $f : S^3 \longrightarrow M$ to 1-skeleton of M and then by assigning to each 2-simplex of D^4 a singular 2-chain on M, (see fig. 1). To extend to 0-skeleton, we will assign to the center of the disc p, an arbitrary point $\tilde{p} \in M$. Next we extend to 1-skeleton by assigning to each arbitrary edge $[p, v_{i_1}]$ connecting the center of D^4 with a vertex v_{i_1}

and directed from p to v_{i_1}, a minimal geodesic that connects $\tilde{p}$ with a vertex $\tilde{v}_{i_1} = f(v_{i_1})$, directed from $\tilde{p}$ to $\tilde{v}_{i_1}$ and denoted $[\tilde{p}, \tilde{v}_{i_1}]$. Now, consider a simplex $\sigma_i^2 = [v_{i_0}, v_{i_1}, v_{i_2}]$, where $v_{i_0} = p$. Its boundary is mapped to the curve $[\tilde{v}_{i_1}, \tilde{v}_{i_2}] - [\tilde{v}_{i_0}, \tilde{v}_{i_1}] + [\tilde{v}_{i_0}, \tilde{v}_{i_1}]$ of length $\leq 2d+\delta$. Let s_i^2 (also denoted as $s^2_{i_0,i_1,i_2}$) be a singular 2-chain of the smallest area, such that $\partial s_i^2 = f(\partial\sigma_i^2)$. Then its area is $\leq FH(2d+\delta)$. We will assign to σ_i^2 this surface s_i^2.

Now we will slightly change our tactics. Consider a 3-simplex σ_i^3. There is a preassigned chain $S_i^2 = \Sigma_{j=0}^3 (-1)^j s^2_{i_0,\ldots,\hat{i}_j,\ldots,i_3}$ of area $\leq 3FH(2d+\delta)+\delta$ that corresponds to the boundary of this simplex. (We can assume without loss of generality that the area of $s^2_{i_1,i_2,i_3} \leq \delta$). S_i^2 is an element in the space of integral 2-cycles $Z_2(M^n, \mathbf{Z})$. Since by our assumption there are no minimal surfaces of "small" area, S_i^2 can be connected with the zero cycle by a curve that passes through cycles of area $\leq 3FH(2d+\delta)+\delta$, i. e. there exists $h_i^1 : [0,1] \longrightarrow Z_2(M, \mathbf{Z})$, (sometimes denoted $h^1_{i_0,i_1,i_3}$), such that $h_i^1(0) = S_i^2$ and $h_i^1(1)$ is the 0-cycle. We will assign the above path to the simplex σ_i^3. Finally, take a 4-simplex σ_i^4, (see fig. 2). Consider its boundary $\partial\sigma_i^4 = \Sigma_{j=0}^4 (-1)^j [v_{i_0}, \ldots, \hat{v}_{i_j}, \ldots, v_{i_4}]$. Each of its faces $(-1)^j [v_{i_0}, \ldots, \hat{v}_{i_j}, \ldots, v_{i_4}]$ corresponds to the map $(-1)^j h^1_{i_0,\ldots,\hat{i}_j,\ldots,i_4}(t)$, where $-h^1_{i_0,\ldots,\hat{i}_j,\ldots,i_4}(t)$ is a cycle that is geometrically the same as $h^1_{i_0,\ldots,\hat{i}_j,\ldots,i_4}(t)$, but is oppositely oriented.

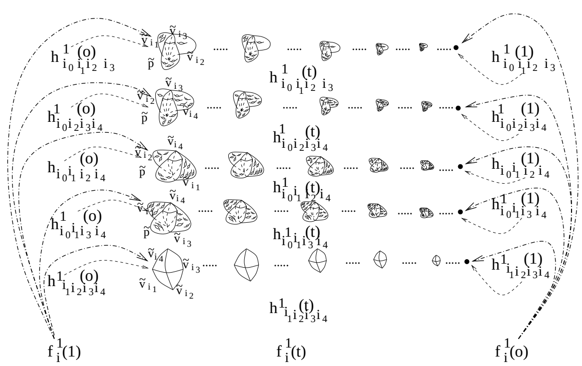

Figure 2: The loop $f_i^1(t)$.

Now we will perform the trick that we will use throughout the paper. Consider the map $f_i^1 : [0,1] \longrightarrow Z_2(M, \mathbf{Z})$, such that $f_i^1(t) = \Sigma_{j=0}^4 (-1)^j h_{i_0,\ldots,\hat{i}_j,\ldots,i_4}(1-t)$. Note that $f_i^1(0)$ is the zero cycle and that $f_i^1(1)$ is also the zero cycle represented by 10 pairs of singular chains, where each pair contains two copies of one chain with opposite orientations. Thus, $f_i^1(t)$ is a loop in the space of 2-cycles. Note also that the area of $f_i^1(t)$ is bounded from above by $12FH(2d+\delta)+5\delta$. By our assumption $f_i^1(t)$ is contractible in $Z_2(M,\mathbf{Z})$ along the cycles of area $\leq 12FH(2d+\delta)+5\delta$. (Otherwise the proof of Theorem 4.10 in [P] implies the existence of a minimal surface (=a stationary integral varifold) of area $\leq 12FH(2d+\delta)+5\delta$, which is impossible because of our assumption.) Thus, we obtain a disc $h_i^2 : D^2 \longrightarrow Z_2(M,\mathbf{Z})$. We will assign this disc to the simplex σ_i^4.

From the geometric measure theory we know that

$$\pi_k(Z_2(M,\mathbf{Z})) = H_{k+2}(M).$$

So, the disc of the form $h_i^2 : D^2 \longrightarrow Z_2(M,\mathbf{Z})$ corresponds to a singular chain s_i^4 on M. (However the construction of the correspondence is technical. Our intention to avoid the technicalities of this construction is responsible for the fact that the proofs of Theorems 1.3 and 1.4 given in the next sections are more awkward than this sketch). Now consider $S^4 = \Sigma_{i=1}^Q s_i^4$, where Q is the number of simplices of dimension 4 in the triangulation of D^4. $\partial S^4 = f_*([S^3])$. Thus, we obtain a contadiction. Therefore, we can conclude that $A(M) \leq 12FH(2d+\delta)+5\delta$. Now let δ go to zero.

The proof of Theorem 1.3 will be given in Section 2. The proof of Theorem 1.4 will be given in Section 3. Section 1 will be devoted to the discussion of the regularity of minimal surfaces the area of which we estimate. As we have mentioned before the arguments of the proofs will be somewhat more technical than the explanations above, and will run as follows.

Once again, for the sake of simplicity of the explanation, assume that our manifold M is diffeomorphic to the standard 3-sphere. Let $f : S^3 \longrightarrow M$ be a diffeomorphism, and suppose that the triangulation of S^3 and the induced triangulation on M is very fine, (i.e. the diameter of simplices smaller than δ). Let D^4 be triangulated as a cone over S^3.

The proof will consist of the following three steps:

Step 1 (Easy). Corresponding to the map $f : S^3 \longrightarrow M$ one can construct a non-contractible map $\tilde{f} : S^1 \longrightarrow Z_2(M,\mathbf{Z})$, a loop in the space of the integral 2-cycles on a manifold M. We do not have any control over the masses of $f(t), t \in S^1$.

Step 2 (Main Step). The loop $\tilde{f}$ is homotopic to the sum of Q loops $\tilde{g}_i$, where the number Q equals to the number of simplices in the triangulation of S^3, and the masses of all 2-cycles $\tilde{g}_i(t), t \in S^1$ satisfy upper bounds as in the right hand side of the inequality (1) in the text of Theorem 1.3.

Step 3 (An application of [P]). We show that if there is no minimal surface with "small" area than each loop in step 2 can be contracted to a point, thus obtaining a contradiction.

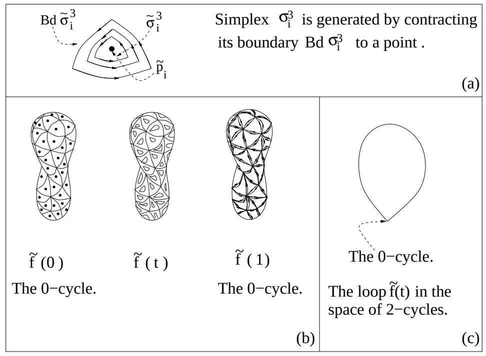

Figure 3: The loop $\tilde{f}$.

Step 1. Let $\{\sigma_i^3\}_{i=1}^Q$ be the set of simplices that constitute the fundamental class $[S^3]$ of S^3. Then $\{\tilde{\sigma}_i^3\}_{i=1}^Q$ are the corresponding simplices of M.

Without any loss of generality we can assume that each $\tilde{\sigma}_i^3$ can be obtained from its boundary $\partial\tilde{\sigma}_i^3$ by contracting it to the "center" of the simplex, the point $\tilde{p}_i$ by a homotopy $\tilde{h}_i : [0,1] \longrightarrow M$ along the 2-spheres $\tilde{h}_i(t)$, (see fig. 3 (a). This figure schematically depicts $\tilde{\sigma}_i^3$ as a simplex of dimension 2).

Then $\tilde{f}$ can be constructed as follows. We will begin with the zero cycle that consists of the sum of the "centers" of simplices: $\Sigma_{i=1}^Q \tilde{p}_i$. We will then follow the images of $\partial\tilde{\sigma}_i^3$'s under the area decreasing homotopies $\tilde{h}_i(t)$'s, that is $\tilde{f}(t) = \Sigma_{i=1}^Q \tilde{h}_i(1-t)$. Note that $\tilde{f}(1) = \Sigma_{i=1}^Q \partial\tilde{\sigma}_i^3$, which is also the zero cycle. Thus, we obtain a (non-contractible) loop in the space of 2-cycles, (see fig. 3 (b). There $f(S^3)$ is schematically depicted as a 2-dimensional sphere).

Step 2. Each loop $\tilde{g}_i$ is constructed as follows. Consider a disc D^4, such that $\partial D^4 = S^3$, triangulated as a cone over S^3. Let $p \in D^4$ be the center of this disc. We can assign to this point an arbitrary point $\tilde{p} \in M$, thus extending the map $f : S^3 \longrightarrow M$ to the 0-skeleton of D^4. Next consider a line segment of the form $[p, v_i]$ directed from p to v_i. We can assign to it a minimal geodesic segment of length smaller than the diameter d of M joining the point $\tilde{p}$ with the vertex $\tilde{v}_i = f(v_i)$. This segment will be directed from $\tilde{p}$ to $\tilde{v}_i$ and denoted as $[\tilde{p}, \tilde{v}_i]$. This extends f to 1-skeleton of D^4. Next consider a 2-simplex $\sigma_i^2 = [p, v_{i_1}, v_{i_2}]$, (it will sometimes be denoted as $\sigma^2_{i_0,i_1,i_2}$. Its boundary is mapped to a closed curve $[\tilde{v}_{i_1}, \tilde{v}_{i_2}] - [\tilde{p}, \tilde{v}_{i_2}] + [\tilde{p}, \tilde{v}_{i_1}]$. Let s_i^2 (sometimes

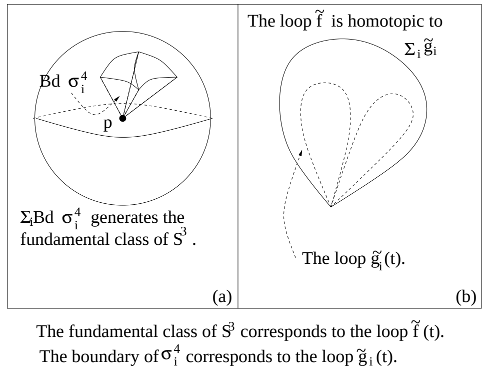

Figure 4:

denoted as $s^2_{i_0,i_1,i_2}$) be a singular 2-chain of the smallest area filling this curve. We can assign s^2_i to simplex σ^2_i, thus we obtain a map from the 2-skeleton of D^4 to the space of integral 2-currents. Now let us take an arbitrary 3-simplex $\sigma^3_i = [p, v_{i_1}, v_{i_2}, v_{i_3}]$. Its boundary $\partial\sigma^3_i = \sum_{j=0}^3(-1)^j[v_{i_0}, ..., \hat{v}_{i_j}, ..., v_{i_3}]$, where $v_{i_0} = p$. For each face $(-1)^j[v_{i_0}, ..., \hat{v}_{i_j}, ..., v_{i_3}]$ we have pre-assigned a singular 2-chain $(-1)^j s^2_{i_0,...,\hat{i}_j,...,i_3}$. Consider $\Sigma_{j=0}^3(-1)^j s^2_{i_0,...,\hat{i}_j,...,i_3}$. This is a 2-cycle of area smaller than $3FH(2d+\delta)+\delta$, (without loss of generality we can assume that the area of $s^2_{i_1,i_2,i_3}$ is smaller than δ). Thus, it is an element in $Z_2(M, \mathbf{Z})$. Assuming there is no minimal 2-cycle that locally minimizes the area of area smaller than that, this cycle can be connected with the zero cycle, that is there exists a map $h^1_{i_0,...,i_3} : [0, 1] \longrightarrow Z_2(M, \mathbf{Z})$ that begins with our cycle and ends with the zero cycle. Finally, let us construct $\tilde{g}_i$. Consider a 4-simplex $\sigma^4_i = [v_{i_0}, v_{i_1}, ..., v_{i_4}]$. Its boundary, $\partial\sigma^4_i = \Sigma_{j=1}^4(-1)^j[p, v_{i_1}, ..., \hat{v}_{i_j}, ..., v_{i_4}]$. For each face in the boundary $(-1)^j[v_{i_0}, v_{i_1}, ..., \hat{v}_{i_j}, ..., v_{i_4}]$ there was constructed a map $(-1)^j h^3_{i_0,...,\hat{i}_j,...,i_4}$: $[0, 1] \longrightarrow Z_2(M, \mathbf{Z})$, where $-h(t)$ is the same cycle as $h(t)$, but taken with an opposite orientation. We will define the map $\tilde{g}_i(t)$ as the sum $\Sigma_{j=0}^4(-1)^j h^1_{i_0,...,\hat{i}_j,...,i_4}(1-t)$. Note that $\tilde{g}_i(0)$ is the zero cycle represented by $\tilde{p}_{i_0,...,\hat{i}_j,...,i_4}$ and that $\tilde{g}_i(1)$ is also the zero cycle, because of the cancellations due to the fact that each surface that corresponds to a 2-dimensional face of the simplex σ^4_i enters twice with the opposite orientation. Thus, $\tilde{g}_i(t)$ is a loop. Area of $\tilde{g}_i(t)$ is bounded from above by $12FH(2d+\delta)+5\delta$. Moreover, $\tilde{f}(t)$ is homotopic to the sum of the above loops.

Step 3. Since $\tilde{f}(t)$ is homotopic to the sum of the loops $\tilde{g}_i$, (see fig.

4), at least one of those loops in not contractible. Therefore, if we try to contract it using a mass decreasing flow described in ch. 4 of [P] (and introduced earlier by F. Almgren), it should get stuck on a critical point, which, as it was shown by J. Pitts in chapter 4 of [P] would be a stationary integral varifold. In ch. 7 of [P] J. Pitts uses the results of [SSY] imply that in the three-dimensional case this stationary varifold will turn out to be an embbedded minimal surface.

Similarly, one can show that $A(M^n) \leq \frac{(n+2)!}{6} FH(6FillRad(M^n))$. Recall that the filling radius of a Riemannian manifold M^n was introduced by Gromov in [Gr] as follows:

Definition 2.1 (Filling Radius) *Let M^n be a Riemannian manifold topologicaly imbedded into an arbitrary metric space X. Then its filling radius, denoted $FillRad(M^n \subset X)$, is the infimum of $\varepsilon > 0$, such that M^n bounds in the ε- neighborhood $N_\varepsilon(M^n)$, i.e. homomorphism $H_n(M^n, \mathbf{Z_2}) \longrightarrow H_n(N_\varepsilon(M^n), \mathbf{Z_2})$ induced by the inclusion map vanishes. Let M^n be an abstract manifold. Then its filling radius, denoted $FillRad(M^n)$ will be $FillRad(M^n \subset X)$, where $X = L^\infty(M^n)$, i.e. the Banach space of bounded Borel functions f on M^n and the imbedding of M^n into X is the map that assigns to each point p of M^n the distance function $p \longrightarrow f_p = d(p, q)$.*

In the same paper Gromov poved the following important inequality relating the filling radius and the volume:

Theorem 2.2 ((M. Gromov))

$$FillRad(M^n) \leq \sqrt{(n+1)!}n^n(n+1)vol(M^n)^{\frac{1}{n}}.$$

Now, suppose once again that M^n is diffeomorphic to S^3. The definition of the Filling Radius implies that M^n bounds in the $(FillRad(M^n) + \delta)$ - neighborhood of M^n in $L^\infty(M^n)$. Let W fill M^n in the $(FillRad(M^n) + \delta)$-neighborhood of M, (that is $M^n = \partial W$). Without loss of generality we can assume that W is a polyhedron. Suppose that W together with M^n is endowed with a very fine triangulation. As in (i) the proof will consist of the three steps: constructing a non-contractible loop $\tilde{f} : S^1 \longrightarrow Z_2(M^n, \mathbf{Z})$; constructing a family of loops $\tilde{g}_i : S^1 \longrightarrow Z_2(M^n, \mathbf{Z})$; concluding that one of those loops must be non-contractible, and therefore there exists an almost minimizing integral varifold of area smaller than the area of integral cycles through which this loop passes.

Step 1. The first step is analogous to that of (i).

Step 2. Let v_i be a vertex of W. Assign to it a closest vertex in M^n, denoted $\tilde{v}_i$. Thus, $d(v_i, \tilde{v}_i) \leq FillRad(M^n) + \delta$. This extends the identity map $Id : 2004-006.tex, v1.12004/05/2717:35:19levyExplevy$ on M^n to 0-skeleton of W. Now to any 1-simplex $[v_{i_1}, v_{i_2}] \subset W \setminus M^n$ we can assign a minimal geodesic segment connecting $\tilde{v}_{i_1}$ and $\tilde{v}_{i_2}$ of length smaller than $2FillRad(M^n) + 3\delta$. This segment will be denoted as $[\tilde{v}_{i_1}, \tilde{v}_{i_2}]$. This extends $Id : 2004-006.tex, v1.12004/05/2717:35:19levyExplevy$ to 1-skeleton of M^n. Next consider an arbitrary 2-simplex $\sigma_i^3 = [v_{i_1}, v_{i_2}, v_{i_3}]$. Its boundary is mapped to a curve of length smaller than $6FillRad(M^n) + 9\delta$. Let $s^2_{i_1,i_2,i_3}$ be a singular 2-chain of the smallest area that has this curve as its boundary. Its area is going to be smaller than $FH(6FillRad(M^n) + 9\delta)$. The rest of the procedure (including Step 3) is the same as in the proof of (1).

3 The proof of Theorem 1.3

In this section we will prove 1.3. We will begin by proving statements (i) and (ii). Statement (iii) follows from (ii).

Proof. We are going to prove the theorem by contradiction. Assume that there is no stationary integral varifold as in the text of the theorem and then prove that it exists.

Suppose M^n is a $(q-1)$-connected manifold with $\pi_q(M^n) \neq \{0\}$. Let $f : S^q \longrightarrow M^n$ be a non-contractible map in case (i), and in case (ii) chooser a (singular) W that fills M (in $L^\infty(M^n)$) such that for each $x \in W$ $dist(x, M^n) \leq (n+1)n^n\sqrt{(n+1)!}vol(M^n)^{\frac{1}{n}}$. (As it had been already mentioned, in [Gr] M. Gromov proved that such a filling exists.) Let S^q be triangulated into simplices σ_i^q of diameter $d(\sigma_i^q) \leq \delta$ for some small δ in case (i). In case (ii) assume that W (and $M^n = \partial W$) has been triangulated into simplices σ_i^n of diameter at most δ. In both cases let Q denote the number of simplices. The proof will consist of three steps:

Step 1. We will begin by constructing a non-contractible map $\tilde{f} : S^{q-2} \longrightarrow Z_2(M^n, \mathbf{Z})$ in case (i) that corresponds to the original map $f : S^q \longrightarrow M^n$ under the Almgren correspondance in case (i) and the map $\tilde{f} : S^{n-2} \longrightarrow Z_2(M^n, \mathbf{Z})$ that corresponds to the fundamental class of M^n in case (ii).

Step 2. We will construct maps $\tilde{g}_i$ from S^{q-2} in case (i) and from S^{n-2} in case (ii) to the space $Z_2(M^n, \mathbf{Z}), i \in \{1, ..., Q\}$, where Q is a number of simplices of dimension $(q+1)$ in the triangulation of $D^{(q+1)}$ in case (i) and it is a number of simplices of dimension $(n+1)$ in the triangulation of W in case (ii). It will turn out that $\tilde{f}$ is homotopic to the sum of $\tilde{g}_i' s$. Therefore, at least one of the maps $\tilde{g}_i' s$ is not contractible. The most important feature of $\tilde{g}_i$ is that the mass of $\tilde{g}_i(t)$ does not exceed the right hand side of (1) (or (2)) for every t, i.

Step 3. We can conclude that there exists an almost minimizing integral varifold of area smaller than that of the cycles through which passes the non-contractible map of Step 2.

Step 1. Let us begin by considering simplices $\tilde{\sigma}_i^3 = [\tilde{v}_{i_0}, ..., \tilde{v}_{i_3}]$ of dimension 3, (we will sometimes denote it as $\tilde{\sigma}^3_{i_0,...,i_3}$, in general, k-simplices will be sometimes denoted as $\tilde{\sigma}^k_{i_0,...,i_k}$). Without loss of generality we can assume that this simplex can be generated by contracting its boundary $\partial\tilde{\sigma}_i^3$ to the "center" of the simplex $\tilde{p}_i^3$, (sometimes denoted $\tilde{p}_{i_0,...,i_3}$) with a homotopy $\tilde{h}_i^1(t) = \tilde{h}^1_{i_0,...,i_3}$. This homotopy can be chosen, for example, to be the radial homotopy of the boundary of the (very small and, therefore, almost Euclidean) simplex to its center. Note that we can consider this homotopy as a path in the space of integral 2-cycles that starts with the boundary of a simplex and ends with the zero cycle. Next consider a 4-dimensional simplex $\tilde{\sigma}_i^4 = [\tilde{v}_{i_0}, ..., \tilde{v}_{i_4}]$. Its boundary $\partial\tilde{\sigma}_i^4 = \Sigma_{j=0}^4 (-1)^j [\tilde{v}_{i_0}, ..., \hat{\tilde{v}}_{i_j}, ..., \tilde{v}_{i_4}]$. For each 3-face $(-1)^j [\tilde{v}_{i_0}, ..., \hat{\tilde{v}}_{i_j}, ..., \tilde{v}_{i_4}]$ we have constructed a path $(-1)^j \tilde{h}^1_{i_0,...,\hat{i}_j,...,i_4}$: $[0,1] \longrightarrow Z_2(M^n, \mathbf{Z})$. Now, let us construct a map $\tilde{f}_i^1 : [0,1] \longrightarrow Z_2(M^n, \mathbf{Z})$ as follows: let $\tilde{f}_i^1(t) = \Sigma_{j=0}^4 (-1)^j \tilde{h}^1_{i_0,...,\hat{i}_j,...,i_4}(1-t)$. Then, note that $\tilde{f}_i^1(0)$ is the zero cycle that corresponds to the sum of the "centers" of the 3-faces of $\tilde{\sigma}_i^4$ and that $\tilde{f}_i^1(0)$ is also the zero cycle due to the cancellation of each pair of the two faces, that have the same geometric image, but are taken with the opposite orientation. Thus, this map is really a "small" loop in the space of the integral 2-cycles. We can contract it there and obtain a "small" disc $\tilde{h}_i^2 : D^2 \longrightarrow Z_2(M^n, \mathbf{Z})$, (this map will also be sometimes denoted as $\tilde{h}^2_{i_0,...,i_4}$). Geometrically, this discs corresponds to the considered 4-dimensional simplex regarded as the chain filling its boundary. The construction of this disc can be made absolutely explicit: It is just a slicing of the simplex into 2-surfaces that extends the slicing of its boundary. It can be obtained just by coning of the slicing of the boundary. Now we can just proceed by induction until we will get a slicing of each q-dimensional singular simplex of M^n obtained from f and the considered very fine triangulation of S^q (case (i)) or n-dimensional simplex of M^n (case (ii)). As the result, we obtain maps $\tilde{h}_i^q$ of D^{q-2} into $Z_2(M, \mathbf{Z})$ (or $\tilde{h}_i^n$ of D^{n-2} into $Z_2(M^n, \mathbf{Z})$, where the mass of each 2-cycle in the image of each of these maps can be made arbitrarily small.

We finish by defining the map $\tilde{f}$ as the sum of those maps over all simplices in the triangulation. Note that this map turns out to be a map from the $(q-2)$- (Case (i)) or $(n-2)$- (Case (ii)) dimensional sphere. Note that by doing so we lose any control over the mass of 2-cycles in the image of f.

Step 2. We will now construct maps $\tilde{g}_i$s, such that $\tilde{f} = \Sigma_{i=1}^{Q} \tilde{g}_i$. The beginning of procedure of constructing those maps is somewhat different for case (i) and for case (ii).

Case (i). Take a point p, the center of the disc D^{q+1}, which has been triangulated as a cone over S^q. Assign to this point an arbitrary point in a manifold, that will be denoted $\tilde{p}$. Note that this extends the map f to 0-skeleton of D^{q+1}. Next consider an edge of the form $[p, v_i]$. We will assign to it a minimal geodesic segment $[\tilde{p}, \tilde{v}_i]$ joining the point $\tilde{p}$ with the corresponding vertex $\tilde{v}_i = f(v_i)$. This extends the map f to 1-skeleton of D^{q+1}.

Case (ii). Let v_i be an arbitrary vertex of W. Let $\tilde{v}_i$ be a vertex of M^n that minimizes the distance between v_i and M^n. Then $d(v_i, \tilde{v}_i) \leq FillRadM^n + \delta$. We will assign $\tilde{v}_i$ to the vertex v_i. This extends the identity map $Id : M^n \longrightarrow M^n$ to W. Now consider an arbitrary edge of W of the form $[v_i, v_j]$. We can assign to this edge a minimal geodesic segment joining the corresponding vertices $\tilde{v}_i$ and $\tilde{v}_j$. It will be denoted $[\tilde{v}_i, \tilde{v}_j]$. This segment will have the length of at most $2FillRadM^n + 3\delta$.

The rest of the procedure will be the same in the two cases. Consider an arbitrary 2-simplex of the form $\sigma_i^2 = [v_{i_0}, v_{i_1}, v_{i_2}]$ of D^{q+1}, where (v_{i_0} denotes here and later the center of the disc p) in Case (i). Its boundary is mapped to a closed curve of length at most $2d + \delta$ in Case (i) (of $6FillRadM^n + 9\delta$ in Case (ii)). Let $s_i^2 = s_{i_0,i_1,i_2}^2$ be a singular surface (=singular 2-chain) of area smaller than $FH(2d + \delta)$ in Case (i) ($FH(6FillRadM^n + 6\delta)$ in Case (ii)). We will assign this surface to the simplex σ_i^2.

Next consider a 3-simplex $\sigma_i^3 = [v_{i_0}, ..., v_{i_3}]$. Its boundary corresponds to the singular surface $\Sigma_{j=0}^{3}(-1)^j s_{i_0,...\hat{i}_j,...,i_3}$. Its area is $\leq 3FH(2d+\delta) + \tilde{\delta}$, (since we can assume that all simplices of $f(S^q)$ are small) in Case (i) ($\leq 4FH(6FillRadM^n + 6\delta)$ in Case (ii)). This surface can be considered as an element in the space of integral 2-cycles on M^n. Now, either there exists a minimal 2-cycle that locally minimize the mass of non-zero mass smaller than that of this surface, or this surface, viewed as a point in $Z_2(M^n, \mathbf{Z})$ can be connected with the zero cycle with a path $h_i^1 : [0, 1] \longrightarrow Z_2(M^n, \mathbf{Z})$.

Here we are using the proof of Theorem 4.10 in [P]. This theorem does notamention the mass of the stationary varifold but its proof implies that the mass does not exceed the maximal mass of cycles in the image of the considered map of the sphere.)

Therefore, we can establish a correspondence between the 3- simplices and the above maps.

Now consider a 4-simplex $\sigma_i^4 = [v_{i_0}, ..., v_{i_4}]$. For each face $(-1)^j[v_{i_0}, ..., \hat{v}_{i_j}, ..., v_{i_4}]$ there was preassigned a map $(-1)^j h^1_{i_0,...,\hat{i}_j,...,i_4} : [0,1] \longrightarrow Z_2(M^n, \mathbf{Z})$. Now, let $f_i^1(t) = \Sigma_{j=0}^1 (-1)^j h^1_{i_0,...,\hat{i}_j,...,i_4}(1-t)$. Each $f_i^1(t)$ is an integral 2-cycle of area $\leq 4 \cdot 3FH(2d+\delta) + 2\delta$ in Case (i) ($\leq 5 \cdot 4FH(6FillRadM^n + 9\delta)$). Note that $f_i(0)$ is the zero cycle and that $f_i(1)$ is also the zero cycle due to the cancellations of the pairs of surfaces that enter with the opposite orientation. Thus, in reality, $f_i^1(t)$ is a loop in the space of the integral 2-cycles. Since we have assumed that there are no almost minimizing integral varifolds of "small" area the map $f_i^1(t)$ is contractible over the disc $h_i^2(s)$. Therefore, we can assign this map of the disc to a 4-simplex σ_i^4.

Now suppose that for each simplex of dimension k we have constructed the corresponding map $h_i^{k-2} : D^{k-2} \longrightarrow Z_2(M^n, \mathbf{Z})$. Let us consider an arbitrary $(k+1)$-simplex $\sigma_i^k = [v_{i_0}, ..., v_{i_{k+1}}]$. Consider a face in the boundary of this simplex $(-1)^j[v_{i_0}, ..., \hat{v}_{i_j}, ..., v_{i_{k+1}}]$. To this face there corresponds a map $h^{k-2}_{i_0,....,\hat{i}_j,...,i_{k+1}} : D^{k-2} \longrightarrow Z_2(M^n, \mathbf{Z})$. Now consider a map $f_i^{k-2} : D^{k-2} \longrightarrow Z_2(M^n, \mathbf{Z})$, such that $f_i^{k-2}(r,\theta) = \Sigma_{j=0}^{k+1}(-1)^j h^{k-2}_{i_0,....,\hat{i}_j,...,i_{k+1}}(1-r,\theta)$. Note that $f_i^{k-2}(1,\theta)$ is the zero cycle. Therefore, in reality this map is a map from S^{k-2} to the space of cycles. By our assumption this map is contractible, thus we obtain $h_i^{k-1} : D^{k-1} \longrightarrow Z_2(M^n, \mathbf{Z})$. We will assign this map to simplex σ_i^{k-1}.

We will continue in the above manner until we construct maps f_i^{q-2} in Case (i) and maps f_i^{n-2}s in Case (ii). We will call these maps $\tilde{g}_i$. The sum of these maps is homotopic to the map f constructed in Step 1.

Step 3. We can conlude that one of the maps $\tilde{f}_i$ is not contractible. Therefore, another application of Theorem 4.10 of [P] (or, more precisely, of its proof) implies that there exists a non-trivial stationary integral varifold with the mass as claimed in Theorem 1.3.

If $n = 3$, then the regularity assertion follows from Theorem 7.12 in [P]. □

4 Proof of Theorem 1.4

The proof is similar to that of Theorem 1.3.

Proof. The proof is by contradiction. We will assume that there is no stationary varifold as in Theorem 1.4. As in the 1.3 we are going to discuss cases (i) and (ii) at the same time. Theorem will be proved in three steps.
Step 1. Let $f : S^q \longrightarrow M^n$ be a non-contractible map in Case (i) and let $Id : 2004-006.tex, v1.12004/05/2717:35:19 levyExplevy$ be the identity map in Case (ii). We are going to construct a non-contractible map $\tilde{f} : S^{q-k} \longrightarrow Z_k(M^n, \mathbf{Z})$ in Case (i) and the map $\tilde{f} : S^{n-k} \longrightarrow Z_k(M^n, \mathbf{Z})$ in Case (ii). (Here and below $Z_k(M^n, \mathbf{Z})$ denotes the space of integral k-cycles on M^n.)
Step 2. We will construct maps $\tilde{g}_i$ from S^{q-k} (or S^{n-k}) to the space of k-dimensional cycles on M^n, such that the map $\tilde{f}$ of Step 1 will be homotopic to their sum. The masses of k-cycles $\tilde{g}_i(t)$ do not exceed the right hand side in the inequality (i) (or (ii)) in Theorem 1.4.
Step 3. We will conclude that, one of the maps $\tilde{g}_i$ constructed on Step 2 is not contractible, and therefore the results of J. Pitts imply the existence of a stationary varifold as in Theorem 1.4 thereby obtaining a contradiction.

Here is a detailed description of the first two steps of the proof.

Step 1. Consider an arbitrary simplex $\tilde{\sigma}_i^{k+1} = \tilde{v}_{i_0}, ..., \tilde{v}_{i_{k+1}}]$, (in general an arbitrary simplex of dimension l will be sometimes denoted as $\sigma^l_{i_0,...,i_l}$ to keep track of the vertices that generate it). This simplex lies in the $(k+1)$-skeleton of $f(S^q)$ in Case (i) and in the $(k+1)$-skeleton of M^n in Case (ii). Without any loss of generality we can assume that this simplex is generated by a volume decreasing homotopy $\tilde{h}_i^1 : [0,1] \longrightarrow Z_k(M^n, \mathbf{Z})$ that connects its boundary with a point $\tilde{p}_i^{k+1}$, (the homotopy will be sometimes denoted as $\tilde{h}^1_{i_0,...,i_{k+1}}$, and the point will be sometimes denoted as $\tilde{p}^{k+1}_{i_0,...,i_{k+1}}$). We can define a correspondence between $\tilde{\sigma}_i^{k+1}$ and the map h_i^1. Now consider a $(k+2)$-dimensional simplex $\tilde{\sigma}_i^{(k+2)} = [\tilde{v}_{i_0}, ..., \tilde{v}_{i_{k+2}}]$. Each face $(-1)^j[\tilde{v}_{i_0}, ..., \hat{\tilde{v}}_{i_j}, ..., \tilde{v}_{i_{k+2}}]$ of its boundary corresponds to the map $(-1)^j \tilde{h}^1_{i_0,...,\hat{i}_j,...,i_{k+2}}$. Define a new map $\tilde{f}_i^1 : [0,1] \longrightarrow Z_k(M^n, \mathbf{Z})$ by letting $\tilde{f}_i^1(t) = \Sigma_{j=0}^{k+2}(-1)^j \tilde{h}^1_{i_0,...,\hat{i}_j,...,i_{k+2}}(1-t)$. Note that $\tilde{f}_i^1(0) = \tilde{f}_i^1(1)$ and is the zero cycle, thus the newly constructed map is a loop in the space of k-cycles. Since all the simplices in the triangulation of S^q or M are small, this loop is contractible. The contraction amounts to slicing the $(k+2)$-dimensional simplex into k-cycles, so that this slicing extends the slicing of the boundary.

(It can be explicitly defined using a coning of the slicing of the boundary.) That allows us to obtain a disc $\tilde{h}_i^2 : D^2 \longrightarrow Z_k(M^n, \mathbf{Z})$, and so we can define a correspondence between simplex $\tilde{\sigma}_i^{k+2}$ and the map $\tilde{h}_i^2$.

Now suppose we have constructed maps $\tilde{h}_i^{l-k} : D^{l-k} \longrightarrow Z_k(M^n, \mathbf{Z})$ that correspond to l-dimensional simplices $\tilde{\sigma}_i^l$. Consider an arbitrary $(l+1)$-dimensional simplex $\tilde{\sigma}_i^{l+1} = [\tilde{v}_{i_0}, ..., \tilde{v}_{i_{l+1}}]$. Define the map $\tilde{f}_i^{l-k} : D^{l-k} \longrightarrow Z_k(M^n, \mathbf{Z})$ by letting $\tilde{f}_i^{l-k}(r,\theta) = \Sigma_{j=0}^{l+1}(-1)^j \tilde{h}^{l-k}_{i_0,...,\hat{i}_j,...,i_{l+1}}(1-r,\theta)$. This map corresponds to the boundary of $\tilde{\sigma}_i^{l+1}$. Note that $\tilde{f}_i^{l-k}(1,\theta)$ is the zero cycle, and thus it is a map from the $(l-k)$-dimensional sphere to the space of integral k-cycles. This sphere is contractible over the disc $\tilde{h}_i^{l+1-k} : D^{l+1-k} \longrightarrow Z_k(M^n, \mathbf{Z})$ in the space of "small" integral k-cycles. This contraction amounts to extending the slicing of the boundary of the considered $(l+1)$-dimensional simplex to its interior, and can be explicitly constructed just by the coning. We continue in the above manner until we construct the maps $\tilde{h}_i^{q-k}$s in case (i) and the maps $\tilde{h}_i^{n-k}$s in case (ii). Take the sum of those maps over all the simplices in the triangulation to obtain sphere $\tilde{f} : S^{q-k} \longrightarrow Z_k(M^n, \mathbf{Z})$ in the space of integral k-cycles in Case (i) and $\tilde{f} : S^{n-k} \longrightarrow Z_k(M^n, \mathbf{Z})$ in Case (ii).

Step 2. (The main step.) We will begin by extending the map $f : S^q \longrightarrow M^n$ to the 2-skeleton of D^3 (Case (i)) or by extending the map $Id : 2004-006.tex, v1.12004/05/2717:35:19levyExplevy$ to the 1-skeleton of W (Case (ii)). In both of those cases the procedure is identical to that of Step 2 in the proof of Theorem 1.3. After that we will establish a correspondence between simplices σ_i^l, where $2 \leq l \leq k$ and singular chains of the corresponding dimension on M^n. Let us consider an arbitrary 2-simplex $\sigma_i^2 = [v_{i_0}, v_{i_1}, v_{i_2}]$. Its boundary is mapped to a curve of length $\leq 2d + \delta$ (Case (i)) or of length $\leq 6FillRadM^n + 9\delta$ (Case (ii)). In the future we will denote it l_c.

Let s_i^2 be a singular surface of smallest area that has this curve as its boundary. We will assign this surface to this simplex. The area of this surface will be $\leq FH_2(l_c)$. Now consider an arbitrary 3-simplex σ_i^3. Its boundary is assigned a 2-cycle $S_i^2 = \Sigma_{j=0}^3(-1)^j s^2_{i_0,..,\hat{i}_j,...,i_3}$ of area $\leq 3FH_2(l_c) + \delta$ in Case (i) and of area $\leq 4FH_2(l_c)$ in Case (ii). Let s_i^3 be a singular 3-chain of smallest area that has S_i^2 as its boundary. Its volume is at most $FH_3(3FH_2(l_c)+\delta)$ in case (i) and at most $FH_3(4FH_2(l_c))$ in case (ii). Next suppose that for each arbitrary simplex σ_i^l, where $l \leq k-1$ we have constructed a chain s_i^l of volume $\leq FH_l(lFH_{l-1}((l-1)FH_{l-2}(.....)+\delta)+\delta)$ in Case (i) and $\leq FH_l((l+1)FH_{l-1}(lFH_{l-2}((l-1)...)))$ in Case (ii). Consider an arbitrary $(l+1)$-simplex σ_i^l. For each face $(-1)^j[v_{i_0},...,\hat{v}_{i_j},...,v_{i_{l+1}}]$ in its boundary there exists a preassigned singular chain $(-1)^j s_{i_0,...,\hat{i}_j,...,i_{l+1}}$. Consider a cycle $S_i^l = \Sigma_{j=0}^{l+1}(-1)^j s_{i_0,...,\hat{i}_j,...,i_{l+1}}$. Find a singular $(l+1)$-chain of s_i^{l+1}, such that $\partial s_i^{l+1} = S_i^l$. This chain will be assigned to σ_i^l. We should continue in the above manner until we reach the k-skeleton of D^q or respectively of W.

Now consider an arbitrary $(k+1)$-simplex σ_i^{k+1}. Each face in its boundary $(-1)^j[v_{i_0,...,\hat{i}_j,...,i_{k+1}}]$ corresponds to the singular chain $(-1)^j s_{i_0,...,\hat{i}_j,...,i_{k+1}}$. Consider the following cycle $S_i^{k+1} = \Sigma_{j+0}^{k+1}(-1)^j s_{i_0,...,\hat{i}_j,...,i_{k+1}}$. This is an element of $Z_k(M^n, \mathbf{Z})$ of volume $\leq (k+1)FH_k(kFH_{k-1}((k-1)...)+\delta)+\delta$ in Case (i) and of volume $\leq (k+2)FH_k((k+1)FH_{k_1}(k...))$. By our assumption this cycle can be connected with the zero cycle with a path that will only pass through the cycles of smaller volume. Let us denote this path by $h_i^1 : [0,1] \longrightarrow Z_k(M^n, \mathbf{Z})$. We will assign this path to the above simplex σ_i^{k+1}. Now suppose we have constructed the maps $h_i^{l-k} : D^{l-k} \longrightarrow Z_k(M^n, \mathbf{Z})$ corresponding to simplices σ_i^l, where $l \leq q-1$ in Case (i) and $\leq n-1$ in Case (ii). Consider an arbitrary $(l+1)$-dimensional simplex σ_i^{l+1}. Each face $(-1)^j[v_{i_0},...,\hat{v}_{i_j},...,v_{i_{l+1}}]$ corresponds to a map $(-1)^j h^{l-k}_{i_0,...,\hat{i}_j,...,i_{l+1}}(r,\theta)$. Define a new map $f_i^{k-l}(r,\theta) = \Sigma_{j=0}^{l+1}(-1)^j h^{l-k}_{i_0,...,\hat{i}_j,...,i_{l+1}}(1-r,\theta)$. Note that $f(1,\theta)$ is the zero cycle, and thus, this is a map from S^{l-k} to the space of k-cycles. By our assumption this map is contractible and this is how we obtain $h_i^{l-k+1} : D^{l-k+1} \longrightarrow Z_k(M^n, \mathbf{Z})$. We continue in the above manner until we construct maps f_i^{q-k} in case (i) or the maps f_i^{n-k} in case (ii). Those are our maps $\tilde{g}_i' s$. Note that $\tilde{f}$ is homotopic to their sum.

$\square$

Acknowledgements: A part of this work was done at MSRI, Berkeley during our visit in April-May, 2004. We would like to thank MSRI for its warm hospitality.

References

[A] F. Almgren, The homotopy groups of the integral cycle groups, Topology 1(1962), 257-289.

[CD] T. Colding, C. De Lellis, The min-max construction of minimal surfaces, Surv. in Diff. Geom., 8(2003), 75-107.

[Gr] M. Gromov, Filling Riemannian Manifolds, J. Diff. Geom. 18(1983), 1-147.

[NR1] A. Nabutovsky, R. Rotman, Volume, diameter and the minimal mass of a stationary 1-cycle, to appear in GAFA.

[NR2] A. Nabutovsky, R. Rotman, The area of a minimal embedded 2-sphere in a manifold diffeomeorphic to S^3, IMRN, 2003, no. 39, 2121-2129.

[P] J. Pitts, Existence and regularity of minimal surfaces on Riemannian manifolds, Math. Notes, 27, Princeton Univ. Press, 1981.

[Si] L. Simon, Survey lectures on minimal submanifolds, in "Seminar on Minimal submanifolds", ed. by E. Bombieri, Ann. Math. Studies, 103, Princeton University Press, 1983, pp. 3-52.

[SS] L. Simon, R. Schoen, Regularity of stable minimal hypersurfaces, Comm. Pure Appl. Math. 34(1981), 741-797.

[SSY] L. Simon, R. Schoen, S.T. Yau, Curvature estimates for minimal hypersurfaces", Acta Math. 134(1975), 276-288.

[S] F. Smith, On the existence of embedded minimal 2-sphere in the 3-sphere, endowed with arbitrary metric, Ph. D. Thesis, University of Melbourne, Melbourne.

A. Nabutovsky
Department of Mathematics
The Pennsylvania State University,
University Park, PA 16802,
USA
e-mail: nabutov@math.psu.edu

and
Department of Mathematics
University of Toronto
Toronto, Ontario M5P 3G3
Canada
e-mail: alex@math.toronto.edu

R. Rotman
Department of Mathematics
The Pennsylvania State University,
University Park, PA 16802,
USA
e-mail: rotman@math.psu.edu

and
Department of Mathematics
University of Toronto
Toronto, Ontario M5P 3G3
Canada
e-mail: rina@math.toronto.edu