SMOOTHLY PARAMETERISED ČECH COHOMOLOGY OF COMPLEX MANIFOLDS

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Abstract. A Stein covering of a complex manifold may be used to realise its analytic cohomology in accordance with the Čech theory. If, however, the Stein covering is parameterised by a smooth manifold rather than just a discrete set, then we construct a cohomology theory in which an exterior derivative replaces the usual combinatorial Čech differential. Our construction is motivated by integral geometry and the representation theory of Lie groups.

1. Introduction

The usual language of Čech cohomology is adapted for discrete coverings $\{U_i\}_{i\in I}$: the parameter space I has no particular structure. In complex analysis, however, it is typical to consider infinite coverings $\{U_\xi\}_{\xi\in\Xi}$ of a complex manifold Z by open Stein subsets which are themselves parameterised by points of an auxiliary smooth manifold Ξ. In such a setting, it is unnatural to forget the nature of the parameter space and treat the parameters as discrete. Instead, it is natural to consider de Rham cohomology on Ξ depending holomorphically on points in elements of the covering. The exact details are in the following section. It is the smoothly parameterised Čech cohomology introduced in [12] and further discussed in [13]. In suitable very general circumstances, we obtain the analytic cohomology $H^p(Z,\mathcal{O})$ of Z. This transfer from the usual combinatorial Čech language to de Rham cohomology on the parameter space Ξ with holomorphic coefficients in U_ξ is akin to the transition from integral sums to integrals.

Smoothly parameterised Čech cohomology provides an alternative to the classical Čech and Dolbeault realisations of $H^p(Z,\mathcal{O})$. In fact, not only is it one more natural way to present analytic cohomology but, as we will see in examples, it is a very effective way to connect analytic cohomology with the geometry of complex manifolds.

Let us emphasise that using continuous Čech cohomology instead of discrete, we have a very explicit way to construct an operator from Čech cohomology to Dolbeault cohomology using a smooth 'section', viz. $\gamma : Z \to \Xi$ such that $z \in U_{\gamma(z)}$, $\forall z \in Z$. This direct link with Dolbeault cohomology is unavailable in usual Čech theory.

The first aim of this article is to make this informal discussion completely precise and to state the technical conditions under which the p^{th} cohomology of the resulting 'de Rham complex with holomorphic coefficients' is $H^p(Z,\mathcal{O})$. In fact, the proof

2000 *Mathematics Subject Classification.* Primary 32L10; Secondary 22E46, 32L25, 43A85.

Support from the Australian Research Council and the Mathematical Sciences Research Institute is gratefully acknowledged. Research at MSRI is supported in part by NSF grant DMS-9810361. Simon Gindikin is partially supported by NSF grant DMS-0070816.

extends to a considerably more general setting. In effect, the covering need not be by open subsets but, rather, by Stein submanifolds. At the same time, the parameter space Ξ is not strictly necessary: the appropriate submersion is replaced by a foliation. Our general formulation also deals with the case when the parameter space Ξ is itself a complex manifold: we obtain the theory established in [6, 7]. The main analytic input for our general proof is due to Jurchescu [19]. Other ingredients for our construction are taken from [4]. The results of this article were roughly sketched in [3]—here we provide complete details.

We state our basic result and prove it in a more general setting than for smoothly parameterised Čech cohomology. There are two reasons. Firstly, the general proof is more natural. Secondly, we expect the general machinery to apply to examples arising naturally in representation theory where there is no smoothly parameterised Stein covering. In the interests of readers only concerned with the smoothly parameterised Čech theory, we carefully define this basic theory in §2 and outline its proof in §5. The full proof is a consequence of the general formulation and reasoning in §4.

Our primary motivation is the study of cohomology of tubes over non-convex cones and the explicit representations this cohomology provides. We provide some examples of our construction in this setting arising from pseudo-Hermitian symmetric spaces.

2. Formulation of the smooth Čech theory

This section contains a formulation of smoothly parameterised Čech cohomology and a statement of our main result, Theorem 1. It provides sufficient conditions that the smoothly parameterised Čech theory compute the usual analytic cohomology.

Let Z be a complex manifold whose cohomology $H^p(Z,\mathcal{O})$ we wish to compute. Suppose we are given an open covering $\{U_\xi\}_{\xi\in\Xi}$ of Z where Ξ is a smooth manifold. We need a sense in which U_ξ depends smoothly on ξ. For this, let us introduce

$$M=\{(\xi,z)\in\Xi\times Z \text{ s.t. } z\in U_\xi\}\subseteq\Xi\times Z$$

and simply insist that M be an open subset of $\Xi\times Z$.

2.1. **First example.** A useful example to bear in mind is $Z=\mathbb{C}^n\setminus\mathbb{R}^n$. In this case, we may take

$$U_\xi=\{z=x+iy\in\mathbb{C}^n \text{ s.t. } \langle\xi,y\rangle>0\},\quad \text{for } \xi\in\Xi=\{\xi\in\mathbb{R}^n \text{ s.t. } |\xi|=1\} \tag{1}$$

as an especially natural covering. Notice that each U_ξ is convex and hence Stein. Also, for each $z\in Z$ the set $\{\xi\in\Xi \text{ s.t. } z\in U_\xi\}$ is a hemisphere. Thus, M is an open subset of $S^{n-1}\times Z$ and the natural projection $M\to Z$ has contractible fibres. A link between the smoothly parameterised Čech theory in this case and Sato's cohomological description of hyperfunctions on $\mathbb{R}^n$ is provided in [3, 13].

2.2. **Second example.** Let $Z=\mathbb{CP}_n\setminus B$, the complement of a closed ball in complex projective space:–

$$Z=\{[z_0,z_1,\dots,z_n]\in\mathbb{CP}_n \text{ s.t. } |z_0|^2<|z_1|^2+\cdots+|z_n|^2\}.$$

Consider $\xi \in \mathrm{Gr}_{n-1}(\mathbb{C}^{n+1})$ as a linearly embedded $\mathbb{CP}_{n-2} \hookrightarrow \mathbb{CP}_n$ and let Ξ denote those that avoid B. Then, for each $\xi \in \Xi$ we may take

$$U_\xi = \{z \in Z \text{ s.t. the join of } z \text{ and } \xi \text{ continues to avoid } B\}.$$

This is a smoothly parameterised Stein covering of Z. Even better, Ξ is itself a complex manifold and the holomorphic language of [6, 7, 11] may be employed. This language may be employed easily to invert the Penrose-Radon transform (the case $n = 2$ is detailed in [3]).

2.3. **Third example.** Let $V \subset \mathbb{R}^3$ be the following non-convex cone:–

$$V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \text{ s.t. } {x_1}^2 + {x_2}^2 > {x_3}^2\}.$$

It is one of the open orbits of the standard action of $\mathrm{SO}(2,1)$ on $\mathbb{R}^3$ (the other two being convex cones). Let $Z = V + i\mathbb{R}^3 \subset \mathbb{C}^3$ be the corresponding tube domain. Let Ξ be the unit circle and for $\theta \in \Xi$ take

$$U_\theta = \{z = x + iy \in \mathbb{C}^3 \text{ s.t. } x_1 \cos\theta + x_2 \sin\theta > |x_3|\}.$$

It is a tube over a convex cone. Thus, we have a Stein covering smoothly parameterised by Ξ. Notice, however, that this covering is not preserved by the action of $\mathrm{SO}(2,1)$. The Stein covering

$$U_{\theta,\phi} = \{x + iy \text{ s.t. } x_1 \cos\theta + x_2 \sin\theta > x_3 \text{ and } x_1 \cos\phi + x_2 \sin\phi > -x_3\}$$

for $\theta \neq \phi$ remedies this.

These examples illustrate various points. In §2.1 a finite Stein covering by half spaces is available but cannot respect the evident rotational symmetry of Z. The smoothly parameterised covering is certainly more natural from this point of view. The holomorphically parameterised covering in §2.2 is perhaps the best choice. But, for the complex manifolds in §2.1 and §2.3, the smoothly parameterised coverings seem to be optimal. Thus, we are obliged to extend the holomorphic machinery of [6, 7] to the smooth case.

Returning to the general discussion of the smoothly parameterised case, it is convenient to introduce submersions η and τ

$$\begin{tikzcd} & M \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ Z & & \Xi \end{tikzcd} \tag{2}$$

as restrictions of the natural projections to M. Locally, on $\Xi \times Z$ we may consider complex-valued smooth functions $f = f(\xi, z)$ that are holomorphic in z for fixed ξ. We shall say that such functions are *partially holomorphic* and write Œ for the sheaf of germs thereof. On $\Xi \times Z$ and hence also on M, we may consider p-forms in the ξ-variables alone: precisely, if we let Λ^p_Ξ denote the bundles of p-forms on Ξ and set $B^p = \tau^*\Lambda^p_\Xi$, then such p-forms are smooth sections of B^p. Geometrically, these are p-forms along the fibres of η. We shall also refer to them as p-forms *relative* to η, or simply relative p-forms if η is understood. In smooth local coördinates $(\xi_1, \ldots, \xi_m)$

on Ξ and holomorphic local coördinates $(z_1,\ldots,z_n)$ on Z, relative p-forms may be written

$$\sum_{i_1,\ldots,i_p}\omega_{i_1\cdots i_p}(\xi_1,\ldots,\xi_m,z_1,\ldots,z_n)\,d\xi_{i_1}\wedge\cdots\wedge d\xi_{i_p}$$

but it makes coördinate-free sense to require that all the coefficients be holomorphic in $z_1,\ldots,z_n$. We shall refer to such relative p-forms as partially holomorphic and write $\text{Œ}(B^p)$ for the sheaf of germs thereof. It is easily verified that the exterior derivative in the ξ-variables alone, given by the usual formula with $z\in Z$ as a passenger, is well-defined and takes partially holomorphic relative p-forms to partially holomorphic relative $(p+1)$-forms. This is called the relative exterior derivative and is written d_η. Of course, ${d_\eta}^2=0$. In summary, we have a complex of sheaves on M

$$0\to\text{Œ}\xrightarrow{d_\eta}\text{Œ}(B^1)\xrightarrow{d_\eta}\text{Œ}(B^2)\to\cdots\to\text{Œ}(B^p)\xrightarrow{d_\eta}\text{Œ}(B^{p+1})\to\cdots$$

and we may define the p^{th} smooth Čech cohomology to be

$$\frac{\ker:\Gamma(M,\text{Œ}(B^p))\xrightarrow{d_\eta}\Gamma(M,\text{Œ}(B^{p+1}))}{\text{im}:\ \Gamma(M,\text{Œ}(B^{p-1}))\xrightarrow{d_\eta}\Gamma(M,\text{Œ}(B^p))}.$$

In order that this coincide with the analytic cohomology $H^p(Z,\mathcal{O})$, we need a smooth analogue of the Leray condition from usual (discrete) Čech cohomology. Roughly speaking, we would like each U_ξ to be Stein but in a way that depends smoothly on $\xi\in\Xi$. To make this precise, we define the *partially holomorphic hull* $\widehat{K}_M$ of a compact subset $K\subset M$ to be

$$\widehat{K}_M=\{x\in M\text{ s.t. }|f(x)|\le\sup_K|f|\ \ \forall f\in\Gamma(M,\text{Œ})\}. \tag{3}$$

Then M is a *Cartan manifold* in the sense of Jurchescu [19] if and only if the following three conditions hold.

- if $K\subset M$ is compact, then so is $\widehat{K}_M$;
- the partially holomorphic functions on M separate points;
- the partially holomorphic functions on M provide local coördinates.

Specifically, local coördinates $(\xi_1,\ldots,\xi_m,z_z\ldots,z_n)$ are each partially holomorphic and the third requirement of a Cartan manifold is that charts may be chosen in which the coördinate functions extend to global partially holomorphic functions. If M is an open subset of $\Xi\times\mathbb{C}^n$, then the final two conditions are evidently satisfied. In this case, a Cartan manifold may reasonably be called a 'domain of partial holomorphy'.

For a smoothly parameterised Leray theorem, some topological restriction is also necessary. For each $z\in Z$, the relative de Rham sequence is trying to compute the de Rham cohomology of the fibre $\eta^{-1}(z)$. We need to eliminate this effect:–

Theorem 1. *Let Z be a complex manifold endowed with a smoothly parameterised open cover viewed as* (2)*. Suppose M is a Cartan manifold and η has contractible fibres. Then*

$$H^p(Z,\mathcal{O})\cong H^p(\Gamma(M,\text{Œ}(B^\bullet))),\ \forall p.$$

This theorem is a special case of a much more general result, Theorem 4, whose formulation and proof will occupy the next two sections.

In order for Theorem 1 to be useful, of course, we need a good supply of Cartan manifolds:–

Proposition 1. *Let Ξ be a smooth manifold. If $M \subset \Xi \times \mathbb{C}^n$ is open and the natural projection $\tau : M \to \Xi$ has convex fibres, then M is Cartan.*

Proof. It suffices to check partial holomorphic convexity: the other two requirements are inherited from $\Xi \times \mathbb{C}^n$. For each linear functional $\ell : \mathbb{C}^n \to \mathbb{C}$ and $\phi(\xi)$ a smooth bump function on Ξ, the function

$$(\xi, z) \mapsto \phi(\xi) \exp \ell(z)$$

is partially holomorphic. Hence, the partially holomorphic hull of any subset $K \subset M$ is contained in its fibrewise convex hull. □

Notice that if $M \subset \Xi \times \mathbb{C}^n$ is a Cartan manifold, then the fibres of $\tau : M \to \Xi$ are Stein. The converse, however, is false:–

$$M = \mathbb{R} \times \mathbb{C} \setminus \{(0,0)\} \xrightarrow{\tau} \mathbb{R}$$

has Stein fibres but is not Cartan. The Cauchy integral formula shows that partially holomorphic functions extend across the origin. Consequently,

$$K = \{(\xi, z) \in M \text{ s.t. } |\xi| = 1,\ |z| \leq 1\} \implies \widehat{K}_M = \{(\xi, z) \in M \text{ s.t. } |\xi| \leq 1,\ |z| \leq 1\},$$

which is not compact.

Suppose the hypotheses of Theorem 1 are satisfied and we are given $\gamma : Z \to M$ a smooth section of the submersion $\eta : M \to Z$. We may restrict forms on Z to the image of this section, consider the result as a form on Z, and take the $(0,p)$-part. This procedure gives a more explicit interpretation of Theorem 1 in terms of Dolbeault cohomology. Formally, we may proceed as follows. A smooth section ω of B^p is, in particular, a p-form on M whence its pullback $\gamma^*\omega$ is a p-form on Z. Let $\mathcal{E}_Z^{0,p}$ denote the sheaf of smooth forms on Z of type $(0,p)$ and define

$$\Gamma(M, \text{Œ}(B^p)) \longrightarrow \Gamma(Z, \mathcal{E}_Z^{0,p}) \tag{4}$$

by $\omega \mapsto (\gamma^*\omega)^{0,p}$, the $(0,p)$-component of $\gamma^*\omega$. As M is locally a product, we can split the exterior derivative into vertical and horizontal parts:–

$$d\omega = d_\eta\omega + d_\tau\omega = d_\eta\omega + \partial_\tau\omega + \bar{\partial}_\tau\omega,$$

where d_τ is further split according to type. That ω is partially holomorphic is to say $\bar{\partial}_\tau\omega = 0$. Evidently, $\gamma^*\partial_\tau\omega$ is of type $(1,p)$ on Z. Therefore,

$$(\gamma^* d_\eta\omega)^{(0,p+1)} = (\gamma^* d\omega)^{(0,p+1)} = (d(\gamma^*\omega))^{(0,p+1)} = \bar{\partial}((\gamma^*\omega)^{(0,p)}).$$

In other words, (4) is a homomorphism of complexes. The right hand side is the Dolbeault complex on Z. The following result is a special case of Theorem 6 in §4.

Theorem 2. *If the hypotheses of Theorem 1 are satisfied and $\gamma : Z \to M$ is an arbitrary smooth section of the submersion $\eta : M \to Z$, then the homomorphism* (4) *induces an isomorphism on cohomology.*

To close this discussion, let us revisit our three examples. In §2.1 the covering (1) of $\mathbb{C}^n \setminus \mathbb{R}^n$ satisfies the conditions of Theorem 1 and admits a natural smooth section γ. We have observed already that the fibres of η are, as hemispheres, contractible. That

$$M = \{(\xi, z = x + iy) \in \mathbb{R}^n \times \mathbb{C}^n \text{ s.t. } |\xi| = 1 \text{ and } \langle \xi, y \rangle > 0\}$$

is a Cartan manifold follows immediately from Proposition 1. For $\gamma : \mathbb{C}^n \setminus \mathbb{R}^n \to M$ we may take $z = x + iy \mapsto \xi = y/|y|$.

In §2.2 the corresponding manifold M is Stein, as it should be in the case of a holomorphically parameterised covering. For $\gamma : Z \to M$ we may take

$$z = [z_0, z_1, \ldots, z_n] \xmapsto{\gamma} (\{[0, \xi_1, \ldots, \xi_n] \in \mathbb{CP}_n \text{ s.t. } \bar{z}_1 \xi_1 + \cdots + \bar{z}_n \xi_n = 0\}, z)$$

and the fibres of η are easily seen to contract onto this section. Note, however, that γ is only $\mathrm{SU}(n)$-invariant, whereas the full symmetry group is $\mathrm{SU}(1, n)$.

In §2.3 with either of the two coverings, the relevant manifold M is Cartan from Proposition 1 and for $\gamma : Z \to M$ we may take

$$\gamma(x + iy) = (\arctan(x_2/x_1), x + iy) \quad \text{or} \quad (\arctan(x_2/x_1), \arctan(x_2/x_1), x + iy),$$

easily checking that the fibres of η contract onto its image. This section γ is only $\mathrm{SO}(2)$-invariant, whilst the full symmetry group for the second covering is $\mathrm{SO}(2, 1)$.

3. Mixed manifolds

The reader only interested in smoothly parameterised Čech cohomology and its applications will find a summary of the proof for this case in §5.

A *mixed manifold* M of type (m, n) is a smooth manifold of dimension $m + 2n$ equipped with a Levi flat CR structure of codimension m. More precisely, we are given an integrable distribution $H \subset TM$ of rank $2n$ and an endomorphism $J : H \to H$ whose Nijenhuis tensor,

$$N(X, Y) \equiv [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] \quad \text{for } X, Y \in \Gamma(M, H),$$

vanishes. Thus, M is equipped with a foliation of dimension $2n$ with smoothly varying complex structure on the leaves. If we set

$$T_M^{0,1} = \{X \in \mathbb{C}H \text{ s.t. } (J + i)X = 0\},$$

then

$$T_M^{0,1} \oplus \overline{T_M^{0,1}} = \mathbb{C}H \quad \text{and} \quad [T_M^{0,1}, T_M^{0,1}] = T_M^{0,1},$$

where this second statement means that $T_M^{0,1}$ is closed under Lie bracket. Equivalently, we may set $\Lambda_M^{1,0} = (T_M^{0,1})^\perp$ and ask that this generate a differentially closed ideal.

By the Newlander-Nirenberg theorem [20], mixed manifolds are locally modelled on $\mathbb{R}^m \times \mathbb{C}^n$ with transition functions of the form

$$(\xi, z) \mapsto (s(\xi), w(\xi, z)),$$

where $w(\xi, z)$ is holomorphic in z. It is evident that mixed manifolds provide a natural generalisation of the open subsets $M \subset \Xi \times Z$ that we encountered in the previous section. Various notions extend immediately. *Partially holomorphic* functions on a general mixed manifold M are smooth functions whose restriction to the leaves of the foliation are holomorphic. In other words, these are the smooth complex-valued functions annihilated by vector fields from $T^{0,1}_M$. As before, we shall write Œ for the sheaf of germs of partially holomorphic functions. If we define the bundle of $(0,1)$-forms on M by $\Lambda^{0,1}_M := \Lambda^1_M/\Lambda^{1,0}_M$ and write $\mathcal{E}^{0,1}_M$ for the corresponding sheaf of smooth sections, then we have a resolution

$$0 \to \text{Œ} \to \mathcal{E}^{0,0}_M \xrightarrow{\bar\partial} \mathcal{E}^{0,1}_M \xrightarrow{\bar\partial} \mathcal{E}^{0,2}_M \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{E}^{0,n}_M \to 0, \tag{5}$$

which is the Dolbeault resolution on each leaf.

A mixed manifold of type $(0,n)$ is simply a complex manifold, in which case (5) is the Dolbeault resolution. The *partially holomorphic hull* $\widehat{K}_M$ of a compact subset $K \subset M$ is defined exactly as before by (3) and the definition of a *Cartan manifold* also reads the same. By analogy with the classical Levi problem on a Stein manifold, Jurchescu [19] proved the following vanishing theorem:–

Theorem 3. *If M is a Cartan manifold, then*

$$H^q(M, \text{Œ}) = 0, \ \forall q \geq 1.$$

An alternative proof may be found in [1]. We are grateful to Gennadi Henkin for drawing our attention to these articles.

4. General formulation and proof

Suppose M is a mixed manifold, Z is a complex manifold, and $\eta : M \to Z$ is a partially holomorphic submersion. More precisely, we shall suppose that η is a smooth surjection of maximal rank and that η is holomorphic on the leaves of the mixed manifold structure. We do not assume any particular relationship between the dimension of Z and the dimension of the complex leaves in M. Extreme special cases are when M is itself complex or when M is smooth (in which case it may be simply Z but regarded as a smooth manifold).

That η is maximal rank is to say that $\eta^*\Lambda^1_Z \to \Lambda^1_M$ is an injection. That η is partially holomorphic is to say that $\eta^*\Lambda^{1,0}_Z \subseteq \Lambda^{1,0}_M$. Define a vector bundle B^1 on M by the exact sequence

$$0 \to \eta^*\Lambda^{1,0}_Z \to \Lambda^{1,0}_M \to B^1 \to 0.$$

We shall see, in proving Theorem 4 below, that B^1 is naturally holomorphic on the leaves of the foliation. The same is true for $B^p \equiv \Lambda^p(B^1)$. We shall say that such bundles are partially holomorphic and write $\text{Œ}(B^p)$ for the sheaf of smooth sections of B^p that are holomorphic along the leaves. The following theorem generalises Theorem 1.

Theorem 4. *There is a complex of sheaves on M*

$$0 \to \text{Œ}(B^0) \xrightarrow{d_\eta} \text{Œ}(B^1) \xrightarrow{d_\eta} \text{Œ}(B^2) \to \cdots \to \text{Œ}(B^p) \xrightarrow{d_\eta} \text{Œ}(B^{p+1}) \to \cdots \tag{6}$$

so that if M is a Cartan manifold and the fibres of η are contractible, then

$$H^p(Z, \mathcal{O}) \cong H^p(\Gamma(M, \text{Œ}(B^\bullet))), \ \forall p. \tag{7}$$

Proof. In order to understand the bundle B^1 on M, let us consider the following commutative diagram with exact rows and columns.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \eta^*\Lambda_Z^{0,1} & \to & \Lambda_M^{0,1} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \to & \eta^*\Lambda_Z^1 & \to & \Lambda_M^1 & \to & \Lambda_\eta^1 & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \eta^*\Lambda_Z^{1,0} & \to & \Lambda_M^{1,0} & \to & B^1 & \to & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{8}
$$

In particular, the bundle Λ_η^1 is defined by exactness of the middle row and its sections are 1-forms along the fibres of η. Let us define a vector bundle A^1 on M by the exact sequence

$$0 \to \eta^*\Lambda_Z^{1,0} \to \Lambda_M^1 \to A^1 \to 0. \tag{9}$$

Then (8) gives rise to two short exact sequences involving A^1, namely

$$0 \to \eta^*\Lambda_Z^{0,1} \to A^1 \to \Lambda_\eta^1 \to 0 \tag{10}$$

and

$$0 \to B^1 \to A^1 \to \Lambda_M^{0,1} \to 0. \tag{11}$$

Observe that the short exact sequence (9) together with the de Rham complex $\Lambda_M^\bullet$ on M induces a differential complex $A^\bullet$ on M. Each of (10) and (11) now gives rise to a spectral sequence for computing the cohomology of $A^\bullet$ and Theorem 4 will be a consequence of combining these spectral sequences with appropriate vanishing results, namely Jurchescu's Theorem 3 and a theorem of Buchdahl [5].

The short exact sequence (11) induces a filtering of the complex $A^\bullet$ on M and, in particular, we recover the resolution (5) and, indeed, coupled resolutions

$$0 \to Œ(B^p) \to \mathcal{E}_M^{0,0}(B^p) \xrightarrow{\bar\partial} \mathcal{E}_M^{0,1}(B^p) \xrightarrow{\bar\partial} \mathcal{E}_M^{0,2}(B^p) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{E}_M^{0,n}(B^p) \to 0, \tag{12}$$

for each $p \geq 0$. In particular, we see that B^p is partially holomorphic, as presaged just prior to Theorem 4. The spectral sequence associated to this filtration of $A^\bullet$ is

$$E_1^{p,q} = H^q(M, Œ(B^p)) \Longrightarrow H^{p+q}(\Gamma(M, A^\bullet)). \tag{13}$$

As with Stein manifolds, Theorem 3 applies equally to the cohomology of partially holomorphic vector bundles on a Cartan manifold. Hence, the spectral sequence (13) collapses to an isomorphism

$$H^p(\Gamma(M, A^\bullet)) \cong H^p(\Gamma(M, Œ(B^\bullet))). \tag{14}$$

The short exact sequence (10) induces a different filtering of the complex $A^\bullet$ on M. In particular, the induced complexes $\mathcal{E}_\eta^\bullet(\eta^*\Lambda^{0,p})$ resolve the sheaves $\eta^{-1}\mathcal{E}^{0,p}$ of germs of smooth sections of $\eta^*\Lambda^{0,p}$ locally pulled back from Z. The differential operator

$d_\eta : \eta^*\Lambda^{0,p} \to \Lambda^1_\eta(\eta^*\Lambda^{0,p})$ is sometimes called a Bott partial connection. The spectral sequence associated to this alternative filtration of $A^\bullet$ is

$$E_1^{p,q} = H^q(M, \eta^{-1}\mathcal{E}^{0,p}) \Longrightarrow H^{p+q}(\Gamma(M, A^\bullet)). \tag{15}$$

In [5], Buchdahl shows that if η has connected fibres whose de Rham cohomology $H^q(\eta^{-1}(z), \mathbb{C})$ vanishes for $q \geq 1$, then $H^q(M, \eta^{-1}\mathcal{E}^{0,p}) = 0$ for $q \geq 1$. Hence, the spectral sequence (15) collapses to an isomorphism

$$H^p(\Gamma(M, \eta^{-1}\mathcal{E}^{0,\bullet})) \cong H^p(\Gamma(M, A^\bullet)). \tag{16}$$

Since η has connected fibres, sections of $\eta^*\Lambda^{0,p}$ that are locally pulled back from Z are, in fact, pulled back from Z. Hence, the left hand side of (16) is the Dolbeault cohomology of Z. Combining (16) and (14) gives

$$H^p(Z, \mathcal{O}) \cong H^p(\Gamma(M, \eta^{-1}\mathcal{E}^{0,\bullet})) \cong H^p(\Gamma(M, A^\bullet)) \cong H^p(\Gamma(M, \text{Œ}(B^\bullet))), \ \forall p,$$

as required. □

It is worth recording separately various special cases. First of all, there is the spectral sequence (13) combined with the isomorphism (16) when M is a general mixed manifold:–

Theorem 5. *Suppose that the fibres of η are contractible. Then, there is a spectral sequence*

$$E_1^{p,q} = H^q(M, \text{Œ}(B^p)) \Longrightarrow H^{p+q}(Z, \mathcal{O}).$$

There are also three interesting special cases of the theorem itself:–

- we take M to be Z but regarded as a smooth manifold: in this case $\text{Œ}(B^\bullet)$ reduces to the usual Dolbeault resolution;
- we take M to be a Stein manifold: in this case we obtain the holomorphic language of [6, 7, 11];
- we take $M \subset \Xi \times Z$ as in §2, supposing that M is a Cartan manifold: we obtain the smoothly parameterised Čech cohomology and Theorem 1.

As in §2, it is possible to be more explicit concerning the isomorphism (7) in the presence of a section:–

Theorem 6. *Suppose $\gamma : Z \to M$ is a smooth section of the submersion $\eta : M \to Z$. Then there is a mapping of complexes*

$$\Gamma(M, \text{Œ}(B^\bullet)) \longrightarrow \Gamma(Z, \mathcal{E}^{0,\bullet}) \tag{17}$$

that, if the hypotheses of Theorem 4 are satisfied, induces the isomorphism (7) *on cohomology.*

Proof. Our construction follows (4) except that more care has to be exercised, taking into account that M is no longer locally a product. From (11) and (9) we see that

$$\omega \in \Gamma(M, B^p) \hookrightarrow \Gamma(M, A^p)$$

may be lifted to a p-form $\widetilde{\omega}$ on M and then its pullback $\gamma^*\widetilde{\omega}$ is a p-form on Z. From (9) it follows that the $(0,p)$-component $(\gamma^*\widetilde{\omega})^{0,p}$ is independent of choice of lift. By construction, (17) is a composition

$$\Gamma(M, \text{Œ}(B^\bullet)) \longrightarrow \Gamma(M, A^\bullet) \longrightarrow \Gamma(Z, \mathcal{E}^{0,\bullet})$$

each of is evidently a mapping of complexes. In fact, the first of these has nothing to do with γ and, in the proof of Theorem 4, induces the isomorphism (14). On the other hand, (10) induces a chain mapping $\eta^* : \Gamma(Z, \mathcal{E}^{0,\bullet}) \to \Gamma(M, A^\bullet)$, which is the source of the isomorphism with Dolbeault cohomology when the appropriate topological conditions on the fibres of η are satisfied. Since $\eta \circ \gamma = \mathrm{Id}$, we see that $\gamma^* \circ \eta^* = \mathrm{Id}$ and, in this case, taking the $(0,p)$-component is evidently vacuous. $\square$

5. Summary

For those readers concerned only with the case of smoothly parameterised Čech cohomology, it is perhaps worthwhile to summarise the events of §§3–4 as they apply to this case. In effect, the proof boils down to the following. Let $\eta^{-1}\mathcal{O}$ denote the inverse image sheaf of $\mathcal{O}$. It is the sheaf of smooth functions on M locally of the form $f \circ \eta$ for f a holomorphic function on Z. Equivalently, they are locally constant along the fibres of f or, in other words, annihilated by d_η, the exterior derivative along the fibres of η. We obtain a resolution

$$0 \to \eta^{-1}\mathcal{O} \to \text{Œ}(B^0) \xrightarrow{d_\eta} \text{Œ}(B^1) \xrightarrow{d_\eta} \text{Œ}(B^2) \to \cdots$$

of $\eta^{-1}\mathcal{O}$ by sheaves that, by dint of Jurchescu's Theorem 3, are acyclic when M is Cartan. We conclude that

$$\frac{\ker : \Gamma(M, \text{Œ}(B^p)) \xrightarrow{d_\eta} \Gamma(M, \text{Œ}(B^{p+1}))}{\mathrm{im} : \Gamma(M, \text{Œ}(B^{p-1})) \xrightarrow{d_\eta} \Gamma(M, \text{Œ}(B^p))} \cong H^p(M, \eta^{-1}\mathcal{O}).$$

On the other hand, there is always a tautological homomorphism

$$H^p(Z, \mathcal{O}) \to H^p(M, \eta^{-1}\mathcal{O}),$$

which, by Buchdahl's theorem [5], is an isomorphism when η has contractible fibres.

6. Non-convex tube domains

In this section we discuss the application of Theorem 4 to describe the cohomology of tubes over non-convex cones, cf. [10]. Let $V \subset \mathbb{R}^n$ be an open cone (so $x \in V \implies \lambda x \in V$ for $\lambda > 0$). The V we have in mind need not be convex. In contrast to convex cones, non-convex cones can display very pathological behaviour and we need to restrict the class of cones under consideration. We fix a natural number s and consider cones which are unions of s-planes. More specifically, we shall employ smoothly parameterised Stein coverings obtained as tubes over some special convex subcones of V as follows. Let us call a cone W an s-wedge if it is the direct product of an s-subspace $d(W) \in \mathrm{Gr}_s(\mathbb{R}^n)$ and an $(n-s)$-dimensional sharp convex cone. More precisely, C should be a convex cone that does not contain a line in an $(n-s)$-dimensional linear subspace complementary to $d(W)$ and W should consist of those vectors in $\mathbb{R}^n$ whose projection onto this complementary subspace lie in C, in which case we shall write $W = d(W) \times C$. Let us say that a cone V is s-convex if it is the union of s-wedges W_ξ for $\xi \in \Xi$ some connected smooth manifold and

- the mapping $d : \Xi \to \mathrm{Gr}_s(\mathbb{R}^n)$ is a finite-to-one smooth covering with connected open range;
- for each $x \in V$, the set $\{\xi \in \Xi \text{ s.t. } W_\xi \ni x\}$ is non-empty contractible.

It is natural to expect for s-convex tubes that the interesting cohomology occurs in degree s. The notion of 0-convex coincides with convex and, certainly, it is natural to consider holomorphic functions since they provide an infinite-dimensional function space whilst higher cohomology vanishes. Under suitable conditions, it is possible to develop a very advanced theory of the cohomology $H^s(T, \mathcal{O})$ of the corresponding tube domain $Z = V + i\mathbb{R}^n$ analogous to Bochner's theory in convex tubes [13].

Definition *If $V \subset \mathbb{R}^n$ is an s-convex cone, we shall say that $V + i\mathbb{R}^n \subset \mathbb{C}^n$ is an <u>s-convex tube</u>.*

Theorem 7. *Suppose $Z = V + i\mathbb{R}^n$ is an s-convex tube. Define $M \subset \Xi \times Z$ by*

$$M = \{(\xi, z) \in \Xi \times Z \text{ s.t. } z \in W_\xi \times i\mathbb{R}^n\}.$$

Then M is a Cartan manifold and the natural projection $\eta : M \to Z$ has contractible fibres.

Proof. The fibre of the natural projection $\tau : M \to \Xi$ over ξ is $W_\xi + i\mathbb{R}^n$. This is convex, so Proposition 1 implies M is Cartan. The fibre of η over $z = x + iy \in Z$ is $\{\xi \in \Xi \text{ s.t. } W_\xi \ni x\}$. This is contractible by definition of s-convex. □

Immediately from Theorem 1, we conclude:–

Corollary 1. *The analytic cohomology of an s-convex tube may be computed by smooth Čech cohomology.*

Examples of s-convex tubes are provided by pseudo-Hermitian symmetric spaces of tube type [9]. Here we describe three such spaces:–

6.1. **First example.** Generalising §2.3, let $n = p + q$ with $p \geq 2$, introduce on $\mathbb{R}^n$ the non-degenerate non-degenerate symmetric form $\langle\ ,\ \rangle$ of type (p, q)

$$\langle x, y\rangle = x_1y_1 + \cdots + x_py_p - x_{p+1}y_{p+1} - \cdots - x_ny_n,$$

and set

$$V = \{x \in \mathbb{R}^n \text{ s.t. } \langle x, x\rangle > 0\}.$$

This is a connected non-convex cone and we maintain that is $(p-1)$-convex, as follows. Choose an orientation on one, and hence all, p-dimensional linear subspaces on which $\langle\ ,\ \rangle$ is positive definite. Let ξ denote an oriented $(p-1)$-dimensional linear subspace of $\mathbb{R}^n$ on which $\langle\ ,\ \rangle$ is positive definite. Then $\langle\ ,\ \rangle$ restricted to $\xi^\perp$ is a non-degenerate symmetric form of type $(1, q)$. Therefore,

$$C_\xi = \{x \in \mathbb{R}^n \text{ s.t. } \langle x, \xi\rangle = 0,\ \langle x, x\rangle > 0, \text{ and } \xi \oplus \mathbb{R}x \text{ has chosen orientation}\}$$

is a sharp convex cone, whence $W_\xi = \xi \times C_\xi$ is a $(p-1)$-wedge. In this way, V is covered by $(p-1)$-wedges parameterised by

$$\Xi = \left\{\xi \in \mathrm{Gr}^+_{p-1}(\mathbb{R}^n) \text{ s.t. } \langle\ ,\ \rangle|_\xi \text{ is positive definite}\right\}$$

where Gr^+ denotes the oriented Grassmannian. The mapping $d : \Xi \to \mathrm{Gr}_{p-1}(\mathbb{R}^n)$ simply forgets the orientation. Notice that the corresponding smoothly parameterised Čech covering of the $(s-1)$-convex tube $V + i\mathbb{R}^n$ respects the action of the natural symmetry group $\mathrm{SO}(p, q)$.

6.2. **Second example.** Let $n = m(m + 1)/2$ and identify $\mathbb{R}^n$ with the $m \times m$ real symmetric matrices. Let $m = p + q$ and set

$$V = \{X \in \mathbb{R}^n \text{ s.t. } X \text{ has signature } (p, q)\}.$$

Then $Z = V + i\mathbb{R}^n$ is a pq-convex tube. Matrices of the form

$$X = \begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix} + \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix},$$

where B is an arbitrary $p \times q$ matrix, P is $p \times p$ symmetric positive definite, and Q is $q \times q$ symmetric negative definite, form a pq-wedge and a covering by pq-wedges may be obtained by moving this one under the action $X \mapsto AXA^t$ of $\mathrm{SL}(m, \mathbb{R})$.

6.3. **Third example.** Let $n = m^2$, identify $\mathbb{R}^n$ with the $m \times m$ real matrices, and set

$$V = \{X \in \mathbb{R}^n \text{ s.t. } \det X > 0\}.$$

Then $Z = V + i\mathbb{R}^n$ is an $m(m - 1)/2$-convex tube. There is a basic wedge in V:–

$$W = \{X = S + P \text{ s.t. } S \text{ is skew and } P \text{ is symmetric positive definite}\}$$

and the general wedge in the smoothly parameterised covering is obtained from this one under the action $X \mapsto AXA^{-1}$ of $\mathrm{SL}(m, \mathbb{R})$.

6.4. **Example §2.3 revisited.** Whilst the tube over the non-convex cone

$$V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \text{ s.t. } {x_1}^2 + {x_2}^2 > {x_3}^2\}$$

admits a smoothly parameterised Stein covering, it also admits a foliation by closed Stein submanifolds as follows. Firstly, define $\pi : V \to S^1$ by

$$\begin{array}{rcl} (x_1, x_2, x_3) & \xrightarrow{\pi} & \left(\dfrac{x_1x_3 + x_2\sqrt{{x_1}^2 + {x_2}^2 - {x_3}^2}}{{x_1}^2 + {x_2}^2}, \dfrac{x_2x_3 - x_1\sqrt{{x_1}^2 + {x_2}^2 - {x_3}^2}}{{x_1}^2 + {x_2}^2}\right) \\ & = & (p, q) \quad \text{characterised by} \left\{ \begin{array}{l} p^2 + q^2 = 1 \\ px_1 + qx_2 = x_3 \\ px_2 - qx_1 > 0. \end{array}\right. \end{array}$$

For each $(p, q) \in S^1$, the inverse image $\pi^{-1}(p, q)$ is a half plane tangent to the null cone $\{{x_1}^2 + {x_2}^2 = {x_3}^2\}$. Now define $\tau : V + i\mathbb{R}^3 = Z \to \Xi = S^1 \times \mathbb{R}$ by

$$\tau(x_1 + iy_1, x_2 + iy_2, x_3 + iy_3) = (p, q, py_1 + qy_2 - y_3), \quad \text{where } (p, q) = \pi(x_1, x_2, x_3).$$

Then, $\tau^{-1}(p, q, 0)$ is a half-plane in the complex linear subspace $\{pz_1 + qz_2 = z_3\}$ and, more generally, $\tau^{-1}(p, q, r)$ is an imaginary translate thereof. In particular, the fibres of τ are all Stein. The submersion $\tau : Z \to \Xi$ evidently endows Z with the structure of a mixed manifold of type $(2, 2)$. Let us write M for Z endowed with this structure. Arguing as in the proof of Proposition 1, we see that M is a Cartan manifold. If we denote by $\eta : M \to Z$ the identity mapping, then we may use Theorem 4 to realise $H^p(Z, \mathcal{O})$ as follows. Firstly, let us write $\Lambda^{0,1}_\tau$ instead of $\Lambda^{0,1}_M$ for the $(0, 1)$-forms along the fibres of τ. Then, because the bundle Λ^1_η vanishes, the two short exact sequences (10) and (11) reduce to the single short exact sequence

$$0 \to B^1 \to \Lambda^{0,1}_Z \to \Lambda^{0,1}_\tau \to 0.$$

In our case, B^1 is a line bundle. The $\bar\partial$-operator on Z induces $\bar\partial_\tau : B^1 \to \Lambda^{0,1}_\tau \otimes B^1$ defining the partially holomorphic structure on B^1. Also $\bar\partial$ induces $d_\eta : \text{Œ} \to \text{Œ}(B^1)$ and Theorem 4 says that $\Gamma(Z, \mathcal{O}) = \ker d_\eta : \Gamma(Z, \text{Œ}) \to \Gamma(Z, \text{Œ}(B^1))$ and

$$H^1(Z, \mathcal{O}) = \operatorname{coker} d_\eta : \Gamma(Z, \text{Œ}) \to \Gamma(Z, \text{Œ}(B^1)).$$

This last statement implies that we may always find a Dolbeault representative ω on Z that restricts to zero on every leaf of our Stein foliation. Another immediate consequence of Theorem 4 is that all higher cohomology vanishes.

7. Twistor spaces

In order to have a true example of Theorem 4, other than smoothly parameterised Čech cohomology, in this section we sketch how the cohomology of the twistor space of a self-dual Riemannian manifold may be described and how this description may be used to invert the Penrose transform. The twistor space Z of a self-dual Riemannian manifold Ξ is constructed in [2]. Its salient features are as follows. The sphere bundle in the anti-self-dual 2-forms is naturally a 3-dimensional complex manifold Z and the fibres of the mapping $\pi : Z \to \Xi$ are $\mathbb{CP}_1$'s. There is an anti-holomorphic involution $\sigma : Z \to Z$, which is the antipodal map on each fibre of π. There is a holomorphic line-bundle $\mathcal{O}(1)$ on Z, which is the hyperplane section bundle on each fibre of π. The Penrose transform [8, 18, 21] identifies the cohomology $H^1(Z, \mathcal{O}(k))$ with solutions of certain conformally invariant systems of partial differential equations on Ξ, the so-called 'massless field equations'. In [21] Woodhouse establishes this result by utilising canonical representatives of the Dolbeault cohomology, characterised as being harmonic on each fibre of π. In fact, in the conformally flat case, these canonical representatives coincide with those already constructed by Gindikin and Henkin via complexification and the classical κ-operator from integral geometry in [14, 15, 16, 17].

Define $M \equiv \{(z, w) \in Z \times Z \text{ s.t. } \pi(z) = \pi(w) \text{ but } z \neq w\}$. Then, we have the commutative diagram:–

$$\begin{tikzcd} & M \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ Z \arrow[rr, "\pi"] & & \Xi \end{tikzcd}$$

where $\eta(z, w) = z$. The fibres of τ are intrinsically $\mathbb{CP}_1 \times \mathbb{CP}_1 \setminus \Delta$ where Δ is the diagonal. In particular, they are holomorphic and, indeed, Stein. In fact, M is a Cartan manifold. The mapping $\eta : M \to Z$ is a submersion with contractible fibres (intrinsically each fibre is $\mathbb{CP}_1 \setminus \{\text{point}\}$). Therefore, we are in a position to use Theorem 4. From a massless field on Ξ, it is straightforward to identify the bundles B^p on M and, at least for right-handed fields, write down a specific representative in the cohomology $H^1(\Gamma(M, \text{Œ}(B^\bullet)))$ inverting the Penrose transform. Pulling back this representative under $\gamma : Z \to M$ given by $\gamma(z) = (z, \sigma(z))$ and taking the $(0,1)$-component in accordance with Theorem 6 gives the Woodhouse representative [21]. From the point of view of this article, Theorem 4 inverts the Penrose transform in the Riemannian setting (directly going between cohomology on Z and fields on Ξ), whereas the holomorphically parameterised Čech cohomology of [6, 7, 11] (which Theorem 4 generalises) inverts the Penrose transform via the complexification of Ξ, utilising that the massless field equations on Ξ are elliptic and whose solutions are thereby real-analytic. This is the inversion via the κ-operator detailed in [14, 15, 16, 17].

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