WHEN IS GALOIS COHOMOLOGY FREE OR TRIVIAL?

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ABSTRACT. Let p be a prime and F a field, perfect if $p > 2$, containing a primitive pth root of unity. Let E/F be a cyclic extension of degree p and $G_E \lhd G_F$ the associated absolute Galois groups. We determine precise conditions for the cohomology group $H^n(E) = H^n(G_E, \mathsf{F}_p)$ to be free or trivial as an $\mathsf{F}_p[\mathrm{Gal}(E/F)]$-module. We examine when these properties for $H^n(E)$ are inherited by $H^k(E)$, $k > n$, and, by analogy with cohomological dimension, we introduce notions of cohomological freeness and cohomological triviality. We give examples of $H^n(E)$ free or trivial for each $n \in \mathsf{N}$ with prescribed cohomological dimension.

Let p be a prime and F a field containing a primitive pth root of unity ξ_p, and if $p > 2$ suppose additionally that F is perfect. Let E/F be a cyclic extension of degree p and G_E the absolute Galois group of E. In our previous paper [LMS] we determined the structure of $H^n(G_E, \mathsf{F}_p)$, $n \in \mathsf{N}$, as an $\mathsf{F}_p[G]$-module. In this paper we study more closely the question of when $H^n(G_E, \mathsf{F}_p)$ is free or trivial as an $\mathsf{F}_p[G]$-module.

Let $a \in F$ satisfy $E = F(\sqrt[p]{a})$. We write $H^n(F)$ for $H^n(G_F, \mathsf{F}_p)$ and $\mathrm{ann}_n\, x$ for the annihilator of x under the cup-product operation on $H^n(F)$. (Thus $\mathrm{ann}_n\, x \subset H^n(F)$.) Let $(f) \in H^1(F)$ denote the class of f under the Kummer isomorphism of $H^1(F)$ with the pth-power classes of $F^\times := F \setminus \{0\}$, and let $(f, g) \in H^2(F)$ denote the cup-product of (f) and $(g) \in H^1(F)$.

We first give precise conditions for free $\mathsf{F}_p[G]$-module cohomology.

Theorem 1. *Let $n \in \mathsf{N}$.*

Suppose $p > 2$ and F is perfect. Then the following are equivalent:

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(1) $H^n(E)$ *is a free* $\mathsf{F}_p[G]$*-module*
(2) $H^{n-1}(F) = \operatorname{ann}_{n-1}(a)$
(3) $\operatorname{res}\colon H^n(F) \to H^n(E)$ *is injective*
(4) $\operatorname{cor}\colon H^{n-1}(E) \to H^{n-1}(F)$ *is surjective.*

Suppose $p = 2$*. Then the following are equivalent:*

(1) $H^n(E)$ *is a free* $\mathsf{F}_2[G]$*-module*
(2) $\operatorname{ann}_{n-1}(a) = \operatorname{ann}_{n-1}(a, -1)$ *and*
$H^n(F) = \operatorname{cor} H^n(E) + (a) \cup H^{n-1}(F)$
(3) $\operatorname{ann}_{n-1}(a) = \operatorname{ann}_{n-1}(a, -1)$ *and*
$H^n(F) = \operatorname{ann}_n(a) + (a) \cup H^{n-1}(F)$
(4) $H^n(F) = \operatorname{ann}_n(a) \oplus (a) \cup H^{n-1}(F)$.

In the following theorem we examine to what extent free cohomology is hereditary.

Theorem 2. *Suppose that either*

- $p > 2$ *or*
- $p = 2$ *and* $a \in (F^{\times 2} + F^{\times 2}) \setminus F^2$.

Then free cohomology is hereditary: if $n \in \mathsf{N}$*, then for all* $m \geq n$*,*

$$H^n(E) \text{ is a free } \mathsf{F}_p[G]\text{-module} \implies H^m(E) \text{ is a free } \mathsf{F}_p[G]\text{-module.}$$

Moreover, if $H^m(E)$*,* $m \in \mathsf{N}$*, is a free* $\mathsf{F}_p[G]$*-module, then the sequence*

$$0 \to H^m(F) \xrightarrow{\operatorname{res}} H^m(E) \xrightarrow{\operatorname{cor}} H^m(F) \to 0.$$

is exact in the first and third terms.

We consider Theorems 1 and 2 in section 3. We moreover show that when $p > 2$, $H^1(E)$ is never free. When $p = 2$ we show that free cohomology is not generally hereditary and establish a condition for hereditary freeness that is more general than the one given above.

We next give precise conditions for trivial $\mathsf{F}_p[G]$-module cohomology.

Theorem 3. *Let* $n \in \mathsf{N}$*.*

Suppose $p > 2$ *and* F *is perfect. Then the following are equivalent:*

(1) $H^n(E)$ *is a trivial* $\mathsf{F}_p[G]$*-module*

(2) $(\xi_p) \cup H^{n-1}(F) \subset (a) \cup H^{n-1}(F)$ *and* $\mathrm{ann}_n(a) = (a) \cup H^{n-1}(F)$
(3) $(\xi_p) \cup H^{n-1}(F) \subset (a) \cup H^{n-1}(F)$ *and* $H^n(E) = \mathrm{res}\, H^n(F) + (\sqrt[p]{a}) \cup \mathrm{res}\, H^{n-1}(F)$.

Suppose $p = 2$. Then the following are equivalent:

(1) $H^n(E)$ *is a trivial* $\mathsf{F}_2[G]$*-module*
(2) $\mathrm{ann}_n(a) \subset (a) \cup \mathrm{ann}_{n-1}(a, -1)$.

In the $p = 2$ case, suppose additionally that $a \in (F^{\times 2} + F^{\times 2}) \setminus F^2$. Then the conditions above are also equivalent to

(3) $H^n(E) = \mathrm{res}\, H^n(F) + (\delta) \cup \mathrm{res}\, H^{n-1}(F)$ *where* $(\delta) \in H^1(E)^G$ *satisfies* $N_{E/F}(\delta) = (a)$.

For $p > 2$ and $n = 1$ the second condition in (3) was observed in [War, Lemma 3].

We deduce that trivial $\mathsf{F}_p[G]$-module cohomology is a hereditary property.

Theorem 4. *Trivial $\mathsf{F}_p[G]$-module cohomology is hereditary: if $n \in \mathsf{N}$, then for all $m \geq n$,*

$$H^n(E)^G = H^n(E) \implies H^m(E)^G = H^m(E).$$

Moreover, if $H^m(E)^G = H^m(E)$, $m \in \mathsf{N}$, then the following sequence is exact:

$$0 \to \mathrm{ann}_{m-1}(a) \to H^{m-1}(F) \xrightarrow{(a)\cup -} H^m(F) \xrightarrow{\mathrm{res}}$$
$$H^m(E) \xrightarrow{\mathrm{cor}} (a) \cup \mathrm{ann}_{m-1}\left((a) \cup (\xi_p)\right) \to 0,$$

where the map $\mathrm{ann}_{m-1}(a) \to H^{m-1}(F)$ is the natural inclusion.

We consider Theorems 3 and 4 in section 5.

In section 4 we introduce $\mathrm{cf}(E/F)$, the largest degree $n \in \mathsf{N}$ for which $H^n(E)$ is not free or ∞ if $H^n(E)$ is never free, and we give examples, for each $m \geq n \geq 1$, of extensions E/F with $\mathrm{cf}(E/F) = n$ and G_E a pro-p-group of cohomological dimension m.

In section 6 we introduce $\mathrm{ct}(E/F)$, the largest degree $n \in \mathsf{N}$ for which $H^n(E)$ is not a trivial $\mathsf{F}_p[G]$-module or ∞ if $H^n(E)$ is never trivial, and we give examples, for each $m \geq n \geq 1$, of extensions E/F with $\mathrm{ct}(E/F) = n$ and G_E a pro-p-group of cohomological dimension m.

Our proof relies on two recent results of Voevodsky in his proof of the Bloch-Kato Conjecture. (Before Voevodsky's proof these results were standard conjectures in Galois cohomology and they were proved in important special cases.) In section 1 we recall these results and present two corollaries deducing collections of equivalent statements in Milnor K-theory. In section 2 we introduce various lemmas that give sufficient conditions for our $\mathsf{F}_p[G]$-modules to be free or trivial, demonstrate that some properties in Milnor K-theory are hereditary, and establish some basic facts about certain p-henselian fields we will use to construct our examples in sections 4 and 6. For the convenience of the reader we have made our paper quite independent of [LMS].

1. Bloch-Kato and Milnor K-theory

The main ingredient for our determination of the G-module structure of $H^n(E)$ is Milnor K-theory. (See [M] and [FV, Chap. IX].) For $i \geq 0$, let K_iF denote the ith Milnor K-group of the field F, with standard generators denoted by $\{f_1, \dots, f_i\}$, $f_1, \dots, f_i \in F \setminus \{0\}$. For $\alpha \in K_iF$, we denote by $\bar{\alpha}$ the class of α modulo p, and we use the usual abbreviation k_nF for K_nF/pK_nF. The image of an element $\alpha \in K_iF$ in $H^i(F)$ we also denote by α. Because we will often use the elements $\overline{\{a\}}$, $\overline{\{\xi_p\}}$, $\overline{\{a, a\}}$, and $\overline{\{a, \xi_p\}}$, we omit the bars for these elements. We also omit the bar in the element $\overline{\{\sqrt[p]{a}\}}$.

We write $N_{E/F}$ for the norm map $K_nE \to K_nF$, and we use the same notation for the induced map modulo p. We write $\mathrm{cor} = \mathrm{cor}_{E/F}$ for the corresponding map of cohomology $H^n(E) \to H^n(F)$. We denote by i_E the natural homomorphism from K_nF to K_nE, and we use the same notation for the induced map modulo p. We denote by $\mathrm{res} = \mathrm{res}_{E/F}$ the corresponding map of cohomology $H^n(F) \to H^n(E)$. We use a well-known projection formula in Milnor K-theory several times. (See [FW, page 81].)

Our proof relies on the following two results in Voevodsky's proof of the Bloch-Kato Conjecture. The first is the Bloch-Kato Conjecture itself:

Theorem 5 ([V1, Definition 5.1] and [V2, Theorem 7.1])**.**

(1) *Let F be a field of characteristic not p and $m \in \mathsf{N}$. Then the norm residue homomorphism*

$$k_m F \to H^m(G_F, \mu_p^{\otimes m})$$

is an isomorphism.

(2) *For any cyclic extension E/F of degree p, the sequence*

$$K_m E \xrightarrow{1-\sigma} K_m E \xrightarrow{N_{E/F}} K_m F$$

is exact.

The second result establishes an exact sequence connecting $k_m F$ and $k_m E$ for consecutive m. (We translate the statement of the original result to K-theory using the previous theorem.) In the following result a is chosen to satisfy $E = F(\sqrt[p]{a})$.

Theorem 6 ([V1, Proposition 5.2])**.** *Let F be a field of characteristic not p with no extensions of degree prime to p. Then for any cyclic extension E/F of degree p and $m \geq 1$, the sequence*

$$k_{m-1}E \xrightarrow{N_{E/F}} k_{m-1}F \xrightarrow{\{a\}\cdot -} k_m F \xrightarrow{i_E} k_m E$$

is exact.

Now if F is a perfect field containing a primitive pth root of unity, we observe that we may remove the hypothesis that the field F has no extensions of degree prime to p.

Theorem 7 (Modification of Theorem 6: [LMS, Theorem 5])**.** *Let F be a field containing a primitive pth root of unity, and if $p > 2$ assume that F is perfect. Then for any cyclic extension E/F of degree p and $m \geq 1$ the sequence*

$$k_{m-1}E \xrightarrow{N_{E/F}} k_{m-1}F \xrightarrow{\{a\}\cdot -} k_m F \xrightarrow{i_E} k_m E$$

is exact.

We have the following corollaries of Theorem 7. For an element $\bar{\alpha}$ of $k_i F$, let

$$\operatorname{ann}_{n-1} \bar{\alpha} = \operatorname{ann}_{k_{n-1}F} \bar{\alpha} = \operatorname{ann}\left(k_{n-1}F \xrightarrow{\bar{\alpha}\cdot -} k_{n-1+i}F\right)$$

denote the annihilator of the product with $\bar{\alpha}$.

Corollary 1. *Assume the same hypotheses. The following are equivalent for $n \in \mathsf{N}$:*

(1) $k_{n-1}F = \mathrm{ann}_{n-1}\{a\}$
(2) $k_{n-1}F = \mathrm{ann}_{n-1}\{a\} = \mathrm{ann}_{n-1}\{a, \xi_p\}$
(3) $i_E \colon k_nF \to k_nE$ *is injective*
(4) $N_{E/F} \colon k_{n-1}E \to k_{n-1}F$ *is surjective.*

Proof. The equivalence of the items (1), (3), and (4) follows directly from the exact sequence. Assuming (1) we see that

$$k_{n-1}F = \mathrm{ann}_{n-1}\{a\} \subset \mathrm{ann}_{n-1}\{a, \xi_p\} \subset k_{n-1}F,$$

whence (2) follows, and (2) implies (1) trivially.

In Lemma 4 we show that all of the properties in Corollary 1 are hereditary.

Corollary 2. *Assume the same hypotheses. The following are equivalent for* $n \in \mathsf{N}$*:*

(1) $\mathrm{ann}_{n-1}\{a\} = \mathrm{ann}_{n-1}\{a, -1\}$ *and*
$k_nF = N_{E/F}k_nE + \{a\} \cdot k_{n-1}F$
(2) $\mathrm{ann}_{n-1}\{a\} = \mathrm{ann}_{n-1}\{a, -1\}$ *and*
$k_nF = \mathrm{ann}_n\{a\} + \{a\} \cdot k_{n-1}F$
(3) $k_nF = \mathrm{ann}_n\{a\} \oplus \{a\} \cdot k_{n-1}F.$

Proof. (1) $\Longrightarrow$ (2). This implication follows directly from $N_{E/F}k_nE = \mathrm{ann}_n\{a\}$.

(2) $\Longrightarrow$ (3). Let $\bar{\alpha} \in (\{a\} \cdot k_{n-1}F) \cap \mathrm{ann}_n\{a\}$. Then $\bar{\alpha} = \{a\} \cdot \bar{f}$ for some $f \in K_{n-1}F$. Since $\{a\} \cdot \bar{\alpha} = 0$, $\{a, a\} \cdot \bar{f} = 0$. Because $\{a, a\} = \{a, -1\}$, we have $\{a, -1\} \cdot \bar{f} = 0$, and by the first hypothesis, $\{a\} \cdot \bar{f} = 0$. Then $\bar{\alpha} = 0$ and the sum is direct.

(3) $\Longrightarrow$ (1). The second claim follows from the fact that $\mathrm{ann}_n\{a\} = N_{E/F}k_nE$. For the first, suppose $\{a, -1\} \cdot \bar{f} = 0$ for $f \in K_{n-1}F$. Because $\{a, -1\} = \{a, a\}$, we have

$$\{a\} \cdot \bar{f} \in (\mathrm{ann}_n\{a\}) \cap (\{a\} \cdot k_{n-1}F) = \{0\}.$$

Hence $\bar{f} \in \mathrm{ann}_{n-1}\{a\}$ and $\mathrm{ann}_{n-1}\{a\} = \mathrm{ann}_{n-1}\{a, -1\}$ as required.

2. Notation and Lemmas

For a field F, we let $F^\times$ denote its multiplicative group $F \setminus \{0\}$. We assume throughout the entire paper that F is perfect if $p = [E : F] > 2$ and F is arbitrary of characteristic not 2 if $p = [E : F] = 2$.

For the remainder of the paper $n \in \mathsf{N}$ denotes an arbitrary natural number, E/F a cyclic extension of fields of degree p with a primitive pth root of unity $\xi_p \in F$, and $a \in F^\times$ an element such that $E = F(\sqrt[p]{a})$. Let $G = \mathrm{Gal}(E/F)$, and choose $\sigma \in G$ to satisfy $\sqrt[p]{a}^{\sigma-1} = \xi_p$. For $f, g \in F^\times$, we write (f) for the class of f in $H^1(F) \cong F^\times/F^{\times p}$ and (f,g) for $(f) \cup (g) \in H^2(F)$.

2.1. **Module Structure.**

For $\gamma \in K_nE$, let $l(\gamma)$ denote the dimension of the cyclic $\mathsf{F}_p[G]$-submodule $\langle\bar{\gamma}\rangle$ of k_nE generated by $\bar{\gamma}$. Then we have, for $l(\gamma) \geq 1$,

$$(\sigma-1)^{l(\gamma)-1}\langle\bar{\gamma}\rangle = \langle\bar{\gamma}\rangle^G \neq 0 \quad \text{and} \quad (\sigma-1)^{l(\gamma)}\langle\bar{\gamma}\rangle = 0.$$

We denote by N the map $(\sigma-1)^{p-1}$ on k_nE. Because $(\sigma-1)^{p-1} = 1+\sigma+\cdots+\sigma^{p-1}$ in $\mathsf{F}_p[G]$, we may use $i_E N_{E/F}$ and N interchangeably on k_nE.

Our first lemma establishes that in certain situations, all elements in $(k_nE)^G$ are norm classes.

Lemma 1. *Let $n \in \mathsf{N}$. Suppose that either*

- *$p > 2$ and $N_{E/F}\colon k_{n-1}E \to k_{n-1}F$ is surjective, or*
- *$p = 2$, $\mathrm{ann}_{n-1}\{a\} = \mathrm{ann}_{n-1}\{a,-1\}$, and $k_nF = N_{E/F}k_nE + \{a\}\cdot k_{n-1}F$.*

Then we have

(1) *For each $\gamma \in K_nE$, there exists $\alpha \in K_nE$ such that*
$$\langle N\bar{\alpha}\rangle = \langle\bar{\gamma}\rangle^G.$$
(2) $(k_nE)^G = i_E N_{E/F} k_nE = (\sigma-1)^{p-1}k_nE = i_E k_nF$.

Proof. (1). Assume first that $p > 2$. By hypothesis, $N_{E/F}\colon k_{n-1}E \to k_{n-1}F$ is surjective, and then using the projection formula ([FW, p. 81]) we see that $N_{E/F}\colon k_nE \to k_nF$ is also surjective. Hence if $\bar{\gamma} \in i_E k_nF$ then there exists $\bar{\alpha} \in k_nE$ such that $N\bar{\alpha} = \bar{\gamma}$ and we are done. Otherwise, let $l = l(\gamma)$ and suppose $\bar{\gamma} \notin i_E k_nF$ and $1 \leq l \leq i \leq p$.

If $l \geq 2$ we show by induction on i that there exists $\alpha_i \in K_nE$ such that $\langle(\sigma-1)^{i-1}\bar{\alpha}_i\rangle = \langle\bar{\gamma}\rangle^G$. Then setting $\alpha := \alpha_p$, the proof will be complete in the case when $2 \leq l$. The case $l = 1$ we then handle at the end of the proof, using the case $2 \leq l$.

Assume then that $l \geq 2$. If $i = l$ then $\alpha_i = \gamma$ suffices. Assume now that $1 \leq l \leq i < p$ and that our statement is true for i. Set $c = N_{E/F}\alpha_i$. Since $i_E\bar{c} = N\bar{\alpha}_i = (\sigma - 1)^{p-1}\bar{\alpha}_i$ and $i < p$, $i_E\bar{c} = 0$.

By Corollary 1, we have $\bar{c} = 0$, that is, $c = pf$ for some $f \in K_nF$. Hence

$$N_{E/F}\big(\alpha_i - i_E(f)\big) = 0.$$

By Theorem 5, there exists $\omega \in K_nE$ such that

$$(\sigma - 1)\omega = \alpha_i - i_E(f).$$

If $i > 1$, $l(\alpha_i) > 1$ and so $l(\alpha_i - i_E(f)) > 1$. Hence $(\sigma - 1)^2\bar{\omega} = (\sigma - 1)\bar{\alpha}_i \neq 0$. Therefore $\langle(\sigma - 1)^i\bar{\omega}\rangle = \langle\bar{\gamma}\rangle^G$ and we can set $\alpha_{i+1} = \omega$.

Therefore we have proved that if $l(\gamma) \geq 2$ then there exists $\alpha \in K_nE$ such that $N\bar{\alpha} = (\sigma - 1)^{l(\gamma)-1}\bar{\gamma}$.

Now assume that $l(\gamma) = 1$ but $\bar{\gamma} \notin i_Ek_nF$. Then $\bar{\gamma} = \bar{\alpha}_1$ and $(\sigma - 1)\bar{\omega} = \bar{\alpha}_1 - i_E(\bar{f}) \neq 0$. Thus $l(\omega) = 2$ and our argument above shows that there exists $\beta \in K_nE$ such that $N\bar{\beta} = (\sigma - 1)\bar{\omega} = \bar{\alpha}_1 - i_E(\bar{f})$. As we observed at the beginning of our proof there exists an element $\delta \in K_nE$ such that $N\bar{\delta} = i_E(\bar{f})$. Therefore we have:

$$N(\bar{\beta} + \bar{\delta}) = \bar{\alpha}_1 = \bar{\gamma}.$$

Thus we have established in all cases that for each $\gamma \in K_nE$ there exists $\alpha \in K_nE$ such that $\langle N\bar{\alpha}\rangle = \langle\bar{\gamma}\rangle^G$.

Now consider the case $p = 2$. In this case from our hypothesis $k_nF = N_{E/F}k_nE + \{a\} \cdot k_{n-1}F$ we again have $i_EN_{E/F}k_nE = i_Ek_nF$. Therefore if $\bar{\gamma} \in i_Ek_nF$ our statement follows. Assume that $\bar{\gamma} \in k_nE \setminus i_Ek_nF$. Then $l(\gamma) \leq 2$, and if $l(\gamma) = 2$ we may set $\alpha = \gamma$ and (1) follows again. Next we shall assume that $l(\gamma) = 1$ and therefore $\bar{\gamma} \in (k_nE)^G$. Set $c = N_{E/F}\gamma$. Then $i_E\bar{c} = 0$.

From Theorem 7, we conclude that $c = \{a\} \cdot g + 2f$ for $g \in K_{n-1}F$ and $f \in K_nF$. Hence from the projection formula,

$$\begin{aligned} N_{E/F}\left(\gamma - \{\sqrt{a}\} \cdot i_E(g) - i_E(f)\right) &= (\{a\} \cdot g + 2f) - \{-a\} \cdot g - 2f \\ &= \{-1\} \cdot g. \end{aligned}$$

Using Theorem 7 again, we obtain that $\{a, -1\} \cdot \bar{g} = 0$. Our hypothesis $\mathrm{ann}_{n-1}\{a\} = \mathrm{ann}_{n-1}\{a, -1\}$ gives us that $\{a\} \cdot \bar{g} = 0$. Hence $\{a\} \cdot g = 2h$ for some $h \in K_nF$ and $N_{E/F}\gamma = 2(h + f)$. Thus

$$N_{E/F}\left(\gamma - i_E(h + f)\right) = 0.$$

Then by Theorem 5 there exists $\alpha \in K_nE$ such that

$$(\sigma - 1)\alpha = \gamma - i_E(h + f).$$

Observe that since $\bar{\gamma} \notin i_E k_n F$ we have $\bar{\gamma} - i_E(\overline{h+f}) \neq 0$. Hence

$$\bar{\gamma} = (\sigma - 1)\bar{\alpha} + i_E(\overline{h+f}) \in N k_n E,$$

as required.

(2). Suppose $\bar{\gamma} \in (k_nE)^G$. Then $l(\gamma) = 1$ and the preceding part of our proof shows that $\langle\bar{\gamma}\rangle = \langle N\bar{\alpha}\rangle$ for $\alpha \in K_nE$. Hence $(k_nE)^G \subset i_E N_{E/F} k_n E$. Then

$$i_E N_{E/F} k_n E \subset i_E k_n F \subset (k_nE)^G \subset i_E N_{E/F} k_n E,$$

and so all inclusions are equalities.

Our second lemma establishes a situation in which all elements in k_nE are fixed by G.

Lemma 2. *Let $n \in \mathsf{N}$. Suppose that*

$$i_E(\{\xi_p\} \cdot k_{n-1}F) = i_E N_{E/F} k_n E = \{0\}.$$

Then $(k_nE)^G = k_nE$.

Remark. The hypothesis $i_E(\{\xi_p\} \cdot k_{n-1}F) = \{0\}$ can be omitted in the case $p = 2$.

Proof. Let $\gamma \in K_nE$. We show that $l(\gamma) > 1$ leads to a contradiction, whence we will have the result.

Suppose that $l = l(\gamma) \geq 2$ and $1 \leq l \leq i \leq p$. We show by induction on i that there exists $\alpha_i \in K_nE$ such that $\langle(\sigma-1)^{i-1}\bar{\alpha}_i\rangle = \langle(\sigma-1)^{l-1}\bar{\gamma}\rangle$. If $i = l$ then $\alpha_i = \gamma$ suffices. Assume now that $l \leq i < p$ and that our statement is true for i. Set $c = N_{E/F}\alpha_i$. Since $i_E\bar{c} = N\bar{\alpha}_i = (\sigma-1)^{p-1}\bar{\alpha}_i$ and $i < p$, $i_E\bar{c} = 0$.

By Theorem 7, $\bar{c} = \{a\} \cdot \bar{b}$ for some $b \in K_nF$. Hence $c = \{a\} \cdot b + pf$ for $f \in K_nF$. Then since $2 \leq i < p$ in this case,

$$N_{E/F}\left(\alpha_i - \{\sqrt[p]{a}\} \cdot i_E(b) - i_E(f)\right) = 0.$$

By Theorem 5, there exists $\omega \in K_nE$ such that

$$(\sigma - 1)\omega = \alpha_i - \{\sqrt[p]{a}\} \cdot i_E(b) - i_E(f).$$

Then $(\sigma - 1)^2\omega = (\sigma - 1)\alpha_i - i_E(\{\xi_p\} \cdot i_E(b)) = (\sigma - 1)\alpha_i \neq 0$, and we can set $\alpha_{i+1} = \omega$. Observe that here we use our hypothesis

$$i_E(\{\xi_p\} \cdot k_{n-1}F) = \{0\}.$$

Hence by induction there exists $\alpha_p \in K_nE$ such that

$$\langle N\bar{\alpha}_p\rangle = \langle(\sigma-1)^{l-1}\bar{\gamma}\rangle.$$

But $i_E N_{E/F}\bar{\alpha}_p = 0$, whence $(\sigma-1)^{l-1}\bar{\gamma} = 0$, a contradiction.

Finally, we record a necessary and sufficient condition for an $\mathsf{F}_p[G]$-module to be free.

Lemma 3. *Let M be an $\mathsf{F}_p[G]$-module. Then the following are equivalent:*

(1) *M is a free $\mathsf{F}_p[G]$-module*
(2) *$M^G = (\sigma-1)^{p-1}M$.*

Proof. Condition (2) is equivalent to

$$H^2(G, M) = \{0\}.$$

But this condition is known to be equivalent with (1) (for any p-group G!). (See for example [Lg2, p. 63].)

2.2. **Hereditary Properties.**

We say that a property of Milnor k-groups k_nE and k_nF is *hereditary* if the validity of the property for a given n implies the validity of the property for all integers greater than n.

The next lemma establishes various hereditary properties, including the properties in Corollary 1.

Lemma 4. *Let $n \in \mathsf{N}$.*

The following are hereditary properties:

(1) $k_{n-1}F = \operatorname{ann}_{n-1}\{a\} = \operatorname{ann}_{n-1}\{a, \xi_p\}$.
(2) $i_E\colon k_nF \to k_nE$ *is injective*
(3) $N_{E/F}\colon k_{n-1}E \to k_{n-1}F$ *is surjective*
(4) *for some fixed $\alpha_1, \alpha_2 \in K_1F$, $\bar{\alpha}_1 \cdot k_{n-1}F \subset \bar{\alpha}_2 \cdot k_{n-1}F$*
(5) *for some fixed $\alpha \in K_1E$, $k_nE = i_Ek_nF + \bar{\alpha} \cdot i_Ek_{n-1}F$*

Proof. (1). $k_nF = k_{n-1}F \cdot k_1F$, and since $\operatorname{ann}_{n-1}\{a\} = k_{n-1}F$, we have $\operatorname{ann}_n\{a\} = k_nF$ as well. The other equality follows from $\operatorname{ann}_n\{a\} \subset \operatorname{ann}_n\{a, \xi_p\}$. The result follows by induction.

(2-3). By Corollary 1, the first three properties are equivalent, hence (2) and (3) are hereditary.

(4). $K_nF = K_{n-1}F \cdot K_1F$, so K_nF is generated by elements of the form

$$\{f_1, f_2, \ldots, f_n\} = \{f_1, \ldots, f_{n-1}\} \cdot \{f_n\}, \quad f_i \in F^\times.$$

For each such generator, we calculate

$$\bar{\alpha}_1 \cdot \overline{\{f_1, \ldots, f_n\}} = \bar{\alpha}_2 \cdot \bar{g} \cdot \overline{\{f_n\}}$$

for some $g \in K_{n-1}F$, whence $\bar{\alpha}_1 \cdot k_nF \subset \bar{\alpha}_2 \cdot k_nF$. The result follows by induction.

(5). $k_{n+1}E = k_1E \cdot k_nE$, so the condition on k_nE gives us that $k_{n+1}E$ is generated by elements of the form

$$\bar{\gamma}_1 = \overline{\{\delta\}} \cdot i_E(\overline{\{f_1, \ldots, f_n\}}), \quad \delta \in E^\times, \ f_i \in F^\times$$

and

$$\bar{\gamma}_2 = \overline{\{\delta\}} \cdot \bar{\alpha} \cdot i_E(\overline{\{f_1, \ldots, f_{n-1}\}}), \quad \delta \in E^\times, \ f_i \in F^\times.$$

If $n-1 \geq 1$ then we see that $k_{n+1}E$ is generated by the elements in $k_nE \cdot i_Ek_1F$. By hypothesis $k_nE = i_Ek_nF + \bar{\alpha} \cdot i_Ek_{n-1}F$ and therefore $k_{n+1}E$ is generated by elements in $i_Ek_{n+1}F + \bar{\alpha} \cdot i_Ek_nF$.

If $n = 1$ then using our hypothesis $k_1E = i_Ek_1F + \bar{\alpha} \cdot i_Ek_0F$ we may write the generators $\bar{\gamma}_2$ of k_2E as

$$\bar{\gamma}_2 = \left(i_E(\overline{\{f\}}) + c\bar{\alpha}\right) \cdot \bar{\alpha}, \quad f \in F^\times, \ c \in \mathsf{Z}.$$

Since $\bar{\alpha} \cdot \bar{\alpha} = \{-1\} \cdot \bar{\alpha}$,

$$\bar{\gamma}_2 = i_E(\overline{\{f\}}) \cdot \bar{\alpha} + c\ i_E(\{-1\}) \cdot \bar{\alpha} = -\bar{\alpha} \cdot i_E(\overline{\{f\}}) - \bar{\alpha} \cdot i_E(c\{-1\}).$$

Thus in this case both types of generators of k_2E have the required form of elements in $i_Ek_2F + \bar{\alpha} \cdot i_Ek_1F$.

The result now follows by induction.

2.3. Fields of the Form $\mathsf{C}((\oplus_I \mathsf{Z}_{(p)}))$.

For our examples in sections 4 and 6 we introduce the following notation and results.

Let

$$\mathsf{Z}_{(p)} := \left\{ \frac{c}{d} \in \mathsf{Q} \;\middle|\; c, d \in \mathsf{Z}, d \neq 0; \text{ if } c \neq 0 \text{ then } (c,d) = 1, p \nmid d \right\}.$$

Observe that $\mathsf{Z}_{(p)}$ carries a natural ordering induced from Q. Let I be a well-ordered set of cardinality m, and let Γ be a direct sum of m copies of $\mathsf{Z}_{(p)}$, indexed by I. Then $m = \dim_{\mathsf{F}_p} \Gamma/p\Gamma$. Order Γ lexicographically.

Then Γ is a linearly ordered abelian group. (Recall that each non-empty set can be well-ordered (see [Lg1, Appendix 2, Theorem 4.1]).) Now it is well-known that since Γ is a totally ordered abelian group, the field

$$F_m := \mathsf{C}((\Gamma)) := \{f\colon \Gamma \to \mathsf{C} \mid \operatorname{supp}(f) \text{ is well-ordered}\}$$

is a henselian valued field with value group Γ and residue field C. (See [R1, Chapitre D, Théorèmes 2 et 3, page 103, et Chapitre F, Théorème 4, page 198].) Thus a typical element $f \in F_m$ may be written as a formal sum

$$f = \sum_{g\in\Gamma} a_g t^g$$

such that the set $\operatorname{supp}(f) := \{g \in \Gamma \mid a_g \neq 0\}$ is a well-ordered subset of Γ. The absolute Galois group of F_m is known to be Z_p^m, the topological product of m copies of Z_p [K, pages 3 and 4]. We record one property of F_m in the following lemma.

Lemma 5. *For $m, n \in \mathsf{N} \cup \{\aleph_0\}$,*

$$H^n(F_m) \cong \bigwedge^n H^1(\mathsf{Z}_p^m) \cong \bigwedge^n \oplus_m \mathsf{F}_p$$

where the cup-product is sent to the wedge product.

Proof. Since F_m is a henselian valued field, the second result follows from [Wad, Theorem 3.6], observing that under the Kummer isomorphism $F_m^\times/F_m^{\times p} \cong H^1(F_m)$, $0 = (-1) \in H^1(F_m)$, and $H^j(\mathsf{C}) = \{0\}$ for all $j \in \mathsf{N}$.

Of particular interest to us will be certain fields with absolute Galois groups which are pro-p free products of groups of the form Z_p^m.

Lemma 6. *Suppose that m_1, m_2 are non-zero cardinal numbers, and let F_{m_1} and F_{m_2} be as above. There exists a field F_{m_1,m_2} of characteristic 0, containing a primitive p^2th root of unity ξ_{p^2}, such that the absolute Galois group*

$$G_{F_{m_1,m_2}} \quad\cong\quad G_{F_{m_1}} \star_{\textit{pro-}p} G_{F_{m_2}} \quad\cong\quad \mathsf{Z}_p^{m_1} \star_{\textit{pro-}p} \mathsf{Z}_p^{m_2},$$

where the free products are taken in the category of pro-p-groups, and the natural restriction maps

$$\mathrm{res}_\star \colon H^n(F_{m_1,m_2}) \to H^n(F_{m_1}) \oplus H^n(F_{m_2})$$

are isomorphisms.

Note that we use the notation $\mathrm{res}_\star$ to distinguish this restriction map from restriction maps $H^n(F) \to H^n(E)$. For $h \in H^n(F_{m_1,m_2})$, we will write $\mathrm{res}_\star h = h_1 \oplus h_2$.

Proof. The existence of a field F_{m_1,m_2} with $\mathrm{char}(F_{m_1,m_2}) = \mathrm{char}(F_{m_1}) = \mathrm{char}(F_{m_2}) = 0$ and the given absolute Galois group follows from [EH, Proposition 1.3].

Additionally using the construction of F_{m_1,m_2} following [EH, proof of Proposition 1.3] we assume that F_{m_1,m_2} is the intersection of two henselian valued fields $(L_i, V_i), i = 1, 2$, with residue fields isomorphic to F_{m_1} and F_{m_2} respectively. Here V_i is a henselian valuation on L_i. Then by Hensel's Lemma (see [R2, pages 12 and 13, condition (3)]) and by the fact that F_{m_1} and F_{m_2} have characteristic 0 and both contain a primitive p^2th root of unity, we see that F_{m_1,m_2} also contains a primitive p^2th root of unity. The fact that the restriction maps are isomorphisms follows from [N, Sätze (4.1) und (4.2)].

Remark. From the proof above it follows that F_{m_1,m_2} contains all p^kth primitive roots, $k \in \mathsf{N}$. However we shall not need this observation.

3. When is Galois Cohomology Free?

Proof of Theorem 1. Here and elsewhere we use Theorem 5 to translate between Galois cohomology H^n and K-theory k_n.

First we show that for all p, k_nE free implies that

$$i_E N_{E/F} k_n E = i_E k_n F = (k_n E)^G.$$

If k_nE is free, then by Lemma 3, $(\sigma - 1)^{p-1} k_n E = (k_n E)^G$. Observing that $(\sigma - 1)^{p-1}$ and $i_E N_{E/F}$ are equivalent,

$$i_E N_{E/F} k_n E = (\sigma - 1)^{p-1} k_n E = (k_n E)^G.$$

Then since $i_E k_n F \subset (k_n E)^G$ and $i_E N_{E/F} k_n E \subset i_E k_n F$, we have established our claim.

Assume first that $p > 2$. First we show (1) $\Longrightarrow$ (2). Let $f \in K_{n-1}F$ be arbitrary, and set $\alpha = \{\sqrt[p]{a}\} \cdot f$. Now because $(k_nE)^G = i_E N_{E/F} k_n E$, there exists $\beta \in K_n E$ such that $i_E N_{E/F} \bar{\beta} = i_E(\{\xi_p\} \cdot \bar{f}) \in (k_n E)^G$. Set $\gamma = (\sigma - 1)^{p-2}\beta$. Since $p > 2$, γ is in the image of $\sigma - 1$ and hence has trivial norm. We calculate

$$N_{E/F}(\bar{\alpha} - \bar{\gamma}) = \{a\} \cdot \bar{f}.$$

On the other hand, observing that $i_E N_{E/F} = (\sigma - 1)^{p-1}$ on $k_n E$,

$$(\sigma - 1)(\bar{\alpha} - \bar{\gamma}) = i_E(\{\xi_p\} \cdot \bar{f}) - i_E(\{\xi_p\} \cdot \bar{f}) = 0.$$

Hence $\bar{\alpha} - \bar{\gamma} \in (k_n E)^G = i_E k_n F$. But on $i_E k_n F$ the norm map $N_{E/F}$ is trivial. Hence $\{a\} \cdot \bar{f} = 0$ and $\operatorname{ann}_{n-1}\{a\} = k_{n-1}F$.

By Corollary 1, (2), (3), and (4) are all equivalent. Now we show (4) $\Longrightarrow$ (1). Assume that $N_{E/F} \colon k_{n-1}E \to k_{n-1}F$ is surjective. By Lemma 1 we have $(k_n E)^G = (\sigma - 1)^{p-1} k_n E$. Hence by Lemma 3, $k_n E$ is free.

Now suppose that $p = 2$. By Corollary 2, we need only show that (1) and (2) are equivalent. We show first that (1) $\Longrightarrow$ (2). We established that (1) implies $i_E N_{E/F} k_n E = i_E k_n F$. Since $\ker i_E = \{a\} \cdot k_{n-1}F$, this equality is equivalent to $k_n F = N_{E/F} k_n E + \{a\} \cdot k_{n-1} F$, so we have the second part of (2). Clearly $\operatorname{ann}_{n-1}\{a\} \subset \operatorname{ann}_{n-1}\{a, -1\}$, so we show that $\operatorname{ann}_{n-1}\{a, -1\} \subset \operatorname{ann}_{n-1}\{a\}$.

We adapt the argument above. Let $\bar{f} \in \operatorname{ann}_{n-1}\{a, -1\}$. Set $\bar{\alpha} = \{\sqrt{a}\} \cdot \bar{f}$. Since $\{a\} \cdot \{-1\} \cdot \bar{f} = 0$, Theorem 7 tells us that there exists $\beta \in K_n E$ such that $N_{E/F}\bar{\beta} = \{-1\} \cdot \bar{f}$. Now we calculate by the projection formula

$$N_{E/F}(\bar{\alpha} - \bar{\beta}) = \{-a\} \cdot \bar{f} - \{-1\} \cdot \bar{f} = \{a\} \cdot \bar{f}.$$

On the other hand, using the fact that $\sigma - 1 = \sigma + 1$ when $p = 2$,

$$(\sigma - 1)(\bar{\alpha} - \bar{\beta}) = \{-1\} \cdot \bar{f} - \{-1\} \cdot \bar{f} = 0.$$

Hence $\bar{\alpha} - \bar{\beta} \in (k_n E)^G$. By Lemma 3, $(k_n E)^G = (\sigma - 1) k_n E = i_E N_{E/F} k_n E \subset i_E k_n F$. Therefore $\bar{\alpha} - \bar{\beta} \in i_E k_n F$. But on $i_E k_n F$, the norm map $N_{E/F}$ is trivial. Hence $\{a\} \cdot \bar{f} = 0$, and $\bar{f} \in \operatorname{ann}_{n-1}\{a\}$, so $\operatorname{ann}_{n-1}\{a, -1\} \subset \operatorname{ann}_{n-1}\{a\}$, as required.

Now we show that (2) $\Longrightarrow$ (1). Assume that $\operatorname{ann}_{n-1}\{a, -1\} = \operatorname{ann}_{n-1}\{a\}$ and that $k_n F = N_{E/F} k_n E + \{a\} \cdot k_{n-1} F$. We use Lemmas 1 and 3 to deduce that $k_n E$ is free.

It follows easily that

Corollary 3. *For $p > 2$, k_1E is never free.*

Proof. Since G acts trivially on $k_0E \cong \mathsf{F}_p$,

$$N_{E/F}k_0E = 0 \neq k_0F \cong \mathsf{F}_p.$$

Alternatively, $i_E\colon k_1F \to k_1E$ is not injective, since $\{a\} \in k_1F$ is a nontrivial element of the kernel.

With Theorem 1 in hand, Lemma 4 is enough to establish hereditary freeness in the $p > 2$ case, and for the $p = 2$ case we show that an additional condition, analogous to the $p > 2$ case, is sufficient:

Corollary 4. *Suppose that $p = 2$ and for some $n \in \mathsf{N}$,*

$$\mathrm{ann}_{n-1}\{a\} = k_{n-1}F.$$

Then k_mE is a free $\mathsf{F}_2[G]$-module for all $m \geq n$.

Proof. We show that the two conditions of part (2) of the $p = 2$ portion of Theorem 1 hold for K-theory degree at least n. From Lemma 4, part (1), we deduce that $k_mF = \mathrm{ann}_m\{a\} = \mathrm{ann}_m\{a, -1\}$ for all $m \geq n-1$.

By Theorem 7 and Lemma 4, we have $k_mF = N_{E/F}k_mE$ for all $m \geq n-1$ and therefore we see that

$$k_mF = N_{E/F}k_mE + \{a\} \cdot k_{m-1}F \quad \text{for all } m \geq n.$$

We conclude that k_mE is a free $\mathsf{F}_2[G]$-module for all $m \geq n$.

Just as before it follows easily that

Corollary 5. *For $p = 2$ and $\sqrt{-1} \in F$, k_1E is never free.*

Proof. Since $-1 \in F^{\times 2}$, we have $\{-1\} = 0 \in k_1F$ and $\{a, -1\} = 0 \in k_2F$, so that $\mathrm{ann}_0\{a, -1\} = k_0F \cong \mathsf{F}_2 \neq \mathrm{ann}_0\{a\} = \{0\}$.

We are now ready to prove Theorem 2.

Proof of Theorem 2. For $p > 2$, the fact that free cohomology is hereditary follows from Lemma 4 and condition (2) in Theorem 1. The exactness of the first term of the sequence follows from Theorem 1, part (3), while the exactness at the third term follows from Theorem 1, part (4) and Lemma 4, part (3).

Assume then that $p = 2$ and $a = x^2 + y^2$ for some $x, y \in F^\times$. If $-1 \in F^{\times 2}$ then $\{a, -1\} \in 2K_2F$ and so $\{a, -1\} = 0 \in k_2F$. Otherwise let $K = F(\sqrt{-1})$, and observe that $a = N_{K/F}(x + y\sqrt{-1})$. Then $\{a, -1\} = 0 \in k_2F$. Hence $\operatorname{ann}_{n-1}\{a, -1\} = k_{n-1}F$.

Now observe that since k_nE is a free $\mathsf{F}_2[G]$-module, by Theorem 1 we have $\operatorname{ann}_{n-1}\{a\} = \operatorname{ann}_{n-1}\{a, -1\}$, and so $\operatorname{ann}_{n-1}\{a\} = \operatorname{ann}_{n-1}\{a, -1\} = k_{n-1}F$. We deduce from Corollary 4 that k_mE is a free $\mathsf{F}_2[G]$-module for all $m \geq n$.

For the exact sequence in the case $p = 2$, we have shown that $k_{m-1}F = \operatorname{ann}_{m-1}\{a\}$, and so by Theorem 7 and Lemma 4, we have $k_nF = N_{E/F}k_nE$ for all $n \geq m - 1$. Hence we have exactness at the third term. Furthermore, we conclude from Corollary 1 that i_E is injective from k_mF to k_mE. Hence we have exactness at the first term as well.

We now provide an example of k_1E free but k_2E nonfree, showing that freeness is not generally hereditary when $p = 2$.

Example. Let $p = 2$, $F = \mathsf{Q}_2$, and $a = -1$, so $E = \mathsf{Q}_2(\sqrt{-1})$. Then

$$k_1F \cong F^\times/F^{\times 2} = \langle[-1], [2], [5]\rangle.$$

and

$$N_{E/F}k_1E \cong N_{E/F}(E^\times)F^{\times 2}/F^{\times 2} = \langle[2], [5]\rangle.$$

(See [Lam, page 162, Corollary 2.24].)

Therefore $k_1F = N_{E/F}k_1E + \{-1\} \cdot k_0F$. Moreover, since $[-1] \notin N_{E/F}(E^\times)F^{\times 2}/F^{\times 2}$, we have $\{-1, -1\} \neq 0 \in k_2F$. (Again see [Lam, page 162, Corollary 2.24].)

Hence $\operatorname{ann}_0\{-1, -1\} = \{0\}$. Since $\{-1\} \neq 0$ in k_1F we see that $\operatorname{ann}_0\{-1\} = \{0\}$. Hence the conditions of part (2) of the $p = 2$ portion of Theorem 1 are satisfied, whence k_1E is a free F_2-module.

Observe, however, that since $\operatorname{ann}_0\{-1, -1\} = \{0\} \neq k_0F$, the first hypothesis in Corollary 4 does not hold. Therefore we cannot conclude that k_2E is a free $\mathsf{F}_2[G]$-module—and of course it is not, as it is well known that $k_2E \cong \mathsf{F}_2$. (See [Lam, page 158, Corollary 2.15] and [M, Theorem 4.1].)

4. Examples of $H^k(E)$ Free for all $n < k$ and $H^n(E)$ Nonfree, with Given Cohomological Dimension

We have shown in Theorem 2 that if $p > 2$ then the property $H^n(E)$ is a free $\mathsf{F}_p[G]$-module is hereditary. Moreover, the same property is hereditary in the case $p = 2$ as well if $i = \sqrt{-1} \in F$, since then $a = ((a+1)/2)^2 + ((a-1)i/2)^2$.

These results lead naturally to the definition of an interesting invariant $\mathrm{cf}(E/F) \in \{0\} \cup \mathsf{N} \cup \{\infty\}$:

$$\mathrm{cf}(E/F) = \sup\{n \in \mathsf{N} \cup \{0\} \mid H^n(E) \text{ is not a free } \mathsf{F}_p[G]\text{-module}\}.$$

We have chosen cf to indicate that after degree $\mathrm{cf}(E/F)$, Galois cohomology is cohomologically free. Of course, if $H^n(E)$ is never free then $\mathrm{cf}(E/F) = \infty$, and otherwise $\mathrm{cf}(E/F) \in \mathsf{N} \cup \{0\}$.

Assume for the moment that either $p > 2$ or $\sqrt{-1} \in F$. If $\mathrm{cf}(E/F) = n$, then by definition $H^m(E)$ is a free $\mathsf{F}_p[G]$-module for all $m > n$. On the other hand, by the hereditary property we also have that $H^k(E)$ is not free for all $k \leq \mathrm{cf}(E/F)$. Finally, Corollaries 3 and 5 tell us that $H^1(E)$ is never free and hence $\mathrm{cf}(E/F) \geq 1$. A natural question arises: can we choose a suitable field extension E/F so that $\mathrm{cf}(E/F)$ is a given natural number or ∞? We show that the answer is affirmative.

Before formulating our result precisely, let us recall that for any pro-p-group T we may define $\mathrm{cd}(T)$, the cohomological dimension of T, as

$$\mathrm{cd}(T) = \sup\left\{k \in \mathsf{N} \mid H^k(T, \mathsf{F}_p) \neq \{0\}\right\} \in \mathsf{N} \cup \{\infty\}.$$

(See [RZ, Chapter 7].) Suppose that the absolute Galois group G_E is a pro-p-group. Then, adopting the convention that $\{0\}$ is considered a free $\mathsf{F}_p[G]$-module, we have:

$$\mathrm{cf}(E/F) \leq \mathrm{cd}(G_E).$$

From Corollaries 3 and 5 above it follows that if $p > 2$ or $p = 2$ and $\sqrt{-1} \in F$ then $\mathrm{cf}(E/F) \geq 1$.

Our result is then the following.

Given $1 \leq n \leq m \in \mathsf{N} \cup \{\infty\}$ and a prime p, there exists a cyclic extension E/F of degree p with $\xi_p \in F$ such that

(1) *G_E is a pro-p-group;*
(2) $\mathrm{cf}(E/F) = n$*; and*
(3) $\mathrm{cd}(G_E) = m$*.*

Observe that if we choose $n \in \mathbb{N}$ and $m > n$, then we have obtained examples as promised in the title of this section.

4.1. **The case $m \in \mathbb{N}$.**

(1). Let $F := F_{n,m}$ be a field of characteristic 0 with $G_F \cong \mathbb{Z}_p^n \star_{\text{pro-}p} \mathbb{Z}_p^m$ and $\xi_{p^2} \in F$, given by Lemma 6. Observe particularly that $\sqrt{-1} \in F$ in the case $p = 2$. Since $\mathrm{char}(F) = 0$, F is perfect. Let

$$E = F(\sqrt[p]{a})$$

for any $a \in F^\times$ such that under the restriction map on H^1,

$$\mathrm{res}_\star(a) = (a)_1 \oplus (a)_2, \quad (a)_1 \neq 0,\ (a)_2 = 0.$$

We use here, and later without mention, the fact that $\mathrm{res}_\star$ is an isomorphism, by Lemma 6. Observe that there exists an a with the required conditions because by Lemma 5, $H^1(F_n) \neq \{0\}$.

(2a). *$H^n(E)$ is not free.* We claim that

$$\mathrm{ann}_{n-1}(a) \neq H^{n-1}(F).$$

If $n = 1$ this inequality is true as $(a) \neq 0 \in H^1(F)$. Assume now that $n > 1$. We shall use Lemma 5 together with the fact that the restriction map in the cohomology ring of a profinite group to the cohomology ring of a closed subgroup is a ring homomorphism. (See for example [RZ, Proposition 7.9.4].)

Let $a_1 \in F_n^\times$ satisfy $(a_1) = (a)_1$, and extend $\{(a_1)\}$ to a basis $\{(a_1), (a_2), \cdots, (a_n)\}$ of $H^1(F_n)$. By Lemma 5, the element

$$(a_1) \cup (a_2) \cup \cdots \cup (a_n) \in H^n(F_n)$$

is nontrivial, so that $0 \neq (a_2) \cup \cdots \cup (a_n) \in H^{n-1}(F_n)$. Let $b \in H^{n-1}(F)$ satisfy

$$b_1 = (a_2) \cup \cdots \cup (a_n) \in H^{n-1}(F_n), \qquad b_2 = 0 \in H^{n-1}(F_m).$$

Then since the cup-product commutes with $\mathrm{res}_\star$,

$$\big((a) \cup b\big)_1 = (a_1) \cup b_1 \neq 0 \in H^n(F_n),$$

so that $(a) \cup b \neq 0 \in H^n(F)$ and hence $\mathrm{ann}_{n-1}(a) \neq H^{n-1}(F)$.

If $p > 2$, we conclude by Theorem 1 that $H^n(F)$ is not free. If $p = 2$, observe that since $\sqrt{-1} \in F$, we have $\operatorname{ann}_{n-1}(a, -1) = \operatorname{ann}_{n-1} 0 = H^{n-1}(F)$, so that $\operatorname{ann}_{n-1}(a, -1) \neq \operatorname{ann}_{n-1}(a)$. We deduce from Theorem 1 that $H^n(F)$ is not free.

(2b). $H^k(E)$ *is free for all* $k \geq n+1$. We claim that

$$\operatorname{ann}_n(a) = H^n(F).$$

Let $c \in H^n(F)$. Then since $H^{n+1}(F_n) = 0$ by Lemma 5,

$$\operatorname{res}_\star(a) \cup c = \big((a_1) \cup c_1\big) \oplus \big(0 \cup c_2\big) = 0 \oplus 0 = \operatorname{res}_\star 0$$

Hence $(a) \cup c = 0$ and $\operatorname{ann}_n(a) = H^n(F)$.

If $p > 2$ then we conclude by Theorem 1 that $H^{n+1}(E)$ is free, and by Theorem 2, $H^k(E)$ is free for all $k \geq n+1$. If $p = 2$, observe that $\operatorname{ann}_n(a, -1) = \operatorname{ann}_n 0 = H^n(F)$. Furthermore, we use Corollary 1 to obtain that $\operatorname{cor}\colon H^n(E) \to H^n(F)$ is surjective. Then by Corollary 4, we have that $H^k(E)$ is free for all $k \geq n+1$.

(3). $\operatorname{cd}(G_E) = m$. First we claim that G_E does not contain an element of order p. By Artin-Schreier's theorem (see for instance [J, Chapter VI, Theorem 17]), finite subgroups of absolute Galois groups are either trivial or of order 2, and since $\sqrt{-1} \in E$ no element of order 2 exists in G_E.

Then, by Serre's well-known theorem [S], we obtain

$$\operatorname{cd}(G_E) = \operatorname{cd}(G_F).$$

From Lemmas 5 and 6 we find that

$$\operatorname{cd}(G_F) = \max\{\operatorname{cd}(F_n), \operatorname{cd}(F_m)\} = m.$$

Thus $\operatorname{cd}(G_E) = m$ as required.

4.2. **The case** $n < m = \infty$**.** Set

$$F_\infty := \mathsf{C}\left(\left(\mathsf{Z}^m_{(p)}\right)\right), \qquad \text{where } m = \aleph_0.$$

With the same argument as in the proof of Lemma 6, there exists a field $F := F_{n,\infty}$ such that $G_F \cong G_{F_n} \star_{\text{pro-}p} G_{F_\infty}$ and $\xi_{p^2} \in F$. Then set

$$E = F(\sqrt[p]{a})$$

for any $a \in F^\times$ such that under the restriction map

$$\operatorname{res}_\star\colon H^1(F) \to H^1\left(F_n\right) \oplus H^1\left(F_\infty\right),$$

we have

$$\mathrm{res}_\star(a) = (a)_1 \oplus 0, \quad (a)_1 \neq 0.$$

Then $\mathrm{cd}(G_F) = \mathrm{cd}(G_E) = \infty$, and with the same argument as above we see that $\mathrm{cf}(E/F) = n$.

4.3. **The case $n = \infty = m$.** As above we let Γ be a direct sum of $\aleph_0$ copies of $\mathsf{Z}_{(p)}$. Then we set $F := F_\infty = \mathsf{C}((\Gamma))$. Let $a \in F^\times$ such that $v(a) \in \Gamma \setminus p\Gamma$, where v is a natural valuation on F. Then from the description of Galois cohomology of p-henselian fields (see [Wad, Theorem 3.6]), we obtain

$$\mathrm{ann}_n(a) = (a) \cup H^{n-1}(F)$$

and

$$(a) \cup H^{n-1}(F) \neq H^n(F) \quad \text{for all } n \in \mathsf{N}.$$

(Observe that when $p = 2$ we use the fact that $\sqrt{-1} \in F$ in the cited result.) Setting $E = F(\sqrt[p]{a})$, just as before we have that

$$\mathrm{cf}(E/F) = \infty,$$

as required.

5. When is Galois Cohomology Trivial?

First we need a lemma.

Lemma 7. *Suppose that $p = 2$. Then*

$$\{a\} \cdot \mathrm{ann}_{n-1}\{a, -1\} \subset N_{E/F} k_n E.$$

Proof. Let $\bar\beta \in \mathrm{ann}_{n-1}\{a, -1\}$. Then $\{-1\} \cdot \bar\beta \in \mathrm{ann}_n\{a\} = N_{E/F} k_n E$ by Theorem 7. Let $\gamma \in K_n E$ such that $\{-1\} \cdot \bar\beta = N_{E/F}(\bar\gamma)$. Then we have

$$\{a\} \cdot \bar\beta = N_{E/F}(\{\sqrt{a}\} \cdot i_E(\bar\beta) + \bar\gamma).$$

Thus $\{a\} \cdot \mathrm{ann}_{n-1}\{a, -1\} \subset N_{E/F} k_n E$ as asserted.

It is worth observing that if $n = 1$, Lemma 7 is equivalent to

$$\{a, -1\} = 0 \quad \text{if and only if} \quad \{a\} \in N_{E/F} k_1 E,$$

and therefore Lemma 7 can be viewed as a generalization of this statement.

Now we are ready to prove Theorem 3.

Proof of Theorem 3. As before, we translate to K-theory using Theorem 5. We first consider the case $p > 2$.

(1) $\implies$ (3). Assume that k_nE is a trivial $\mathsf{F}_p[G]$-module. Suppose $f \in K_{n-1}F$, and set $\beta = \{\sqrt[p]{a}\} \cdot i_E(f)$. Then

$$(\sigma - 1)\bar{\beta} = 0 \implies \{\xi_p\} \cdot i_E(\bar{f}) = i_E(\{\xi_p\} \cdot \bar{f}) = 0.$$

But then by Theorem 7, $\{\xi_p\} \cdot \bar{f} \in \{a\} \cdot k_{n-1}F$.

Now let $\gamma \in K_nE$ be arbitrary. Again, $(\sigma - 1)\bar{\gamma} = 0$. Then

$$i_E N_{E/F}\bar{\gamma} = (\sigma - 1)^{p-1}\bar{\gamma} = 0$$

and so by Theorem 7, $N_{E/F}\bar{\gamma} = \{a\} \cdot \bar{f}$ for $f \in K_{n-1}F$. By the projection formula, $N_{E/F}(\{\sqrt[p]{a}\} \cdot i_E(\bar{f})) = \{a\} \cdot \bar{f}$. Then

$$N_{E/F}(\bar{\gamma} - \{\sqrt[p]{a}\} \cdot i_E(\bar{f})) = 0,$$

and hence

$$N_{E/F}(\gamma - \{\sqrt[p]{a}\} \cdot i_E(f)) = pg,$$

for some $g \in K_{n-1}F$. Set

$$\beta = \gamma - \{\sqrt[p]{a}\} \cdot i_E(f) - i_E(g).$$

Then $N_{E/F}(\beta) = 0$. By Theorem 5, there exists $\alpha \in K_nE$ such that $(\sigma - 1)\alpha = \beta$. But since k_nE is fixed by G, $\bar{\beta} = 0$. Hence $k_nE = i_E(k_nF) + \{\sqrt[p]{a}\} \cdot i_E(k_{n-1}F)$.

(3) $\implies$ (2). Since $p > 2$, $N_{E/F}(\{\sqrt[p]{a}\} \cdot i_E(\bar{f})) = \{a\} \cdot \bar{f}$ for $f \in K_{n-1}F$, and $N_{E/F}(i_E(\bar{g})) = 0$ for $g \in K_nF$. Hence $N_{E/F}k_nE = \{a\} \cdot k_{n-1}F$. Since by Theorem 7, $\mathrm{ann}_n\{a\} = N_{E/F}k_nE$, we are done.

(2) $\implies$ (1). Assume that $\{\xi_p\} \cdot k_{n-1}F \subset \{a\} \cdot k_{n-1}F$ and $\{a\} \cdot k_{n-1}F = \mathrm{ann}_n\{a\}$. By Theorem 7, $\mathrm{ann}_n\{a\} = N_{E/F}k_nE$. Hence $\{\xi_p\} \cdot k_{n-1}F \subset N_{E/F}k_nE = \{a\} \cdot k_{n-1}F$. But by Theorem 7, $\{a\} \cdot k_{n-1}F = \ker i_E$. We then apply Lemma 2 to deduce that $k_nE = (k_nE)^G$.

Now we consider the case $p = 2$.

(1) $\implies$ (2). Assume that k_nE is a trivial $\mathsf{F}_2[G]$-module. Let $\alpha \in K_nE$. Then $i_E N_{E/F}\bar{\alpha} = (\sigma - 1)\bar{\alpha} = 0$ implies that $N_{E/F}\bar{\alpha} = \{a\} \cdot \bar{b}$ for some $b \in K_{n-1}F$, by Theorem 7. Now $\{a, -1\} = \{a, a\}$ in k_2F, and then

$$\{a, -1\} \cdot \bar{b} = \{a\} \cdot N_{E/F}\bar{\alpha} = 0,$$

again by Theorem 7. Hence $N_{E/F}k_nE \subset \{a\} \cdot \mathrm{ann}_{n-1}\{a, -1\}$. By Theorem 7, $N_{E/F}k_nE = \mathrm{ann}_n\{a\}$, whence $\mathrm{ann}_n\{a\} \subset \{a\} \cdot \mathrm{ann}_{n-1}\{a, -1\}$.

(2) $\implies$ (1). Assume $\operatorname{ann}_n\{a\} \subset \{a\} \cdot \operatorname{ann}_{n-1}\{a,-1\}$. By Theorem 7, we have $\operatorname{ann}_n\{a\} = N_{E/F}k_nE$ and therefore $N_{E/F}k_nE \subset \{a\} \cdot \operatorname{ann}_{n-1}\{a,-1\}$. By Lemma 7, $\{a\} \cdot \operatorname{ann}_{n-1}\{a,-1\} \subset N_{E/F}k_nE$ and hence $N_{E/F}k_nE = \{a\} \cdot \operatorname{ann}_{n-1}\{a,-1\}$. Let $\gamma \in K_nE$ be arbitrary. Then $N_{E/F}\bar{\gamma} = \{a\} \cdot \bar{b}$ for some $\bar{b} \in \operatorname{ann}_{n-1}\{a,-1\}$. Hence

$$(\sigma - 1)\bar{\gamma} = i_E N_{E/F}\bar{\gamma} = i_E(\{a\} \cdot \bar{b}).$$

But by Theorem 7, $i_E(\{a\} \cdot \bar{b}) = 0$. Hence $(\sigma - 1)\bar{\gamma} = 0$, and $(k_nE)^G = k_nE$ as required.

Now assume $p = 2$ and $a \in (F^{\times 2} + F^{\times 2}) \setminus F^2$. As in the proof of Theorem 2, we have that $\{a,-1\} = 0 \in k_2F$. Therefore $\{-1\} \in \operatorname{ann}_1\{a\}$. Since $\operatorname{ann}_1\{a\} = N_{E/F}k_1E$ by Theorem 7, we obtain $\{-1\} \in N_{E/F}k_1E$. Equivalently, $-1 \in N_{E/F}(E^\times)$. By [A, Theorem 3], E/F embeds in an extension E'/F cyclic of degree 4 with $E' = E(\sqrt{\delta})$ for $\delta \in E^\times$. Kummer theory tells us that $\overline{\{\delta\}} \in (k_1E)^G$, so $(\sigma - 1)(\overline{\{\delta\}}) = (\sigma + 1)(\overline{\{\delta\}}) = 0$. Therefore $(\sigma + 1)(\{\delta\}) \in 2K_1E$, whence $N_{E/F}(\delta) \in F^\times \cap E^{\times 2}$. On the other hand, Kummer theory gives that $(F^\times \cap E^{\times 2})/F^{\times 2} = \{F^{\times 2}, aF^{\times 2}\}$. If $N_{E/F}(\delta) = f^2$ for $f \in F^\times$ then we have:

$$\begin{aligned}\left(\sqrt{\delta}\right)^{\sigma^2-1} &= \left(\sqrt{\delta}\sqrt{\delta}^{\sigma}\right)^{\sigma-1} = \left(\sqrt{\delta}\sqrt{\delta^{\sigma}}\right)^{\sigma-1} = \left(\sqrt{N_{E/F}\delta}\right)^{\sigma-1} \\ &= (\pm f)^{\sigma-1} = 1.\end{aligned}$$

(The choice of sign in the square roots above is irrelevant as we apply $\sigma - 1$ afterwards.) Hence σ extends to an order 2 automorphism of E'/F, a contradiction. We may conclude that $\overline{\{\delta\}} \in (k_1E)^G$ satisfies $N_{E/F}\overline{\{\delta\}} = \{a\}$, as required.

We now follow the proof of the $p > 2$ case to show that (1) and (3) are equivalent in the $p = 2$ case as well.

(1) $\implies$ (3). Assume that k_nE is a trivial $\mathsf{F}_2[G]$-module. Let $\gamma \in K_nE$ be arbitrary. Then $(\sigma - 1)\bar{\gamma} = 0$. Hence

$$i_E N_{E/F}\bar{\gamma} = (\sigma - 1)\bar{\gamma} = 0$$

and so by Theorem 7, $N_{E/F}\bar{\gamma} = \{a\} \cdot \bar{f}$ for $f \in K_{n-1}F$. By the projection formula, $N_{E/F}(\{\delta\} \cdot i_E(\bar{f})) = \{a\} \cdot \bar{f}$. Then

$$N_{E/F}(\bar{\gamma} - \overline{\{\delta\}} \cdot i_E(\bar{f})) = 0,$$

and hence

$$N_{E/F}(\gamma - \{\delta\} \cdot i_E(f)) = 2g,$$

for some $g \in K_{n-1}F$. Set

$$\beta = \gamma - \{\delta\} \cdot i_E(f) - i_E(g).$$

Then $N_{E/F}(\beta) = 0$. By Theorem 5, there exists $\alpha \in K_nE$ such that $(\sigma - 1)\alpha = \beta$. But since k_nE is fixed by G, $\bar{\beta} = 0$. Hence $k_nE = i_E(k_nF) + \overline{\{\delta\}} \cdot i_E(k_{n-1}F)$.

(3) $\implies$ (1). If $\beta \in K_nE$, then $\bar{\beta} = i_E(\bar{g}) + \overline{\{\delta\}} \cdot i_E(\bar{f})$ for $g \in K_nF$ and $f \in K_{n-1}F$. Clearly $(\sigma - 1)\bar{\beta} = 0$, which implies that $(k_nE)^G = k_nE$.

It is worth looking more closely at the case $n = 1$ of Theorem 3.

Observe that the condition

$$(\xi_p) \cup H^0(F) \subset (a) \cup H^0(F),$$

for $n = 1$ and $p > 2$, is equivalent with the condition $\xi_{p^2} \in E^\times$. (See [MS, Corollary 1].)

In the case $n = 1$ and $p = 2$ the condition

$$\mathrm{ann}_1(a) \subset (a) \cup \mathrm{ann}_0(a, -1)$$

can be reformulated as follows:

$$\text{If } (a, -1) = 0 \text{ then } N_{E/F}(k_1E) \subset \langle\{a\}\rangle$$

and if $(a, -1) \neq 0$ then $\mathrm{ann}_1(a) = \{0\}$.

Since $\{-a, a\} = 0$ we see that $\mathrm{ann}_1(a) = 0$ implies that $\{-a\} = 0$ or equivalently $\{a\} = \{-1\}$. Recall that a field F is called Pythagorean if $F^2 + F^2 \subset F^2$. In the first case when $(a, -1) = 0$ the equality $\{-a, a\} = 0$ and $N_{E/F}(k_1E) \subset \langle\{a\}\rangle$ implies $\sqrt{-1} \in E^\times$.

Summarizing our discussion for $p = 2$ we have:

$H^1(E)$ is a trivial $\mathsf{F}_2[G]$-module if and only if either $(a, -1) = 0 \in H^2(F)$ and $N_{E/F}(k_1E) \subset \langle\{a\}\rangle$ or $a = -1$ and F is a Pythagorean field. In both cases $\sqrt{-1} \in E^\times$. (See [MS, Corollary 1].)

Corollary 6. *Suppose $n \in \mathsf{N}$ and $(k_nE)^G = k_nE$. Then we have the following exact sequence:*

$$0 \to \mathrm{ann}_{n-1}\{a\} \to k_{n-1}F \xrightarrow{\{a\}\cdot -} k_nF \xrightarrow{i_E}$$
$$k_nE \xrightarrow{N_{E/F}} \{a\} \cdot \mathrm{ann}_{n-1}\{a, \xi_p\} \to 0.$$

Here the map $\mathrm{ann}_{n-1}\{a\} \to k_{n-1}F$ is the natural inclusion.

Proof. Exactness at the first and second terms is obvious, and exactness at the third term follows from Theorem 7.

We consider exactness at the fifth term. In the $p = 2$ case, Theorem 3 tells us that $\text{ann}_n\{a\} \subset \{a\} \cdot \text{ann}_{n-1}\{a, -1\}$. By Theorem 7, we have $\text{ann}_n\{a\} = N_{E/F} k_n E$, hence $N_{E/F} k_n E \subset \{a\} \cdot \text{ann}_{n-1}\{a, -1\}$. By Lemma 7 we have the reverse inclusion, so that $N_{E/F} k_n E = \{a\} \cdot \text{ann}_{n-1}\{a, -1\}$ and the sequence is exact at the fifth term.

In the $p > 2$ case, observe that $\{\xi_p\} \cdot k_{n-1}F \subset \{a\} \cdot k_{n-1}F$ implies that $k_{n-1}F = \text{ann}_{n-1}\{a, \xi_p\}$, since $\{a, a\} = 0$. Therefore, by part (2) of Theorem 3, we know $\text{ann}_n\{a\} = \{a\} \cdot \text{ann}_{n-1}\{a, \xi_p\}$. By Theorem 7, we have $\text{ann}_n\{a\} = N_{E/F} k_n E$ and hence the sequence is exact at the fifth term in the $p > 2$ case as well.

Hence it remains to show exactness at the fourth term. Suppose $\gamma \in K_n E$ and $N_{E/F}\bar{\gamma} = 0$. Then there exists $f \in K_n F$ such that $N_{E/F}\gamma = pf$, and then $N_{E/F}(\gamma - i_E(f)) = 0$. By Theorem 5, there exists $\alpha \in K_n E$ such that $(\sigma - 1)\bar{\alpha} = \bar{\gamma} - i_E(\bar{f})$. But $(\sigma - 1)\bar{\alpha} = 0$ because $(k_n E)^G = k_n E$. Hence $\bar{\gamma} = i_E(\bar{f})$ and we are done.

We are now ready to prove Theorem 4.

Proof of Theorem 4. In the $p > 2$ case, the result on heredity follows from Theorem 3, part (3), together with two hereditary properties from Lemma 4: item (4), with $\alpha_1 = \{\xi_p\}$ and $\alpha_2 = \{a\}$, and item (5). The exact sequence, in turn, follows from Corollary 6.

In the case $p = 2$, by Theorem 3, it is sufficient to prove that condition (2) in the $p = 2$ case is also hereditary. Assume (2) holds for n and $m > n$. By a well-known fact in Milnor K-theory, the group $K_m E$ is generated by the symbols

$$\alpha = \{u, f_1, \ldots, f_{n-1}, \ldots, f_{m-1}\},$$
$$\text{if } n > 1 \text{ and by } \alpha = \{u, f_1, \ldots, f_{m-1}\} \text{ if } n = 1,$$

where $u \in E^*$ and $f_i \in F^*$ for all $i = 1, \ldots, m-1$. (See [FV, page 291, Corollary 2].)

Assume now that $n > 1$. By the projection formula, we obtain

$$N_{E/F}\bar{\alpha} = \overline{\{N_{E/F}u, f_1, \ldots, f_{n-1}\}} \cdot \overline{\{f_n, \ldots, f_{m-1}\}}.$$

Since $\text{ann}_n\{a\} = N_{E/F} k_n E$ by Theorem 7, condition (2) gives us that

$$\overline{\{N_{E/F}u, f_1, \ldots, f_{n-1}\}} = N_{E/F}\overline{\{u, f_1, \ldots, f_{n-1}\}} \in \{a\} \cdot \text{ann}_{n-1}\{a, -1\}.$$

Hence we may write

$$\overline{\{N_{E/F}u, f_1, \ldots, f_{n-1}\}} = \{a\} \cdot \bar{c},$$

where $\bar{c} \in \mathrm{ann}_{n-1}\{a, -1\}$. Observe that this last equality holds also in the case when $n = 1$, provided that we interpret the left-hand side as $\overline{\{N_{E/F}u\}}$. Thus

$$N_{E/F}\bar{\alpha} = \{a\} \cdot \bar{c} \cdot \overline{\{f_n, \ldots, f_{m-1}\}}$$

and

$$\bar{c} \cdot \overline{\{f_n, \ldots, f_{m-1}\}} \in \mathrm{ann}_{m-1}\{a, -1\}.$$

Therefore $N_{E/F}k_mE \subset \{a\} \cdot \mathrm{ann}_{m-1}\{a, -1\}$, and we see that condition (2) is indeed hereditary.

6. Examples of $H^k(E)$ Trivial for all $n < k$ and $H^n(E)$ Nontrivial, with Given Cohomological Dimension

We have shown in Theorem 4 that the property $H^n(E)$ is a trivial $\mathsf{F}_p[G]$-module is hereditary. This result leads naturally to the definition of an interesting invariant $\mathrm{ct}(E/F) \in \{0\} \cup \mathsf{N} \cup \{\infty\}$:

$$\mathrm{ct}(E/F) = \sup \{n \in \mathsf{N} \cup \{0\} \mid H^n(E) \text{ is not a trivial } \mathsf{F}_p[G]\text{-module}\}.$$

As with cf, we have chosen ct to indicate that after degree $\mathrm{ct}(E/F)$, Galois cohomology consists of trivial $\mathsf{F}_p[G]$-modules. Of course, if $H^n(E)$ is never trivial for $n \geq 1$ then $\mathrm{ct}(E/F) = \infty$, and otherwise $\mathrm{ct}(E/F) \in \mathsf{N} \cup \{0\}$. (Observe that since $H^0(E) \cong \mathsf{F}_p$ and there are no nontrivial G-actions on F_p, we always have that $H^0(E)$ is a trivial $\mathsf{F}_p[G]$-module. However, Theorem 4 establishes the hereditary property only when $n > 0$.)

If $\mathrm{ct}(E/F) = n \in \mathsf{N}$, then by definition $H^m(E)$ is a trivial $\mathsf{F}_p[G]$-module for all $m > n$. On the other hand, by the hereditary property we also have that $H^k(E)$ is not trivial for all $1 \leq k \leq \mathrm{ct}(E/F)$. A natural question arises: can we choose a suitable field extension E/F so that $\mathrm{ct}(E/F)$ is a given natural number or ∞? We show that the answer is affirmative. In fact, we can arrange that both values $\mathrm{ct}(E/F)$ and $\mathrm{cd}(G_E)$ are any natural numbers or ∞ and the absolute Galois group G_F is a pro-p-group modulo an obvious restriction, the inequality described below.

Suppose that the absolute Galois group G_E is a pro-p-group. Then, observing that $\{0\}^G = \{0\}$, we have:

$$\mathrm{ct}(E/F) \leq \mathrm{cd}(G_E).$$

Our result is then the following.

Given $1 \le n \le m \in \mathbb{N} \cup \{\infty\}$ and a prime p, there exists a cyclic extension E/F of degree p with $\xi_p \in F$ such that

1. *G_E is a pro-p-group;*
2. *$\operatorname{ct}(E/F) = n$; and*
3. *$\operatorname{cd}(G_E) = m$.*

It is quite an interesting feature of our construction that it parallels the construction made in the rather opposite free case dealt with before. The only difference is the choice of a in our field extension of the form $E = F(\sqrt[p]{a})$.

6.1. **The case $m \in \mathbb{N}$.**

(1). Let $F := F_{n,m}$ be a field of characteristic 0 with $G_F \cong \mathbb{Z}_p^n \star_{\text{pro-}p} \mathbb{Z}_p^m$ and $\xi_{p^2} \in F$, given by Lemma 6. Let

$$E = F(\sqrt[p]{a})$$

where $a \in F^\times$ such that under the restriction map on H^1,

$$\operatorname{res}_\star(a) = (a)_1 \oplus (a)_2, \qquad (a)_1 = 0,\ (a)_2 \neq 0.$$

Observe that there exists an a with the required conditions because by Lemma 5, $H^1(F_m) \neq \{0\}$.

(2a). *$H^n(E)$ is not trivial.* We claim that

$$\operatorname{ann}_n(a) \not\subset (a) \cup H^{n-1}(F).$$

By Lemma 5, $H^n(F_n)$ contains a nontrivial element c. Let $b \in H^n(F)$ such that

$$b_1 = c \in H^n(F_n) \text{ and } b_2 = 0 \in H^n(F_m).$$

Then $b \neq 0$ and since the cup-product commutes with $\operatorname{res}_\star$,

$$\operatorname{res}_\star(a) \cup b = \big(0 \cup b_1\big) \oplus \big((a_1) \cup 0\big) = 0 \in H^{n+1}(F).$$

Therefore $b \in \operatorname{ann}_n(a)$.

Not let $f \in H^{n-1}(F)$ be arbitrary. Then

$$\big((a) \cup f\big)_1 = 0 \cup f_1 = 0$$

and therefore $b \notin (a) \cup H^{n-1}(F)$. Thus $\operatorname{ann}_n(a) \not\subset (a) \cup H^{n-1}(F)$.

For the case $p > 2$, Theorem 3, part (2) implies that $H^n(E)$ is not trivial.

In the case $p = 2$ we have $(-1) = 0$ since $\sqrt{-1} \in F^\times$. Therefore $0 = (a, -1) \in H^2(F)$. Thus $\mathrm{ann}_{n-1}(a, -1) = H^{n-1}(F)$ and $(a) \cup \mathrm{ann}_{n-1}(a, -1) = (a) \cup H^{n-1}(F)$. Hence by our claim above $\mathrm{ann}_n(a) \not\subset (a) \cup \mathrm{ann}_{n-1}(a, -1)$, and we can again apply Theorem 3 to conclude that $H^n(E)$ is not trivial.

(2b). *$H^k(E)^G = H^k(E)$ for all $k \geq n+1$.*

Let $a_1 \in F_m^\times$ satisfy $(a_1) = (a)_2$ and extend $\{(a_1)\}$ to a basis $\{(a_1), \ldots, (a_m)\}$ of $H^1(F_m)$. Recall that by Lemma 5, $H^k(F_m)$ is just the kth homogenous summand of the exterior algebra over F_p generated by $H^1(F_m)$. Using this fact and writing each element in $H^k(F_m)$ as a sum of elements of the form

$$(a_{i_1}) \cup \cdots \cup (a_{i_k}), \ 1 \leq i_1 < i_2 < \cdots < i_k \leq m,$$

and also the fact that $H^k(F_n) = \{0\}$ we see that

$$\mathrm{ann}_k(a) = (a) \cup H^{k-1}(F).$$

Now again using Theorem 3 as in the case (2a), we conclude that $H^k(E)^G = H^k(E)$.

(3). $\mathrm{cd}(G_E) = m$. Indeed $\mathrm{cd}(G_E) = \mathrm{cd}(G_F)$ by Serre's theorem. (See [S] and the discussion in section 4.1 which guarantees that the hypothesis of Serre's theorem is valid.)

But from Lemma 5 and Lemma 6 we see that

$$\mathrm{cd}(G_F) = \max\{\mathrm{cd}(G_{F_n}), \mathrm{cd}(G_{F_m})\} = m.$$

Thus we see that in the case when $m < \infty$ we constructed a cyclic field extension E/F of degree p with required properties $(1), (2)$ and (3).

6.2. **The case $m = \infty$.** We first consider the subcase of this case when $n < \infty$. As in section 4.2 set

$$F_\infty := \mathsf{C}((\mathsf{Z}^m_{(p)})), \text{ where } m = \aleph_0.$$

By Lemma 6 we see that there exists a field $F := F_{n,\infty}$ such that $G_F \cong G_{F_n} \star_{\text{pro-}p} G_{F_\infty}$ and $\xi_{p^2} \in F$. Let $a \in F^\times$ such that under the restriction map

$$\mathrm{res}_\star : H^1(F) \to H^1(F_n) \oplus H^1(F_\infty)$$

we have

$$\mathrm{res}_\star(a) = 0 \oplus (a)_2, \ (a)_2 \neq 0.$$

Then $\mathrm{cd}(F) = \infty$ and with the same argument as above we see that $\mathrm{ct}(E/F) = n$.

Finally we consider the case $n = \infty = m$. Set again $F_\infty := \mathsf{C}((\mathsf{Z}^m_{(p)}))$, where $m = \aleph_0$ and $F = F_{\infty,\infty}$. Also let $a \in F^\times$ such that

$$\mathrm{res}_\star(a) = 0 \oplus (a)_2, (a)_2 \neq 0.$$

Then using the same argument as in (2b) we see that $\mathrm{ct}(F) = \infty$.

Our construction is now completed.

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