GAUSS-MANIN CONNECTIONS FOR ARRANGEMENTS, IV NONRESONANT EIGENVALUES

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Abstract. An arrangement is a finite set of hyperplanes in a finite dimensional complex affine space. A complex rank one local system on the arrangement complement is determined by a set of complex weights for the hyperplanes. We study the Gauss-Manin connection for the moduli space of arrangements of fixed combinatorial type in the cohomology of the complement with coefficients in the local system determined by the weights. For nonresonant weights, we solve the eigenvalue problem for the endomorphisms arising in the 1-form associated to the Gauss-Manin connection.

1. Introduction

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of n ordered hyperplanes in $\mathbb{C}^\ell$, with complement $\mathsf{M} = M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{j=1}^n H_j$. Assume that $\mathcal{A}$ contains ℓ linearly independent hyperplanes. A complex rank one local system on M is determined by a collection of weights $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$. Associated to $\boldsymbol{\lambda}$, we have a representation $\rho : \pi_1(\mathsf{M}) \to \mathbb{C}^*$, given by $\gamma_j \mapsto \exp(-2\pi\,\mathrm{i}\,\lambda_j)$ for any meridian loop γ_j about the hyperplane H_j of $\mathcal{A}$, and an associated local system $\mathcal{L}$ on M. For weights which are nonresonant in the sense of Schechtman, Terao, and Varchenko [13], the local system cohomology vanishes in all but one dimension, $H^q(\mathsf{M}; \mathcal{L}) = 0$ for $q \neq \ell$. Parallel translation of fibers over curves in the moduli space of all arrangements combinatorially equivalent to $\mathcal{A}$ gives rise to a Gauss-Manin connection on the vector bundle over this moduli space with fiber $H^\ell(\mathsf{M}; \mathcal{L})$. This connection arises in a variety of applications, including the Aomoto-Gelfand theory of hypergeometric integrals [2, 8, 12], and the representation theory of Lie algebras and quantum groups [14, 16]. As such, it has been studied by a number of authors, including Aomoto [1], Schechtman and Varchenko [14, 16], Kaneko [10], and Kanarek [9].

Denote the combinatorial type of $\mathcal{A}$ by $\mathcal{T}$. The moduli space of all arrangements of type $\mathcal{T}$ is determined by the set of dependent collections of subsets of hyperplanes in the projective closure of $\mathcal{A}$ in $\mathbb{CP}^\ell$, see [15]. Let $\mathsf{B}(\mathcal{T})$ be a smooth, connected component of this moduli space. There is a fiber bundle $\mathsf{p} : \mathsf{M}(\mathcal{T}) \to \mathsf{B}(\mathcal{T})$ whose fibers, $\mathsf{p}^{-1}(\mathsf{b}) = \mathsf{M}_\mathsf{b}$, are complements of arrangements $\mathcal{A}_\mathsf{b}$ of type $\mathcal{T}$. Since $\mathsf{B}(\mathcal{T})$ is connected, M_b is diffeomorphic to M. The fiber bundle $\mathsf{p} : \mathsf{M}(\mathcal{T}) \to \mathsf{B}(\mathcal{T})$ is locally trivial. Consequently, given a local system on the fiber, there is an associated flat

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vector bundle $\mathbf{H} \to \mathsf{B}(\mathcal{T})$, with fiber $H^\ell(\mathsf{M}_\mathsf{b};\mathcal{L}_\mathsf{b})$ at $\mathsf{b} \in \mathsf{B}(\mathcal{T})$. For nonresonant weights, Terao [15] showed that the Gauss-Manin connection on this vector bundle has connection 1-form

$$\nabla = \sum \Theta_{\mathcal{T}'} \otimes \Omega_{\boldsymbol{\lambda}}(\mathcal{T}',\mathcal{T}), \tag{1.1}$$

where $\Theta_{\mathcal{T}'}$ is a logarithmic 1-form on the closure of $\mathsf{B}(\mathcal{T})$ with a simple pole along the divisor corresponding to the codimension one degeneration $\mathcal{T}'$ of $\mathcal{T}$, and $\Omega_{\boldsymbol{\lambda}}(\mathcal{T}',\mathcal{T})$ is an endomorphism of $H^\ell(\mathsf{M};\mathcal{L})$. For general position arrangements, this Gauss-Manin connection was found by Aomoto and Kita [2]. Terao [15] computed this connection for a larger class of arrangements. In [4], we determined the "Gauss-Manin endomorphisms" $\Omega_{\boldsymbol{\lambda}}(\mathcal{T}',\mathcal{T})$ for all arrangements. The aim of this paper is to solve the eigenvalue problem for these endomorphisms.

Identify the hyperplanes of $\mathcal{A}$ with their indices. An edge of $\mathcal{A}$ is a nonempty intersection of hyperplanes in $\mathcal{A}$. An edge is *dense* if the subarrangement of hyperplanes containing it is irreducible: the hyperplanes cannot be partitioned into nonempty sets so that, after a change of coordinates, hyperplanes in different sets are in different coordinates, see [13]. For an edge X, define $\lambda_X = \sum_{X \subseteq H_j} \lambda_j$. Let $\mathcal{A}_\infty = \mathcal{A} \cup H_{n+1}$ be the projective closure of $\mathcal{A}$, the union of $\mathcal{A}$ and the hyperplane at infinity in $\mathbb{CP}^\ell$, see [12]. Set $\lambda_{n+1} = -\sum_{j=1}^n \lambda_j$. Schechtman, Terao, and Varchenko [13], refining work of Esnault, Schechtman, and Viehweg [6], found conditions on the weights which insure that the local system cohomology groups vanish except in the top dimension. They proved that if M is the complement of an arrangement $\mathcal{A}$ in $\mathbb{C}^\ell$ of combinatorial type $\mathcal{T}$ with ℓ linearly independent hyperplanes and $\mathcal{L}$ is a rank one local system on M whose weights $\boldsymbol{\lambda}$ satisfy the condition

$$\lambda_X \notin \mathbb{Z}_{\geq 0} \text{ for every dense edge } X \text{ of } \mathcal{A}_\infty,$$

then $H^q(\mathsf{M};\mathcal{L}) = 0$ for $q \neq \ell$ and $\dim H^\ell(\mathsf{M};\mathcal{L}) = |\chi(\mathsf{M})|$, where $\chi(\mathsf{M})$ is the Euler characteristic of M. These conditions depend only on the type $\mathcal{T}$, so we call weights satisfying them $\mathcal{T}$-nonresonant.

Throughout this paper, we assume that $\mathcal{A}$ contains ℓ linearly independent hyperplanes, hence $n \geq \ell$, and that $\boldsymbol{\lambda}$ is $\mathcal{T}$-nonresonant. We consider only codimension one degenerations of combinatorial types and refer to these as simply degenerations.

Theorem. *Let $\mathcal{T}'$ be a degeneration of $\mathcal{T}$, and let $\boldsymbol{\lambda}$ be a collection of generic $\mathcal{T}$-nonresonant weights for the rank one local system $\mathcal{L}$. Then the Gauss-Manin endomorphism $\Omega_{\boldsymbol{\lambda}}(\mathcal{T}',\mathcal{T})$ is diagonalizable. The spectrum of $\Omega_{\boldsymbol{\lambda}}(\mathcal{T}',\mathcal{T})$ is contained in the set $\{0, \lambda_S\}$, where $\lambda_S = \sum_{j\in S} \lambda_j$ for some $S \subset \{1,\dots,n+1\}$.*

The set S is part of a pair (S,r), called the *principal dependence* of the degeneration $\mathcal{T}'$ of $\mathcal{T}$, see Theorem 3.2. It follows from our results in Sections 2, 4, and 5 that weights $\boldsymbol{\lambda}$ which satisfy $\lambda_S \neq 0$ are sufficiently generic. Our results also yield an algorithm for determining the multiplicities of the eigenvalues, see Remark 5.3.

Let $\mathcal{G}$ denote the combinatorial type of a general position arrangement of n hyperplanes in $\mathbb{C}^\ell$. The cohomology of the complement of an arrangement of type $\mathcal{G}$ is the rank ℓ truncation, $A^\bullet(\mathcal{G})$, of the exterior algebra on n generators e_j, $j \in [n]$, where $[n] = \{1, \ldots, n\}$, corresponding to the hyperplanes. The Orlik-Solomon algebra $A^\bullet(\mathcal{A}) \simeq H^\bullet(M(\mathcal{A}); \mathbb{C})$ is generated by one dimensional classes a_j, $j \in [n]$. It is the quotient of $A^\bullet(\mathcal{G})$ by a homogeneous ideal, $I^\bullet(\mathcal{A})$, hence it is a finite dimensional graded $\mathbb{C}$-algebra [11]. It is known that $A^\bullet(\mathcal{A})$ depends only on the combinatorial type $\mathcal{T}$ of $\mathcal{A}$ so we may write $A^\bullet(\mathcal{T})$.

Weights $\boldsymbol{\lambda}$ yield an element $a_{\boldsymbol{\lambda}} = \sum_{j=1}^n \lambda_j a_j$ in $A^1(\mathcal{T})$, and multiplication by $a_{\boldsymbol{\lambda}}$ gives $A^\bullet(\mathcal{T})$ the structure of a cochain complex. The resulting cohomology $H^\bullet(\mathcal{T}) = H^\bullet(A^\bullet(\mathcal{T}), a_{\boldsymbol{\lambda}})$ is a combinatorial analog of $H^\bullet(M(\mathcal{A}); \mathcal{L})$. If the weights are $\mathcal{T}$-nonresonant, then $H^\bullet(M(\mathcal{A}); \mathcal{L}) \simeq H^\bullet(A^\bullet(\mathcal{T}), a_{\boldsymbol{\lambda}})$ and the only (possibly) nonzero group $H^\ell(\mathcal{T})$ has the $\beta\mathbf{nbc}$ basis of Falk and Terao [7]. This basis provides an explicit surjection $\tau : H^\ell(\mathcal{G}) \to H^\ell(\mathcal{T})$. Our results in [4] yield a commutative diagram of endomorphisms for each degeneration $\mathcal{T}'$ of $\mathcal{T}$:

$$\begin{array}{ccc} H^\ell(\mathcal{G}) & \xrightarrow{\tau} & H^\ell(\mathcal{T}) \\ \downarrow{\scriptstyle \widetilde{\Omega}_{\boldsymbol{\lambda}}(\mathcal{T}', \mathcal{T})} & & \downarrow{\scriptstyle \Omega_{\boldsymbol{\lambda}}(\mathcal{T}', \mathcal{T})} \\ H^\ell(\mathcal{G}) & \xrightarrow{\tau} & H^\ell(\mathcal{T}) \end{array} \tag{1.2}$$

The endomorphism $\widetilde{\Omega}_{\boldsymbol{\lambda}}(\mathcal{T}', \mathcal{T})$ of $H^\ell(\mathcal{G})$ is induced by an endomorphism $\omega^\bullet_{\boldsymbol{\lambda}}(\mathcal{T}', \mathcal{T})$ of $A^\ell(\mathcal{G})$, see [5], (3.3), and Theorem 3.1.

Here is a brief outline of the paper. In Section 2, we recall the Aomoto complex and the "formal Gauss-Manin connection matrices" of [5] which are essential in our arguments. We recall the moduli space of combinatorially equivalent arrangements in Section 3 and identify the principal dependence (S, r) of the degeneration $\mathcal{T}'$ of $\mathcal{T}$. Using the principal dependence, we construct a realizable type $\mathcal{T}(S, r)$ and an endomorphism $\Omega_{\boldsymbol{\lambda}}(S, r)$ of $H^\ell(\mathcal{G})$. In Section 4, we determine the eigenstructure of the endomorphism $\Omega_{\boldsymbol{\lambda}}(S, r)$. In Section 5, we show that $\widetilde{\Omega}_{\boldsymbol{\lambda}}(\mathcal{T}', \mathcal{T})$ may be replaced by $\Omega_{\boldsymbol{\lambda}}(S, r)$ in (1.2) and thereby determine the eigenstructure of $\Omega_{\boldsymbol{\lambda}}(\mathcal{T}', \mathcal{T})$. We conclude with several examples to illustrate the main result.

2. General position

In this section, we record a number of constructions in the Orlik-Solomon complex of a general position arrangement which will be used subsequently.

Let $\mathcal{G} = \mathcal{G}^\ell_n$ be the combinatorial type of a general position arrangement of n hyperplanes in $\mathbb{C}^\ell$, where $n \geq \ell$. The Orlik-Solomon algebra $A^\bullet(\mathcal{G})$ is the rank ℓ truncation of an exterior algebra on n generators. Let $T = \{i_1, \ldots, i_q\} \subset [n]$. If order matters, we call T a q-tuple and write $T = (i_1, \ldots, i_q)$ and $e_T = e_{i_1} \cdots e_{i_q}$. The algebra $A^\bullet(\mathcal{G})$ is generated (as an algebra) by $\{e_j \mid 1 \leq j \leq n\}$, and has (additive) basis $\{e_T\}$, where $e_T = 1$ if $T = \emptyset$, and $T \neq \emptyset$ is an increasingly ordered tuple of cardinality at most ℓ.

Define a map $\partial : A^q(\mathcal{G}) \to A^{q-1}(\mathcal{G})$ by $\partial(e_T) = \sum_{k=1}^q (-1)^{k-1} e_{T_k}$, where $T_k = (i_1, \ldots, \widehat{i_k}, \ldots, i_q)$ if $T = (i_1, \ldots, i_q)$. Then $\partial \circ \partial = 0$, providing $A^\bullet(\mathcal{G})$ with the structure of a chain complex

$$(A^\bullet(\mathcal{G}),\partial): \quad A^0(\mathcal{G}) \xleftarrow{\partial} A^1(\mathcal{G}) \longleftarrow \cdots \longleftarrow A^{\ell-1}(\mathcal{G}) \xleftarrow{\partial} A^\ell(\mathcal{G}) \tag{2.1}$$

It is well known that the homology of this complex is concentrated in the top dimension, $H_q(A(\mathcal{G}),\partial)=0$ for $q\neq\ell$. The dimension of the unique nontrivial homology group is $\beta(n,\ell)=\dim H_\ell(A^\bullet(\mathcal{G}),\partial)=\sum_{k=0}^{\ell}(-1)^k\binom{n}{k}=\binom{n-1}{\ell}$.

Weights $\boldsymbol{\lambda}=(\lambda_1,\dots,\lambda_n)\in\mathbb{C}^n$ determine an element $e_{\boldsymbol{\lambda}}=\sum_{j=1}^n\lambda_j e_j$ in $A^1(\mathcal{G})$. Since $A^\bullet(\mathcal{G})$ is a quotient of an exterior algebra, we have $e_{\boldsymbol{\lambda}}e_{\boldsymbol{\lambda}}=0$. Consequently, multiplication by $e_{\boldsymbol{\lambda}}$ defines a cochain complex

$$(A^\bullet(\mathcal{G}),e_{\boldsymbol{\lambda}}): \quad A^0(\mathcal{G}) \xrightarrow{e_{\boldsymbol{\lambda}}} A^1(\mathcal{G}) \longrightarrow \cdots \longrightarrow A^{\ell-1}(\mathcal{G}) \xrightarrow{e_{\boldsymbol{\lambda}}} A^\ell(\mathcal{G}) \tag{2.2}$$

If $\boldsymbol{\lambda}\neq 0$, it is well known that the cohomology of this complex is concentrated in the top dimension, $H^q(A^\bullet(\mathcal{G}),e_{\boldsymbol{\lambda}})=0$ for $q\neq\ell$, and that $\dim H^\ell(A^\bullet(\mathcal{G}),e_{\boldsymbol{\lambda}})=\beta(n,\ell)$.

The endomorphism $\omega_{\boldsymbol{\lambda}}(\mathcal{T}',\mathcal{T})$ of $A^\ell(\mathcal{G})$ which induces the map $\widetilde{\Omega}_{\boldsymbol{\lambda}}(\mathcal{T}',\mathcal{T})\colon H^\ell(\mathcal{G})\to H^\ell(\mathcal{G})$ of (1.2) is the specialization at $\boldsymbol{\lambda}$ of a "formal Gauss-Manin connection endomorphism" given in [5] and in (3.3). The latter is a linear combination of endomorphisms $\omega_S^\bullet$ of the Aomoto complex $(A_R^\bullet(\mathcal{G}),e_{\mathbf{y}})$ of $\mathcal{G}$, a universal complex for the cohomology $H^\bullet(A^\bullet(\mathcal{G}),e_{\boldsymbol{\lambda}})$. The Aomoto complex has terms $A_R^q(\mathcal{G})=A^q(\mathcal{G})\otimes R$, where $R=\mathbb{C}[y_1,\dots,y_n]$ is the polynomial ring, and the boundary map is given by multiplication by $e_{\mathbf{y}}=\sum_{j=1}^n y_je_j$.

The endomorphisms $\omega_S^\bullet$ correspond to subsets S of $[n+1]$, the index set of the projective closure of the general position arrangement in $\mathbb{CP}^\ell$, $\mathcal{G}_\infty$. The symmetric group Σ_{n+1} on $n+1$ letters acts on $A^\bullet(\mathcal{G})$ by permuting the hyperplanes of $\mathcal{G}_\infty$, and on R by permuting the variables y_j, where $y_{n+1}=-\sum_{j=1}^n y_j$. In the basis $\{e_j \mid 1\leq j\leq n\}$ for the Orlik-Solomon algebra, the action of $\sigma\in\Sigma_{n+1}$ is given by $\sigma(e_i)=e_{\sigma(i)}$ if $\sigma(n+1)=n+1$, and by

$$\sigma(e_i)=\begin{cases}-e_{\sigma(n+1)} & \text{if } \sigma(i)=n+1,\\ e_{\sigma(i)}-e_{\sigma(n+1)} & \text{if } \sigma(i)\neq n+1,\end{cases}$$

if $\sigma(n+1)\neq n+1$. Denote the induced action on the Aomoto complex by $\phi_\sigma: A_R^\bullet(\mathcal{G})\to A_R^\bullet(\mathcal{G})$,

$$\phi_\sigma(e_{i_1}\cdots e_{i_p}\otimes f(y_1,\dots,y_n))=\sigma(e_{i_1})\cdots\sigma(e_{i_p})\otimes f(y_{\sigma(1)},\dots,y_{\sigma(n)}).$$

Lemma 2.1. *For each $\sigma\in\Sigma_{n+1}$, the map ϕ_σ is a cochain automorphism of the Aomoto complex $(A_R^\bullet(\mathcal{G}),e_{\mathbf{y}})$.*

If $T=(i_1,\dots,i_p)\subset[n]$ is a p-tuple, then $(j,T)=(j,i_1,\dots,i_p)$ is the $(p+1)$-tuple which adds j with $1\leq j\leq n$ to T as its first entry. For $S=\{s_1,\dots,s_k\}\subset[n+1]$, let σ_S denote the permutation $\left(\begin{smallmatrix}1 & 2 & \cdots & k\\ s_1 & s_2 & \cdots & s_k\end{smallmatrix}\right)$. Write $S\equiv T$ if S and T are equal sets.

Definition 2.2. *Let T be a p-tuple, S a $q+1$ element subset of $[n+1]$, and $j \in [n]$. If $S = S_0 = [q+1]$, define the endomorphism $\omega^\bullet_{S_0} : (A^\bullet_R(\mathcal{G}), e_{\mathbf{y}}) \to (A^\bullet_R(\mathcal{G}), e_{\mathbf{y}})$ by*

$$\omega^p_{S_0}(e_T) = \begin{cases} y_j \partial e_{(j,T)} & \text{if } p = q \text{ and } S_0 \equiv (j,T), \\ e_{\mathbf{y}} \partial e_T & \text{if } p = q+1 \text{ and } S_0 \equiv T, \\ 0 & \text{otherwise.} \end{cases}$$

If $S \neq S_0$, define $\omega^\bullet_S = \phi_{\sigma_S} \circ \omega^\bullet_{S_0} \circ \phi^{-1}_{\sigma_S}$.

One can check that this agrees with the case by case definition in [5, Def. 4.1].

Proposition 2.3 ([5, Prop. 4.2])**.** *For every subset S of $[n+1]$, the map $\omega^\bullet_S$ is a cochain homomorphism of the Aomoto complex $(A^\bullet_R(\mathcal{G}), e_{\mathbf{y}})$.*

For $S_0 = [s]$, $1 \le q \le \ell$, and $1 \le r \le \min(q, s-1)$, consider the sets $\mathcal{V}^{q,r}_{S_0}$ and $\mathcal{W}^{q,r}_{S_0}$ of elements in $A^q_R(\mathcal{G})$ given by

$$\begin{aligned} \mathcal{V}^{q,r}_{S_0} &= \{e_J e_K \mid |J| \le r-1\} \bigcup \{\eta_{S_0} e_J e_K \mid |J| = r-1\} \quad \text{and} \\ \mathcal{W}^{q,r}_{S_0} &= \{e_{S_0} e_K \text{ (if } q \ge s)\} \bigcup \{(\partial e_J) e_K \mid |J| \ge r+1\} \bigcup \{\eta_{S_0} e_J e_K \mid |J| \ge r\}, \end{aligned}$$

where $J \subset S_0$, $K \subset [n] \setminus S_0$, and $\eta_S = \sum_{i \in S} y_i e_i$. Let $\mathcal{B}^{q,r}_{S_0} = \mathcal{V}^{q,r}_{S_0} \bigcup \mathcal{W}^{q,r}_{S_0}$. If $S \subset [n+1]$ and $S \neq S_0$, define $\mathcal{B}^{q,r}_S = \{\phi_{\sigma_S}(v) \mid v \in \mathcal{B}^{q,r}_{S_0}\}$. Define $\mathcal{V}^{q,r}_S$ and $\mathcal{W}^{q,r}_S$ analogously. Given weights $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$, let $\mathcal{B}^{q,r}_S(\boldsymbol{\lambda}) = \{v|_{y_i \mapsto \lambda_i} \mid v \in \mathcal{B}^{q,r}_S\}$ denote the specialization of $\mathcal{B}^{q,r}_S$ at $\boldsymbol{\lambda}$, a sets of vectors in $A^q(\mathcal{G})$. Define $\mathcal{V}^{q,r}_S(\boldsymbol{\lambda})$ and $\mathcal{W}^{q,r}_S(\boldsymbol{\lambda})$ analogously. We will abuse notation and write $\eta_S = \sum_{i \in S} \lambda_i e_i$ when working in the Orlik-Solomon algebra. Note that $\partial \eta_S = \lambda_S = \sum_{i \in S} \lambda_i$.

Lemma 2.4. *If $\lambda_S \neq 0$, the set of vectors $\mathcal{B}^{q,r}_S(\boldsymbol{\lambda})$ spans the vector space $A^q(\mathcal{G})$.*

Proof. It suffices to consider the case $S = S_0$.

First, we show that the set $\{\partial e_J \mid |J| = r+1\} \bigcup \{\eta_S e_J \mid |J| = r-1\}$ spans $A^r(\mathcal{G}^s_s)$, where $\mathcal{G}^s_s$ is a general position arrangement of s hyperplanes (indexed by S) in $\mathbb{C}^s$. For this arrangement, both the chain complex $(A^\bullet(\mathcal{G}^s_s), \partial)$ of (2.1) and the cochain complex $(A^\bullet(\mathcal{G}^s_s), e_{\boldsymbol{\lambda}}) = (A^\bullet(\mathcal{G}^s_s), \eta_S)$ of (2.2) are acyclic, and

$$\begin{aligned} \dim \operatorname{im}[\partial : A^{r+1}(\mathcal{G}^s_s) \to A^r(\mathcal{G}^s_s)] &= \beta(s,r) = \binom{s-1}{r}, \\ \dim \operatorname{im}[\eta_S : A^{r-1}(\mathcal{G}^s_s) \to A^r(\mathcal{G}^s_s)] &= \binom{s}{r} - \beta(s,r) = \binom{s-1}{r-1}. \end{aligned}$$

Note that $\dim A^r(\mathcal{G}^s_s) = \binom{s}{r} = \binom{s-1}{r} + \binom{s-1}{r-1}$.

Suppose $x \in \operatorname{span}\{\partial e_J \mid |J| = r+1\} \cap \operatorname{span}\{\eta_S e_J \mid |J| = r-1\}$. Then $\partial x = 0$, and $x = \eta_S y$ for some $y \in A^{r-1}(\mathcal{G}^s_s)$. So $\partial x = \partial(\eta_S y) = \lambda_S y - \eta_S \partial y = 0$. Since $\lambda_S \neq 0$, we can write $y = c \eta_S \partial y$, where $c = 1/\lambda_S$. But this implies that $x = \eta_S y = 0$. Consequently, $\{\partial e_J \mid |J| = r+1\} \bigcup \{\eta_S e_J \mid |J| = r-1\}$ spans $A^r(\mathcal{G}^s_s)$.

Using this, a straightforward exercise shows that the set of vectors $\mathcal{B}^{q,r}_S(\boldsymbol{\lambda})$ spans the vector space $A^q(\mathcal{G}) = A^q(\mathcal{G}^\ell_n)$. □

3. Principal dependence

Let $\mathcal{T}$ be the combinatorial type of the arrangement $\mathcal{A}$ of n hyperplanes in $\mathbb{C}^\ell$ with $n \geq \ell \geq 1$. We consider the family of all arrangements of type $\mathcal{T}$. Recall that $\mathcal{A}$ is ordered by the subscripts of its hyperplanes and we assume that $\mathcal{A}$, and hence every arrangement of type $\mathcal{T}$, contains ℓ linearly independent hyperplanes.

Choose coordinates $\mathbf{u} = (u_1, \dots, u_\ell)$ on $\mathbb{C}^\ell$. The hyperplanes of an arrangement of type $\mathcal{T}$ are defined by linear polynomials $\alpha_i = b_{i,0} + \sum_{j=1}^{\ell} b_{i,j} u_j$ $(i = 1, \dots, n)$. We embed the arrangement in projective space and add the hyperplane at infinity as last in the ordering, H_{n+1}. The moduli space of all arrangements of type $\mathcal{T}$ may be viewed as the set of matrices

$$\mathsf{b} = \begin{pmatrix} b_{1,0} & b_{1,1} & \cdots & b_{1,\ell} \\ b_{2,0} & b_{2,1} & \cdots & b_{2,\ell} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,0} & b_{n,1} & \cdots & b_{n,\ell} \\ 1 & 0 & \cdots & 0 \end{pmatrix} \tag{3.1}$$

whose rows are elements of $\mathbb{CP}^\ell$, and whose $(\ell+1) \times (\ell+1)$ minors satisfy certain dependency conditions, see [12, Prop. 9.2.2].

Given $S \subset [n+1]$, let $N_S(\mathcal{T}) = N_S(\mathsf{b})$ denote the submatrix of (3.1) with rows specified by S. Let $\operatorname{rank} N_S(\mathcal{T})$ be the size of the largest minor with nonzero determinant. Define the multiplicity of S in $\mathcal{T}$ by

$$m_S(\mathcal{T}) = |S| - \operatorname{rank} N_S(\mathcal{T}). \tag{3.2}$$

Call S *dependent* (in type $\mathcal{T}$) if $m_S(\mathcal{T}) > 0$. For such S, the linear polynomials $\{\alpha_j \mid j \in S\}$ are dependent. For $q \leq n+1$, let $\operatorname{Dep}(\mathcal{T})_q$ denote the dependent sets of cardinality q, and let $\operatorname{Dep}(\mathcal{T}) = \bigcup_q \operatorname{Dep}(\mathcal{T})_q$. If $\mathcal{T}'$ is a combinatorial type for which $\operatorname{Dep}(\mathcal{T}) \subset \operatorname{Dep}(\mathcal{T}')$, let $\operatorname{Dep}(\mathcal{T}', \mathcal{T}) = \operatorname{Dep}(\mathcal{T}') \setminus \operatorname{Dep}(\mathcal{T})$. Terao [15] showed that the combinatorial type $\mathcal{T}$ is determined by $\operatorname{Dep}(\mathcal{T})_{\ell+1}$, but dependent sets of both smaller and larger cardinality arise in our considerations, see Example 3.4.

Let

$$\operatorname{Dep}(\mathcal{T})_q^* = \{S \in \operatorname{Dep}(\mathcal{T})_q \mid \bigcap_{j \in S} H_j \neq \emptyset\}$$

and let $\operatorname{Dep}(\mathcal{T})^* = \bigcup_q \operatorname{Dep}(\mathcal{T})_q^*$. If $S \in \operatorname{Dep}(\mathcal{T})^*$, then $\operatorname{codim}(\bigcap_{j \in S} H_j) < |S|$. If $\mathcal{T}'$ is a combinatorial type for which $\operatorname{Dep}(\mathcal{T})^* \subset \operatorname{Dep}(\mathcal{T}')^*$, let $\operatorname{Dep}(\mathcal{T}', \mathcal{T})^* = \operatorname{Dep}(\mathcal{T}')^* \setminus \operatorname{Dep}(\mathcal{T})^*$. If $|S| \geq \ell + 2$, then $S \in \operatorname{Dep}(\mathcal{T})$ but $S \in \operatorname{Dep}(\mathcal{T})^*$ if and only if every subset of S of cardinality $\ell + 1$ is dependent. It is convenient to work with these smaller collections of dependent sets.

Define endomorphisms of $A_R^\bullet(\mathcal{G})$ by

$$\omega^\bullet(\mathcal{T}) = \sum_{S \in \operatorname{Dep}(\mathcal{T})} m_S(\mathcal{T}) \cdot \omega_S^\bullet \text{ and } \omega^\bullet(\mathcal{T}', \mathcal{T}) = \sum_{S \in \operatorname{Dep}(\mathcal{T}', \mathcal{T})} m_S(\mathcal{T}') \cdot \omega_S^\bullet. \tag{3.3}$$

These are cochain homomorphisms of the Aomoto complex by Proposition 2.3. Since $\mathrm{Dep}(\mathcal{T})^*_q = \mathrm{Dep}(\mathcal{T})_q$ for $q \le \ell + 1$, we have

$$\omega^\bullet(\mathcal{T}) = \sum_{S \in \mathrm{Dep}(\mathcal{T})^*} m_S(\mathcal{T}) \cdot \omega^\bullet_S \text{ and } \omega^\bullet(\mathcal{T}', \mathcal{T}) = \sum_{S \in \mathrm{Dep}(\mathcal{T}', \mathcal{T})^*} m_S(\mathcal{T}') \cdot \omega^\bullet_S. \tag{3.4}$$

Theorem 3.1 ([5]). *The endomorphism $\widetilde{\Omega}_{\boldsymbol{\lambda}}(\mathcal{T}', \mathcal{T})$ is induced by the specialization $\omega_{\boldsymbol{\lambda}}(\mathcal{T}', \mathcal{T}) := \omega^\ell(\mathcal{T}', \mathcal{T})|_{y_j \mapsto \lambda_j}$ of the endomorphism $\omega^\ell(\mathcal{T}', \mathcal{T})$.*

Denote the cardinality of S by $s = |S|$. For $1 \le r \le \min(\ell, s-1)$, consider the combinatorial type $\mathcal{T}(S, r)$ defined by

$$T \in \mathrm{Dep}(\mathcal{T}(S, r))^* \iff |T \cap S| \ge r + 1.$$

This type is realized by a pencil of hyperplanes indexed by S with a common subspace of codimension r, together with $n - s$ hyperplanes in general position. Note that for $r = 1$ the hyperplanes in S coincide, so $\mathcal{T}(S, r)$ is a multi-arrangement.

Theorem 3.2. *Let $\mathcal{T}'$ be a degeneration of a realizable combinatorial type $\mathcal{T}$. For each set $S_i \in \mathrm{Dep}(\mathcal{T}', \mathcal{T})^*$, let r_i be minimal so that $\mathrm{Dep}(\mathcal{T}(S_i, r_i))^* \subset \mathrm{Dep}(\mathcal{T}')^*$. Given the collection $\{(S_i, r_i)\}$, there is a unique pair (S, r) with $r = \min\{r_i\}$, $S_i \subset S$ for every pair (S_i, r_i) where $r_i = r$, and $\mathrm{Dep}(\mathcal{T}(S, r))^* \subset \mathrm{Dep}(\mathcal{T}')^*$.*

Proof. Terao [15] classified the three codimension one degeneration types in the moduli space of an arrangement whose only dependent set is the minimally dependent set T of size $q + 1$.

I: $|S \cap T| \le q - 1$ for all $S \in \mathrm{Dep}(\mathcal{T}', \mathcal{T})^*$;
II: $\{(m, T_k) \mid m \notin T\}$ for each fixed k, $1 \le k \le |T|$;
III: $\{(m, T_k) \mid 1 \le k \le |T|\}$ for each fixed $m \notin T$.

If $q = 1$, then Type II does not appear. Recall that $T_k = (i_1, \dots, \widehat{i_k}, \dots, i_{q+1})$ if $T = (i_1, \dots, i_{q+1})$, and note that $m \in [n+1]$ in cases II and III above.

It follows from our analysis of the corresponding types in general [5] that if a Type II degeneration is present, then the value of r decreases and there is a unique set of maximal cardinality with minimal r. In the other types, r remains constant, but a unique dependent set of $\mathcal{T}$ increases in $\mathcal{T}'$. $\square$

Definition 3.3. *Let $\mathcal{T}'$ be a degeneration of $\mathcal{T}$. We call the pair (S, r) which satisfies the conditions of Theorem 3.2 the* principal dependence *of the degeneration.*

Example 3.4. Let $\mathcal{T}$ be the combinatorial type of the arrangement $\mathcal{A}$ of 4 lines in $\mathbb{C}^2$ depicted in Figure 1. Here $\mathrm{Dep}(\mathcal{T})^* = \{123\}$.

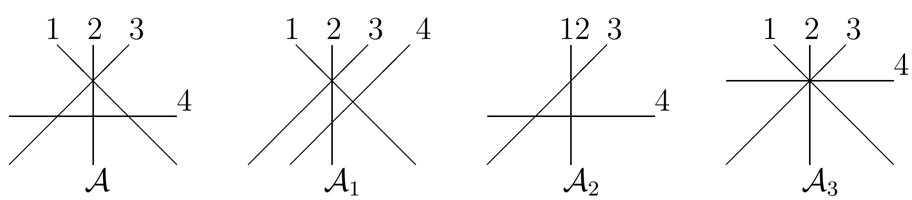

Figure 1. A line arrangement and three degenerations

The combinatorial types $\mathcal{T}_i$ of the (multi)-arrangements $\mathcal{A}_i$ shown in Figure 1 are degenerations of $\mathcal{T}$. For these degenerations, the collections $\{(S_i, r_i)\}$ and corresponding principal dependencies (S, r) are given in the table below.

	$\{(S_i, r_i)\}$	(S, r)
$\mathcal{T}_1$	$(345, 2)$	$(345, 2)$
$\mathcal{T}_2$	$(12, 1), (124, 2), (125, 2)$	$(12, 1)$
$\mathcal{T}_3$	$(124, 2), (134, 2), (234, 2), (1234, 2)$	$(1234, 2)$

For the combinatorial type $\mathcal{T}(S, r)$, write $\omega^\bullet(S, r) = \omega^\bullet(\mathcal{T}(S, r))$, see (3.4). In Theorem 5.1 below, we show that the Gauss-Manin endomorphism $\Omega_\lambda(\mathcal{T}', \mathcal{T})$ of (1.1) is induced by the specialization of $\omega^\ell(S, r)$ at $\mathcal{T}$-nonresonant weights $\boldsymbol{\lambda}$, $\omega^\ell_{\boldsymbol{\lambda}}(S, r)$. First, we solve the eigenvalue problem for the latter endomorphism.

4. Diagonalization

The purpose of this section is to solve the eigenvalue problem for $\omega^q_{\boldsymbol{\lambda}}(S, r)$, the endomorphism of the Orlik-Solomon algebra obtained by specializing $\omega^q(S, r)$ at generic weights $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$. This allows calculation of the eigenstructure of the induced endomorphism in cohomology, $\Omega_{\boldsymbol{\lambda}}(S, r)$, which is related to the Gauss-Manin endomorphism in Theorem 5.1. First, we establish several technical results concerning the endomorphism $\omega^q(S, r)$ of the Aomoto complex itself. Recall that these endomorphisms are given explicitly by

$$\omega^\bullet(S, r) = \sum_{K \in \mathrm{Dep}(\mathcal{T}(S,r))^*} m_K(S, r) \cdot \omega^\bullet_K,$$

where $m_K(S, r)$ is the multiplicity of K in type $\mathcal{T}(S, r)$, see (3.2), and $\omega^\bullet_K$ is given in Definition 2.2. It follows from Proposition 2.3 that $\omega^\bullet(S, r)$ is a chain map. Note that $\omega^q(S, r) = 0$ for $q < r$.

Given (S, r), define

$$\Psi^q_{S,r} = \sum_{\substack{T \subset S \\ |T| = r+1}} \omega^q(T, r).$$

Note that $\Psi^r_{S,r} = \omega^r(S, r)$. For $q \geq r$, the endomorphisms $\omega^q(S, r)$ satisfy the following recursion.

Lemma 4.1. *For $q \geq r$, we have*

$$\Psi^q_{S,r} = \sum_{k=0}^{s-r-1} \binom{r+k-1}{k} \omega^q(S, r+k).$$

Proof. If $T \subset [n]$ satisfies $|T| = r + 1$, then $\mathrm{Dep}(\mathcal{T}(T, r))^* = \{K \mid K \supseteq T\}$, and it is readily checked that $m_K(T, r) = 1$ for each such K. Hence, $\omega^q(T, r) = \sum_{K \supseteq T} \omega^q_K$, and we have

$$\Psi^q_{S,r} = \sum_{\substack{T \subset S \\ |T| = r+1}} \sum_{K \supseteq T} \omega^q_K = \sum_{|K \cap S| \geq r+1} \omega^q_K.$$

If $|K \cap S| = r+p$, then ω^q_K occurs $\binom{r+p}{r+1}$ times in this sum, so

$$\Psi^q_{S,r} = \sum_{|K\cap S|\geq r+1} \omega^q_K = \sum_{p\geq 1}\sum_{|K\cap S|=r+p} \binom{r+p}{r+1}\omega^q_K.$$

If $K \in \mathrm{Dep}(\mathcal{T}(S,j))^*$, then $|K\cap S| \geq j+1$, and $m_K(S,j) = |K\cap S| - j$. It follows that $\omega^q(S,j) = \sum_{|K\cap S|\geq j+1}(|K\cap S|-j)\omega^q_K$. Hence,

$$\sum_{k=0}^{s-r-1}\binom{r+k-1}{k}\omega^q(S,r+k) = \sum_{k=0}^{s-r-1}\sum_{i\geq k+1}\sum_{|K\cap S|=r+i}\binom{r+k-1}{k}(i-k)\omega^q_K$$

Rewriting this last sum, we obtain

$$\sum_{k=0}^{s-r-1}\binom{r+k-1}{k}\omega^q(S,r+k) = \sum_{p\geq 1}\sum_{|K\cap S|=r+p}\sum_{j=0}^{p-1}\binom{r+j-1}{j}(p-j)\omega^q_K.$$

A straightforward inductive argument shows that $\sum_{j=0}^{p-1}\binom{r+j-1}{j}(p-j) = \binom{r+p}{r+1}$, which completes the proof. □

Given S, recall that $\eta_S = \sum_{i\in S} y_i e_i$ and $y_S = \sum_{i\in S} y_i = \partial\eta_S$.

Lemma 4.2. *Let $J \subset S$ and $L \subset [n]\setminus S$. Then*

$$\Psi^q_{S,r}(e_J e_K) = \begin{cases} 0 & \text{if } |J| \leq r-1,\\ \binom{r+p}{r} y_S e_J e_K - \binom{r+p-1}{r-1}\eta_S(\partial e_J)e_K & \text{if } |J| = r+p, \text{ where } p\geq 0.\end{cases}$$

Proof. Given (J,L), it follows from Definition 2.2 that $\omega^q_K(e_J e_L) \neq 0$ only for the following K:

$$\begin{aligned} &(J,L), && (J_k,L,n+1), && (J,L_k,n+1),\\ &(i,J,L), && (J,L,n+1), && (i,J_k,L,n+1), && (i,J,L_k,n+1),\end{aligned} \tag{4.1}$$

where $i \notin (J,L)$.

If $|J| \leq r-1$, then $|K\cap S| \leq r$ for each of the above K, so $T \not\subset K$ for all $T\subset S$ with $|T| = r+1$. It follows that $\omega^q(T,r)(e_J e_L) = 0$ for each such T. Consequently, $\Psi^q_{S,r}(e_J e_L) = 0$.

Let $T \subset S$ be a subset of cardinality $r+1$, and note that $\Psi^q_{T,r} = \sum_{K\supset T}\omega^q_K$, so $\Psi^q_{S,r} = \sum_{T\subset S}\Psi^q_{T,r}$, where the sum is over all $T\subset S$ with $|T| = r+1$. Given such a T, if $|T\cap J| \leq r-1$, then none of the sets K recorded in (4.1) contains T. It follows that $\Psi^q_{T,r}(e_J e_L) = 0$ if $|T\cap J| \leq r-1$.

Suppose $|J| = r$. If $|J\cap T| = r$, then $T \equiv (i,J)$ for some $i \in S\setminus J$, and

$$\begin{aligned}\Psi^q_{T,r}(e_J e_L) &= \omega_{(i,J,L)}(e_J e_L) + \sum_{k=1}^{q-r}\omega_{(i,J,L_k,n+1)}(e_J e_L)\\ &= y_i\partial(e_i e_J e_L) + \sum_{k=1}^{q-r}(-1)^{r+k} y_i e_i e_J e_{L_k}\\ &= y_i e_J e_L - y_i e_i \partial(e_J e_L) + (-1)^r y_i e_i e_J \partial e_L = y_i e_J e_L - y_i e_i(\partial e_J)e_L.\end{aligned}$$

Therefore, using the identity $y_J e_J = \eta_J \partial e_J$, we have

$$\begin{aligned}\Psi^q_{S,r}(e_J e_L) &= \sum_{T\subset S} \Psi^q_{T,r}(e_J e_L) = \sum_{i\in S\setminus J} (y_i e_J e_L - y_i e_i(\partial e_J) e_L) \\ &= (y_S - y_J) e_J e_L - (\eta_S - \eta_J)(\partial e_J) e_L = y_S e_J e_L - \eta_S(\partial e_J) e_L.\end{aligned}$$

Now, assume that $|J| = r+p$ for some $p \geq 1$. As above, we have $\Psi^q_{T,r}(e_J e_L) = 0$ if $|T \cap J| \neq r, r+1$. If $|T \cap J| = r+1$, then $T \subseteq J$ and all of the sets K of (4.1) contain T. In this instance, $\Psi^q_{T,r}(e_J e_L) = \psi(e_J e_L)$, where

$$\psi = \omega_{(J,L)} + \omega_{(J,L,n+1)} + \sum_{k=1}^{q} \Big(\omega_{((J,L)_k,n+1)} + \sum_{i\notin (J,L)} \omega_{(i,(J,L)_k,n+1)}\Big)$$

Writing $J \equiv (T, J')$, a calculation reveals that $\Psi^q_{T,r}(e_J e_L) = \psi(e_J e_L) = y_T e_J e_L$.

If $|T \cap J| = r$, then $T \setminus T \cap J = \{t\}$ for some $t \in S \setminus J$. For such T, of the sets K from (4.1), only (t, J, L), $(t, J_k, L, n+1)$ for $j_k \notin T$, and $(t, J, L_k, n+1)$ contain T. This observation, and a calculation, yields

$$\begin{aligned}\Psi^q_{T,r}(e_J e_L) &= \Big(\omega_{(t,J,L)} + \sum_{j_k\notin T} \omega_{(t,J_k,L,n+1)} + \sum_{k=1}^{q-r-p} \omega_{(t,J,L_k,n+1)}\Big)(e_J e_L) \\ &= \Big(\omega_{(t,J,L)} + \sum_{k=1}^{r+p} \omega_{(t,(J,L)_k,n+1)} - \sum_{j_k\in T} \omega_{(t,J_k,L,n+1)}\Big)(e_j e_L) \\ &= y_t e_J e_L - \sum_{j_k\in T} (-1)^{k-1} y_t e_t e_{J_k} e_L.\end{aligned}$$

Summing over all $T \subset S$ with $|T \cap J| = r$, we obtain

$$\begin{aligned}\sum_{|T\cap J|=r} \Psi^q_{T,r}(e_J e_L) &= \sum_{t\in S\setminus J} \sum_{\substack{A\subset[r+p] \\ |A|=r}} \Big(y_t e_J e_L - \sum_{i=1}^{r} (-1)^{a_i-1} y_t e_t e_{J_{a_i}} e_L\Big) \\ &= \binom{r+p}{r}(y_S - y_J) e_J e_L - \sum_{t\in S\setminus J} \sum_{k=1}^{r+p} (-1)^{k-1} \binom{r+p-1}{r-1} y_t e_t e_{J_k} e_L \\ &= \binom{r+p}{r}(y_S - y_J) e_J e_L - \sum_{t\in S\setminus J} \binom{r+p-1}{r-1} y_t e_t (\partial e_J) e_L \\ &= \binom{r+p}{r} y_S e_J e_L - \binom{r+p-1}{r} y_J e_J e_L - \binom{r+p-1}{r-1} \eta_S (\partial e_J) e_L.\end{aligned}$$

Recall that $\Psi^q_{T,r}(e_J e_L) = y_T e_J e_L$ for $T \subset J$. Summing over all $T \subset J$, we obtain $\sum_{T\subset J} \Psi^q_{T,r}(e_J e_L) = \binom{r+p-1}{r} y_J e_J e_L$. Therefore,

$$\begin{aligned}\Psi^q_{S,r}(e_J e_L) &= \Big(\sum_{T\subset S} \Psi^q_{T,r}\Big)(e_J e_L) = \Big(\sum_{|T|=r} \Psi^q_{T,r} + \sum_{|T|=r+1} \Psi^q_{T,r}\Big)(e_J e_L)\\ &= \binom{r+p}{r} y_S e_J e_L - \binom{r+p-1}{r-1} \eta_S(\partial e_J) e_L\end{aligned}$$

if $|J| = r+p$. $\square$

Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ be a collection of weights, and consider the endomorphism $\omega^q_{\boldsymbol{\lambda}}(S,r) : A^q(\mathcal{G}) \to A^q(\mathcal{G})$ of the Orlik-Solomon algebra obtained by specializing $\omega^q(S,r)$ at $\boldsymbol{\lambda}$. Given S, we abuse notation and write $\eta_S = \sum_{i\in S} \lambda_i e_i$. Recall the spanning set $\mathcal{B}^{q,r}_S(\boldsymbol{\lambda}) = \mathcal{V}^{q,r}_S(\boldsymbol{\lambda}) \bigcup \mathcal{W}^{q,r}_S(\boldsymbol{\lambda})$ of $A^q(\mathcal{G})$ from Lemma 2.4.

Theorem 4.3. *Let $\boldsymbol{\lambda}$ be a collection of weights satisfying $\lambda_S \neq 0$. Then the specialization, $\omega^q_{\boldsymbol{\lambda}}(S,r)$, of $\omega^q(S,r)$ at $\boldsymbol{\lambda}$ is diagonalizable, with eigenvalues 0 and λ_S.*

1. *The 0-eigenspace is spanned by the set of vectors $\mathcal{V}^{q,r}_S(\boldsymbol{\lambda})$ and has dimension*
$$\sum_{p=0}^{r} \binom{s}{p}\binom{n-s}{q-p} - \binom{s-1}{r}\binom{n-s}{q-r}.$$
2. *The λ_S-eigenspace is spanned by the set of vectors $\mathcal{W}^{q,r}_S(\boldsymbol{\lambda})$ and has dimension*
$$\sum_{p=r+1}^{\min(q,s)} \binom{s}{p}\binom{n-s}{q-p} + \binom{s-1}{r}\binom{n-s}{q-r}.$$

Proof. By Lemma 2.4, the set of vectors $\mathcal{B}^{q,r}_S(\boldsymbol{\lambda}) = \mathcal{V}^{q,r}_S(\boldsymbol{\lambda}) \bigcup \mathcal{W}^{q,r}_S(\boldsymbol{\lambda})$ spans the vector space $A^q(\mathcal{G}^\ell_n)$. So to establish this result, it suffices to show that these vectors are eigenvectors of the endomorphism $\omega^q_{\boldsymbol{\lambda}}(S,r)$, and that the dimensions of the eigenspaces are as asserted. We will prove this by induction on $q-r$.

For ease of notation, we will suppress dependence on $\boldsymbol{\lambda}$ in the proof, and, for instance, write simply $\omega^q(S,r) = \omega^q_{\boldsymbol{\lambda}}(S,r)$ and $\Psi^q_{S,r} = \Psi^q_{S,r}|_{y_j\mapsto\lambda_j}$. Using Lemma 2.1, it suffices to consider the case $S \subset [n]$. Let $J \subset S$, $K \subset [n] \setminus S$, and recall that

$$\begin{aligned}\mathcal{V}^{q,r}_S &= \{e_J e_K \mid |J| \leq r-1\} \bigcup \{\eta_S e_J e_K \mid |J| = r-1\} \quad \text{and}\\ \mathcal{W}^{q,r}_S &= \{e_S e_K \ (\text{if } q \geq s)\} \bigcup \{(\partial e_J) e_K \mid |J| \geq r+1\} \bigcup \{\eta_S e_J e_K \mid |J| \geq r\}.\end{aligned}$$

In the case $q-r=0$, we have $\mathcal{V}^{r,r}_S = \{e_J e_K \mid |J| \leq r-1\} \bigcup \{\eta_S e_J \mid |J| = r-1\}$, $\mathcal{W}^{r,r}_S = \{\partial e_J \mid |J| = r+1\}$, and $\omega^r(S,r) = \Psi^r_{S,r}$. By Lemma 4.2, if $|J| \leq r-1$, then $\Psi^r_{S,r}(e_J e_K) = 0$. If $|J| = r-1$, then, using Lemma 4.2 again, we have

$$\begin{aligned}\Psi^r_{S,r}(\eta_S e_J) &= \sum_{i\in S} \lambda_i \Psi^r_{S,r}(e_i e_J) = \sum_{i\in S} \lambda_i(\lambda_S e_i e_J - \eta_S \partial(e_i e_J))\\ &= \lambda_S \eta_S e_J - \sum_{i\in S} \lambda_i \eta_S (e_J - e_i \partial e_J)\\ &= \lambda_S \eta_S e_J - \lambda_S \eta_S e_J + \eta_S \eta_S \partial e_J = 0.\end{aligned}$$

Thus, every element of $E^r(0) = \operatorname{span} \mathcal{V}_S^{r,r}$ is a 0-eigenvector of $\omega^r(S,r)$. A straightforward exercise reveals that $\dim E^r(0) = \sum_{k=0}^{r} \binom{s}{k}\binom{n-s}{r-k} - \binom{s-1}{r}$. If $|J| = r+1$, then, using Lemma 4.2 again,

$$\begin{aligned}\Psi^r_{S,r}(\partial e_J) &= \sum_{k=1}^{r+1}(-1)^{k-1}\Psi^r_{S,r}(e_{J_k}) = \sum_{k=1}^{r+1}(-1)^{k-1}(\lambda_S e_{J_k} - \eta_S \partial e_{J_k}) \\ &= \lambda_S \partial e_J - \eta_S \partial^2 e_J = \lambda_S \partial e_J\end{aligned}$$

Thus, every element of $E^r(\lambda_S) = \operatorname{span} \mathcal{W}_S^{r,r}$ is a λ_S-eigenvector of $\omega^r(S,r)$. Note that $\dim E^r(\lambda_S) = \binom{s-1}{r}$. Since $\dim E^r(0) + \dim E^r(\lambda_S) = \dim A^r(\mathcal{G}_n^\ell)$, the above calculations establish Theorem 4.3 in the case $q - r = 0$.

If $q - r \geq 1$, then by induction, for each $k \geq 1$, $\omega^q(S, r+k)$ is diagonalizable, with eigenvalues 0 and λ_S, and corresponding eigenspaces $E^{r+k}(0) = \operatorname{span} \mathcal{V}_S^{r+k,r}$ and $E^{r+k}(\lambda_S) = \operatorname{span} \mathcal{W}_S^{r+k,r}$. In the determination of the eigenstructure of $\omega^q(S,r)$, we will use the recursion provided by Lemma 4.1 in the following form:

$$\omega^q(S,r) = \Psi^q_{S,r} - \sum_{k=1}^{s-r-1}\binom{r+k-1}{k}\omega^q(S,r+k). \tag{4.2}$$

First, consider the 0-eigenspace of the endomorphism $\omega^q(S,r)$. If $|J| \leq r-1$, then by (4.2), Lemma 4.2, and induction, we have

$$\omega^q(S,r)(e_J e_K) = \Psi^q_{S,r}(e_J e_K) - \sum_{k=1}^{s-r-1}\binom{r+k-1}{k}\omega^q(S,r+k)(e_J e_K) = 0.$$

If $|J| = r-1$, then $\omega^q(S,r+k)(\eta_S e_J e_K) = 0$ for $k \geq 1$ by Lemma 4.2. Using (4.2) and Lemma 4.2, we have

$$\begin{aligned}\omega^q(S,r)(\eta_S e_J e_K) &= \Psi^q_{S,r}(\eta_S e_J e_K) - \sum_{k=1}^{s-r-1}\binom{r+k-1}{k}\omega^q(S,r+k)(\eta_S e_J e_K) \\ &= \Psi^q_{S,r}(\eta_S e_J e_K) = \sum_{i\in S}\lambda_i \Psi^q_{S,r}(e_i e_J e_K) \\ &= \sum_{i\in S}\big[\lambda_i\lambda_S e_i e_J e_K - \lambda_i \eta_S \partial(e_i e_J) e_K\big] \\ &= \lambda_S \eta_S e_J e_K - \sum_{i\in S}\lambda_i \eta_S e_J e_K + \sum_{i\in S}\lambda_i \eta_S e_i (\partial e_J) e_K \\ &= \lambda_S \eta_S e_J e_K - \lambda_S \eta_S e_J e_K + \eta_S \eta_S (\partial e_J) e_K = 0.\end{aligned}$$

Next, consider the λ_S-eigenspace. If $q \geq s$, we must show that $e_S e_K$ is an eigenvector of $\omega^q(S,r)$ corresponding to the eigenvalue λ_S for each $K \subset [n] \setminus S$ with $|K| = q-s$. By induction, we have $\omega^q(S,r+k)(e_S e_K) = \lambda_S e_S e_K$ for each $k \geq 1$. By Lemma 4.2, we have $\Psi^q_{S,r}(e_S e_K) = \binom{s}{r}\lambda_S e_S e_K - \binom{s-1}{r-1}\eta_S \partial e_S e_K$. Since $\eta_S \partial e_S = \lambda_S e_S$, we have $\Psi^q_{S,r}(e_S e_K) = \binom{s-1}{r}\lambda_S e_S e_K$. Hence, by (4.2), we have

$$\begin{aligned}\omega^q(S,r)(e_S e_K) &= \Psi^q_{S,r}(e_S e_K) - \sum_{k=1}^{s-r-1}\binom{r+k-1}{k}\omega^q(S,r+k)(e_S e_K)\\ &= \binom{s-1}{r}\lambda_S e_S e_K - \sum_{k=1}^{s-r-1}\binom{r+k-1}{k}\lambda_S e_S e_K = \lambda_S e_S e_K,\end{aligned}$$

using the binomial identities

$$\sum_{k=0}^{p}\binom{N+k}{k} = \binom{N+p+1}{p} = \binom{N+p+1}{N+1}, \tag{4.3}$$

with $N = r-1$ and $p = s-r-1$.

If $|J| \geq r+1$, we must show that $\omega^q(S,r)(\partial e_J e_K) = \lambda_S \partial e_J e_K$. Suppose $|J| = r+p+1$ for some $p \geq 0$. Then, by Lemma 4.2, we have

$$\begin{aligned}\Psi^q_{S,r}(\partial e_J e_K) &= \sum_{i=1}^{r+p+1}(-1)^{i-1}\Psi^q_{S,r}(e_{J_i} e_K)\\ &= \sum_{i=1}^{r+p+1}(-1)^{i-1}\left[\binom{r+p}{r}\lambda_S e_{J_i} e_K - \binom{r+p-1}{r-1}\eta_S(\partial e_{J_i})e_K\right]\\ &= \binom{r+p}{r}\lambda_S(\partial e_J)e_K - \binom{r+p-1}{r-1}\eta_S(\partial^2 e_J)e_K\\ &= \binom{r+p}{r}\lambda_S(\partial e_J)e_K.\end{aligned}$$

By induction, we have

$$\omega^q(S,r+k)((\partial e_J)e_K) = \begin{cases}\lambda_S(\partial e_J)e_K & \text{if } 1 \leq k \leq p,\\ 0 & \text{if } p+1 \leq k \leq s-r-1.\end{cases}$$

So using the recursion (4.2) and the identities (4.3), we obtain

$$\begin{aligned}\omega^q(S,r)((\partial e_J)e_K) &= \binom{r+p}{r}\lambda_S(\partial e_J)e_K - \sum_{k=1}^{p}\binom{r+k-1}{k}\lambda_S(\partial e_J)e_K\\ &= \lambda_S(\partial e_J)e_K.\end{aligned}$$

If $|J| \geq r$, we must show that $\omega^q(S,r)(\eta_S e_J e_K) = \lambda_S \eta_S e_J e_K$. Suppose $|J| = r+p$ for some $p \geq 0$. Then, by Lemma 4.2, we have

$$\begin{aligned}
\Psi^q_{S,r}(\eta_S e_J e_K) &= \sum_{i\in S} \lambda_i \Psi^q_{S,r}(e_i e_J e_K) \\
&= \sum_{i\in S} y_i \left[\binom{r+p+1}{r} \lambda_S e_i e_J e_K - \binom{r+p}{r-1} \eta_S \partial(e_i e_J) e_K \right] \\
&= \binom{r+p+1}{r} \lambda_S \eta_S e_J e_K - \binom{r+p}{r-1} \sum_{i\in S} \lambda_i \eta_S (e_J - e_i \partial e_J) e_K \\
&= \left[\binom{r+p+1}{r} - \binom{r+p}{r-1} \right] \lambda_S \eta_S e_J e_K + \binom{r+p}{r-1} \eta_S \eta_S (\partial e_J) e_K \\
&= \binom{r+p}{r} \lambda_S \eta_S e_J e_K.
\end{aligned}$$

By induction, we have

$$\omega^q(S, r+k)(\eta_S e_J e_K) = \begin{cases} \lambda_S \eta_S e_J e_K & \text{if } 1 \leq k \leq p, \\ 0 & \text{if } p+1 \leq k \leq s-r-1. \end{cases}$$

So using the recursion (4.2) and the identities (4.3), we obtain $\omega^q(S,r)(\eta_S e_J e_K) = \lambda_S \eta_S e_J e_K$ as above.

Thus the vectors in the sets $\mathcal{V}^{q,r}_S(\boldsymbol{\lambda})$ and $\mathcal{W}^{q,r}_S(\boldsymbol{\lambda})$ are eigenvectors of $\omega^q(S,r)$ corresponding to the eigenvalues 0 and λ_S as asserted. Since these vectors span $A^q(\mathcal{G}^\ell_n)$ by Lemma 2.4, it remains to compute the dimensions of the eigenspaces $E^q(0) = \operatorname{span} \mathcal{V}^{q,r}_S$ and $E^q(\lambda_S) = \operatorname{span} \mathcal{W}^{q,r}_S$ corresponding to these eigenvalues.

If $|J| = p$ and $|J| + |K| = q$, then $\operatorname{span}\{e_J e_K \mid J \subset S, K \subset [n] \setminus S\}$ has dimension

$$\binom{s}{p}\binom{n-s}{q-p}.$$

If $|J| = p+1$ and $|J| - 1 + |K| = q$, then $\operatorname{span}\{(\partial e_J) e_K \mid J \subset S, K \subset [n] \setminus S\}$ has dimension

$$\dim \operatorname{im}[\partial : A^{p+1}(\mathcal{G}^s_s) \to A^p(\mathcal{G}^s_s)] \cdot \binom{n-s}{q-p} = \binom{s-1}{p}\binom{n-s}{q-p}.$$

If $|J| = p-1$ and $|J| + 1 + |K| = q$, then $\operatorname{span}\{\eta_S e_J e_K \mid J \subset S, K \subset [n] \setminus S\}$ has dimension

$$\dim \ker[\eta_S : A^p(\mathcal{G}^s_s) \to A^{p+1}(\mathcal{G}^s_s)] \cdot \binom{n-s}{q-p} = \left[\binom{s}{p} - \binom{s-1}{p} \right] \binom{n-s}{q-p}.$$

Using these calculations, it is readily checked that

$$\dim E^q(0) = \sum_{p=0}^{r} \binom{s}{p}\binom{n-s}{q-p} - \binom{s-1}{r}\binom{n-s}{q-r}, \text{ and}$$

$$\dim E^q(\lambda_S) = \sum_{p=r+1}^{\min(q,s)} \binom{s}{p}\binom{n-s}{q-p} + \binom{s-1}{r}\binom{n-s}{q-r}.$$

The fact that $\dim E^q(0) + \dim E^q(\lambda_S) = \dim A^q(\mathcal{G}_n^\ell) = \binom{n}{q}$ may be checked using the binomial identities

$$\sum_{p=0}^{k} \binom{m}{p}\binom{N}{k-p} = \binom{m+N}{k} \text{ and } \sum_{p=0}^{m} \binom{m}{p}\binom{N}{k+p} = \binom{m+N}{m+r} = \binom{m+N}{N-k}$$

with $m = s$, $N = n - s$, and $k = q$ in the case $q < s$, and $m = s$, $N = n - s$, and $k = n - s - q$ in the case $q \geq s$. $\square$

If $n = \ell$ and $\boldsymbol{\lambda} \neq 0$, the complex $(A^\bullet(\mathcal{G}), e_{\boldsymbol{\lambda}})$ is acyclic. So assume that $n > \ell$. Then, for $\boldsymbol{\lambda} \neq 0$, the cohomology of this complex is concentrated in dimension ℓ, and $\dim H^\ell(\mathcal{G}) = \binom{n-1}{\ell}$. Let $\rho = \rho_{\mathcal{G}} : A^\ell(\mathcal{G}) \to H^\ell(\mathcal{G})$ denote the projection. Since $\omega^\bullet_{\boldsymbol{\lambda}}(S, r)$ is a chain map, the kernel of this projection, $\ker(\rho) \subset A^\ell(\mathcal{G})$, is an invariant subspace for $\omega^\ell_{\boldsymbol{\lambda}}(S, r)$.

Lemma 4.4. *Let $T : V \to V$ be an endomorphism of a finite dimensional (complex) vector space, and V' an invariant subspace. If T is diagonalizable, then the induced endomorphism T'' on the quotient $V'' = V/V'$ is also diagonalizable, and the spectrum of T'' is contained in the spectrum of T.*

Proof. Let T' denote the restriction of T to V', and let $\pi : V \to V''$ be the projection. The vector space V admits a basis $\mathcal{B} = \{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ for which $\mathcal{B}' = \{v_1, \dots, v_k\}$ is a basis for the subspace V' and $\mathcal{B}'' = \{\pi(v_{k+1}), \dots, \pi(v_n)\}$ is a basis for the quotient V''. The matrix of T relative to the basis $\mathcal{B}$ is

$$\mathsf{A} = \begin{pmatrix} \mathsf{A}' & * \\ 0 & \mathsf{A}'' \end{pmatrix},$$

where A' is the matrix of T' relative to $\mathcal{B}'$ and A'' is the matrix of the induced endomorphism T'' relative to $\mathcal{B}''$.

Let $r_1, \dots, r_m$ be the distinct eigenvalues of T. Since T is diagonalizable, the minimal polynomial p of T factors as $p(t) = (t - r_1) \cdots (t - r_m)$. The polynomial p annihilates the matrix A of T, $p(\mathsf{A}) = 0$. Using the block decomposition of A above, it follows that p also annihilates the matrix A'' of T'', $p(\mathsf{A}'') = 0$. Consequently, the minimal polynomial p'' of T'' divides p. Hence, p'' is of the form $(t - r_{i_1}) \cdots (t - r_{i_j})$, T'' is diagonalizable, and the eigenvalues of T'' are among the eigenvalues of T. $\square$

For an arrangement $\mathcal{A}$ of arbitrary combinatorial type $\mathcal{T}$, and $\mathcal{T}$-nonresonant weights $\boldsymbol{\lambda}$, we recall the β**nbc** basis of [7] for the single nonvanishing cohomology group $H^\ell(\mathcal{T}) = H^\ell(\mathsf{M}; \mathcal{L})$. Recall that the hyperplanes of $\mathcal{A} = \{H_j\}_{j=1}^n$ are ordered. A circuit is an inclusion-minimal dependent set of hyperplanes in $\mathcal{A}$, and a broken circuit is a set T for which there exists $H < \min(T)$ so that $T \cup \{H\}$ is a circuit. A frame is a maximal independent set, and an **nbc** frame is a frame which contains no broken circuit. Since $\mathcal{A}$ contains ℓ linearly independent hyperplanes, every frame has cardinality ℓ. The set of **nbc** frames is a basis for $A^\ell(\mathcal{T})$. An **nbc** frame $B = (H_{j_1}, \dots, H_{j_\ell})$ is a β**nbc** frame provided that for each k, $1 \leq k \leq \ell$, there exists $H \in \mathcal{A}$ such that $H < H_{j_k}$ and $(B \setminus \{H_{j_k}\}) \cup \{H\}$ is a frame. Note that these constructions depend only on the combinatorial type $\mathcal{T}$ of $\mathcal{A}$, and let $\beta\mathbf{nbc}(\mathcal{T})$ be the set of all β**nbc** frames of an arrangement of type $\mathcal{T}$.

Definition 4.5. *Given* $B = (H_{j_1}, \dots, H_{j_\ell})$ *in* $\beta\mathbf{nbc}(\mathcal{T})$, define $\xi(B) \in A^\ell(\mathcal{T})$ by $\xi(B) = \wedge_{p=1}^{\ell} a_\lambda(X_p)$, where $X_p = \bigcap_{k=p}^{\ell} H_{j_k}$ and $a_\lambda(X) = \sum_{X \subseteq H_i} \lambda_i a_i$. Denote the cohomology class of $\xi(B)$ in $H^\ell(\mathcal{T}) = H^\ell(A^\bullet(\mathcal{T}), a_{\boldsymbol\lambda})$ by the same symbol. The set $\{\xi(B) \mid B \in \beta\mathbf{nbc}(\mathcal{T})\}$ is the $\beta\mathbf{nbc}$ basis for $H^\ell(\mathcal{T})$.

Theorem 4.6. *Let* $S \subset [n+1]$ *be a subset of cardinality* s*, and fix* r*,* $1 \le r \le \min(\ell, s-1)$*. For* $\mathcal{G}$*-nonresonant weights* $\boldsymbol\lambda$ *satisfying* $\lambda_S \neq 0$*, the endomorphism* $\Omega_{\boldsymbol\lambda}(S,r)$ *of* $H^\ell(\mathcal{G})$ *induced by* $\omega^\ell_{\boldsymbol\lambda}(S,r)$ *is diagonalizable, with eigenvalues* 0 *and* λ_S*. The dimension of the* λ_S*-eigenspace is*

$$\sum_{p=r+1}^{\min(\ell,s)} \binom{s}{p}\binom{n-s-1}{\ell-p} + \binom{s-1}{r}\binom{n-s-1}{\ell-r},$$

and the dimension of the 0*-eigenspace is*

$$\sum_{p=0}^{r} \binom{s}{p}\binom{n-s-1}{\ell-p} - \binom{s-1}{r}\binom{n-s-1}{\ell-r}.$$

Proof. By Theorem 4.3 and Lemma 4.4, the endomorphism $\Omega_{\boldsymbol\lambda}(S,r)$ is diagonalizable, with spectrum contained in $\{0, \lambda_S\}$.

Let $\mathbf{I} = \{I = (i_1, \dots, i_\ell) \mid 1 \le i_1 < i_2 \cdots < i_\ell \le n\}$. Then $\{e_I \mid I \in \mathbf{I}\}$ is the $\mathbf{nbc}$ basis of $A^\ell(\mathcal{G})$ and $\{\xi_I = \lambda_{i_1} \cdots \lambda_{i_\ell} e_I \mid I \in \mathbf{I}, 1 \notin I\}$ is the $\beta\mathbf{nbc}$ basis of $H^\ell(\mathcal{G})$. The projection $\rho : A^\ell(\mathcal{G}) \to H^\ell(\mathcal{G})$ is given by

$$\rho(e_I) = \begin{cases} (\lambda_{i_1} \cdots \lambda_{i_\ell})^{-1} \xi_I & \text{if } 1 \notin I, \\ -(\lambda_{i_1} \cdots \lambda_{i_\ell})^{-1} \sum_{j \notin I} \xi_j \xi_{I_1} & \text{if } 1 \in I. \end{cases}$$

Using Lemma 2.1, we can assume that $S \subset [2,n]$. Since $\rho \circ \omega^\ell_{\boldsymbol\lambda}(S,r) = \Omega_{\boldsymbol\lambda}(S,r) \circ \rho$, if v is an eigenvector of $\omega^\ell_{\boldsymbol\lambda}(S,r)$ and $\rho(v) \neq 0$, then $\rho(v)$ is an eigenvector of $\Omega_{\boldsymbol\lambda}(S,r)$. Let $J \subset S$ and $K \subset [2,n] \setminus S$. Note that $1 \notin K$. Then one can check that the 0-eigenspace of $\Omega_{\boldsymbol\lambda}(S,r)$ is spanned by

$$\{\rho(e_J e_K) \mid |J| \le r-1\} \bigcup \{\rho(\eta_S e_J e_K) \mid |J| = r-1\},$$

that the λ_S-eigenspace of $\Omega_{\boldsymbol\lambda}(S,r)$ is spanned by

$$\{\rho(e_S e_K) \mid \text{if } \ell \ge s\} \bigcup \{\rho((\partial e_J) e_K) \mid |J| \ge r+1\} \bigcup \{\rho(\eta_S e_J e_K) \mid |J| \ge r\},$$

and that the dimensions of these eigenspaces are as asserted. □

Example 4.7. Let $n = 5$, $\ell = 2$, $S = \{3,4,5\}$, and $r = 1$. By Theorem 4.6, for $\mathcal{G}$-nonresonant weights satisfying $\lambda_S \neq 0$, the endomorphism $\Omega_{\boldsymbol\lambda}(S,r)$ of $H^2(\mathcal{G}) \simeq \mathbb{C}^6$ is diagonalizable, the λ_S-eigenspace is 5-dimensional, and the 0-eigenspace is 1-dimensional (note that $\binom{p}{q} = 0$ if $p < q$). Calculating as in the proof of Theorem 4.6, we find that the λ_S-eigenspace has basis

$$\begin{aligned} &\rho(\lambda_2\lambda_3\lambda_5(\partial e_{3,5})e_2) = \lambda_5\xi_{2,3} - \lambda_3\xi_{2,5}, && \rho(-\lambda_3\eta_{3,4,5}e_3) = \xi_{3,4} + \xi_{3,5}, \\ &\rho(\lambda_2\lambda_4\lambda_5(\partial e_{4,5})e_2) = \lambda_5\xi_{2,4} - \lambda_4\xi_{2,5}, && \rho(\lambda_5\eta_{3,4,5}e_5) = \xi_{3,5} + \xi_{4,5}, \\ &\rho(\lambda_3\lambda_4\lambda_5\partial e_{3,4,5}) = \lambda_5\xi_{3,4} - \lambda_4\xi_{3,5} + \lambda_3\xi_{4,5}, \end{aligned}$$

and the 0-eigenspace has basis $\rho(\lambda_1\lambda_2 e_{1,2}) = \xi_{2,3} + \xi_{2,4} + \xi_{2,5}$.

5. Nonresonant eigenvalues

In this section, we prove that the Gauss-Manin endomorphism $\Omega_{\boldsymbol{\lambda}}(\mathcal{T}',\mathcal{T})$ of (1.1) is diagonalizable and determine its eigenvalues. We accomplish this by showing that the endomorphism $\widetilde{\Omega}_{\boldsymbol{\lambda}}(\mathcal{T}',\mathcal{T})$ in the commutative diagram (1.2) may be replaced by the endomorphism $\Omega_{\boldsymbol{\lambda}}(S,r)$, whose eigenstructure was computed in Theorem 4.6.

For an arbitrary type $\mathcal{T}$, let $I^\bullet(\mathcal{T})$ be the corresponding Orlik-Solomon ideal, so that $A^\bullet(\mathcal{T}) \simeq A^\bullet(\mathcal{G})/I^\bullet(\mathcal{T})$. The natural projection of $A^\bullet(\mathcal{G})$ onto $A^\bullet(\mathcal{T})$ is a chain map $\pi : (A^\bullet(\mathcal{G}), e_{\boldsymbol{\lambda}}) \to (A^\bullet(\mathcal{T}), a_{\boldsymbol{\lambda}})$ which, for $\mathcal{T}$-nonresonant weights $\boldsymbol{\lambda}$, induces the projection $\tau : H^\ell(\mathcal{G}) \to H^\ell(\mathcal{T})$ upon passage to cohomology. If $\rho_{\mathcal{G}} : A^\ell(\mathcal{G}) \to H^\ell(\mathcal{G})$ and $\rho_{\mathcal{T}} : A^\ell(\mathcal{T}) \to H^\ell(\mathcal{T})$ are the projections, then $\tau \circ \rho_{\mathcal{G}} = \rho_{\mathcal{T}} \circ \pi$.

Theorem 5.1. *If $\mathcal{T}'$ is a degeneration of $\mathcal{T}$ with principal dependence (S,r), then $\Omega_{\boldsymbol{\lambda}}(\mathcal{T}',\mathcal{T}) \circ \tau = \tau \circ \Omega_{\boldsymbol{\lambda}}(S,r)$. In other words, the following diagram commutes:*

$$\begin{tikzcd}
H^\ell(\mathcal{G}) \arrow[r, "\tau"] \arrow[d, "\Omega_{\boldsymbol{\lambda}}(S,r)"] & H^\ell(\mathcal{T}) \arrow[d, "\Omega_{\boldsymbol{\lambda}}(\mathcal{T}',\mathcal{T})"] \\
H^\ell(\mathcal{G}) \arrow[r, "\tau"] & H^\ell(\mathcal{T})
\end{tikzcd}$$

Proof. As noted in the introduction, the Gauss-Manin endomorphism $\Omega_{\boldsymbol{\lambda}}(\mathcal{T}',\mathcal{T})$ of $H^\ell(\mathcal{T})$ is induced by the endomorphism $\widetilde{\Omega}_{\boldsymbol{\lambda}}(\mathcal{T}',\mathcal{T})$ of $H^\ell(\mathcal{G})$, see [4, Thm. 7.3] and (1.2). In turn, $\widetilde{\Omega}_{\boldsymbol{\lambda}}(\mathcal{T}',\mathcal{T})$ is the map in cohomology induced by the cochain endomorphism $\omega^\bullet_{\boldsymbol{\lambda}}(\mathcal{T}',\mathcal{T})$ of the complex $(A^\bullet(\mathcal{G}), e_{\boldsymbol{\lambda}})$, see Theorem 3.1. The map $\omega^\bullet_{\boldsymbol{\lambda}}(\mathcal{T}',\mathcal{T})$ also induces a cochain endomorphism $\bar{\omega}^\bullet_{\boldsymbol{\lambda}}(\mathcal{T}',\mathcal{T})$ of $(A^\bullet(\mathcal{T}), a_{\boldsymbol{\lambda}})$, and the Gauss-Manin endomorphism $\Omega_{\boldsymbol{\lambda}}(\mathcal{T}',\mathcal{T})$ may be realized as the map in cohomology induced by the latter, see [5, Thm. 7.1]. In summary, we have the following commutative diagram.

$$\begin{tikzcd}
A^\ell(\mathcal{G}) \arrow[rr, "\pi"] \arrow[dr, "\rho_{\mathcal{G}}"] \arrow[dd, "\omega^\ell_{\boldsymbol{\lambda}}(\mathcal{T}',\mathcal{T})"'] & & A^\ell(\mathcal{T}) \arrow[dr, "\rho_{\mathcal{T}}"] \arrow[dd, "\bar{\omega}^\ell_{\boldsymbol{\lambda}}(\mathcal{T}',\mathcal{T})"'] & \\
& H^\ell(\mathcal{G}) \arrow[rr, "\tau"] \arrow[dd, "\widetilde{\Omega}_{\boldsymbol{\lambda}}(\mathcal{T}',\mathcal{T})"'] & & H^\ell(\mathcal{T}) \arrow[dd, "\Omega_{\boldsymbol{\lambda}}(\mathcal{T}',\mathcal{T})"] \\
A^\ell(\mathcal{G}) \arrow[rr, "\pi"] \arrow[dr, "\rho_{\mathcal{G}}"] & & A^\ell(\mathcal{T}) \arrow[dr, "\rho_{\mathcal{T}}"] & \\
& H^\ell(\mathcal{G}) \arrow[rr, "\tau"] & & H^\ell(\mathcal{T})
\end{tikzcd} \tag{5.1}$$

To establish the theorem, it suffices to show that the endomorphisms $\omega^\bullet_{\boldsymbol\lambda}(\mathcal{T}',\mathcal{T})$ and $\omega^\bullet_{\boldsymbol\lambda}(S,r)$ of $A^\bullet(\mathcal{G})$ induce the same endomorphism of $A^\bullet(\mathcal{T})$.

The Orlik-Solomon ideal $I^\bullet(\mathcal{T})$ gives rise to a subcomplex $I^\bullet_R(\mathcal{T}) = I^\bullet(\mathcal{T})\otimes R$ of the Aomoto complex $A^\bullet_R(\mathcal{G})$, with quotient $A^\bullet_R(\mathcal{T})$, the Aomoto complex of type $\mathcal{T}$. Since $\omega^\bullet_{\boldsymbol\lambda}(\mathcal{T}',\mathcal{T})$ and $\omega^\bullet_{\boldsymbol\lambda}(S,r)$ are specializations at $\boldsymbol\lambda$ of the corresponding endomorphims of the Aomoto complex $A^\bullet_R(\mathcal{G})$, it is enough to show that $\omega^\bullet(\mathcal{T}',\mathcal{T})$ and $\omega^\bullet(S,r)$ induce the same endomorphism of $A^\bullet_R(\mathcal{T})$.

By Theorem 3.2, there are dependence pairs (S_i,r_i), $1\le i\le k$, such that $\operatorname{Dep}(\mathcal{T})^*$ contains $\operatorname{Dep}(\mathcal{T}(S_i,r_i))^*$ and $\operatorname{Dep}(\mathcal{T}')^* = \bigcup_{i=0}^k \operatorname{Dep}(\mathcal{T}(S_i,r_i))^*$, where $(S_0,r_0) = (S,r)$ is the pair of principal dependence. It follows that there are constants c_i so that $\omega^\bullet(\mathcal{T}') = \omega^\bullet(S,r) + \sum_{i=1}^k c_i\cdot\omega^\bullet(S_i,r_i)$.

If $\operatorname{Dep}(\mathcal{T}(S_i,r_i))^* \subset \operatorname{Dep}(\mathcal{T})^*$, it follows from Theorem 4.3 that the image of $\omega^\bullet(S_i,r_i) : A^\bullet_R(\mathcal{G}) \to A^\bullet_R(\mathcal{G})$ is contained in $I^\bullet_R(\mathcal{T})$. Consequently, the endomorphisms $\bar\omega^\bullet(\mathcal{T}')$ and $\bar\omega^\bullet(S,r)$ of the Aomoto complex $A^\bullet_R(\mathcal{T})$ induced by $\omega^\bullet(\mathcal{T}')$ and $\omega^\bullet(S,r)$ are equal.

Finally, $\omega^\bullet(\mathcal{T}') = \omega^\bullet(\mathcal{T}',\mathcal{T}) + \omega^\bullet(\mathcal{T})$, see (3.3). It follows from the definitions that the image of $\omega^\bullet(\mathcal{T})$ is also contained in $I^\bullet_R(\mathcal{T})$. Hence, the endomorphisms $\bar\omega^\bullet(\mathcal{T}')$ and $\bar\omega^\bullet(\mathcal{T}',\mathcal{T})$ of $A^\bullet_R(\mathcal{T})$ induced by $\omega^\bullet(\mathcal{T}')$ and $\omega^\bullet(\mathcal{T}',\mathcal{T})$ are equal. □

Theorem 4.6 and Theorem 5.1 yield the result stated in the introduction.

Theorem 5.2. *Let $\mathcal{T}'$ be a degeneration of $\mathcal{T}$ with principal dependence (S,r), and $\boldsymbol\lambda$ a collection of $\mathcal{T}$-nonresonant weights satisfying $\lambda_S\neq 0$. Then the Gauss-Manin endomorphism $\Omega_{\boldsymbol\lambda}(\mathcal{T}',\mathcal{T})$ is diagonalizable, with spectrum contained in $\{0,\lambda_S\}$.*

Proof. By Theorem 4.6, the endomorphism $\Omega_{\boldsymbol\lambda}(S,r)$ of $H^\ell(\mathcal{G})$ is diagonalizable, with eigenvalues 0 and λ_S. By Theorem 5.1, we have $\Omega_{\boldsymbol\lambda}(\mathcal{T}',\mathcal{T})\circ\tau = \tau\circ\Omega_{\boldsymbol\lambda}(S,r)$. Checking that $\ker(\tau)\subset H^\ell(\mathcal{G})$ is an invariant subspace for $\Omega_{\boldsymbol\lambda}(S,r)$, the result follows from Lemma 4.4. □

Remark 5.3. The Gauss-Manin endomorphism $\Omega_{\boldsymbol\lambda}(\mathcal{T}',\mathcal{T})$ of $H^\ell(\mathcal{T})$ is determined by the endomorphism $\Omega_{\boldsymbol\lambda}(S,r)$ of $H^\ell(\mathcal{G})$ and the projection $\tau : H^\ell(\mathcal{G})\to H^\ell(\mathcal{T})$ via the equality $\Omega_{\boldsymbol\lambda}(\mathcal{T}',\mathcal{T})\circ\tau = \tau\circ\Omega_{\boldsymbol\lambda}(S,r)$. Together with the explicit description of the eigenstructure of $\Omega_{\boldsymbol\lambda}(S,r)$ provided by Theorems 4.3 and 4.6, this yields an algorithm for finding the (geometric) multiplicities of the eigenvalues of $\Omega_{\boldsymbol\lambda}(\mathcal{T}',\mathcal{T})$.

The Gauss-Manin connection $\nabla = \sum \Theta_{\mathcal{T}'}\otimes\Omega_{\boldsymbol\lambda}(\mathcal{T}',\mathcal{T})$ on the vector bundle $\mathbf{H}\to \mathsf{B}(\mathcal{T})$ with fiber $H^\ell(\mathcal{T})$ corresponds to a monodromy representation $\Psi : \pi_1(\mathsf{B}(\mathcal{T}))\to \operatorname{Aut}_{\mathbb{C}}\bigl(H^\ell(\mathcal{T})\bigr)$. For a degeneration $\mathcal{T}'$ of $\mathcal{T}$, let $\gamma_{\mathcal{T}'}\in\pi_1(\mathsf{B}(\mathcal{T}))$ be a simple loop in $\mathsf{B}(\mathcal{T})$ around a generic point in $\mathsf{B}(\mathcal{T}')$. Then the automorphism $\Psi(\gamma_{\mathcal{T}'})$ is conjugate to $\exp\bigl(-2\pi\,\mathrm{i}\,\Omega_{\boldsymbol\lambda}(\mathcal{T}',\mathcal{T})\bigr)$, see for instance [3, Prop. 4.1]. Theorem 5.2 yields:

Corollary 5.4. *Let $\mathcal{T}'$ be a degeneration of $\mathcal{T}$ with principal dependence (S,r), and $\boldsymbol\lambda$ a collection of $\mathcal{T}$-nonresonant weights satisfying $\lambda_S\neq 0$. Then the automorphism $\Psi(\gamma_{\mathcal{T}'})$ is diagonalizable, with spectrum contained in $\{1,\exp(-2\pi\,\mathrm{i}\,\lambda_S)\}$.*

We conclude with several examples which illustrate these results.

5.5. **Codimension zero.** Recall that $\mathcal{G}$ denotes the combinatorial type of a general position arrangement of n hyperplanes in $\mathbb{C}^\ell$, and that $n \geq \ell$. Weights $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ are $\mathcal{G}$-nonresonant if $\lambda_j \neq 0$ for each j. If $n = \ell$, then $H^\bullet(\mathcal{G}) = 0$, so we assume that $n > \ell$. Then $\dim H^\ell(\mathcal{G}) = \binom{n-1}{\ell}$. The moduli space $\mathsf{B}(\mathcal{G})$ has codimension zero in $(\mathbb{CP}^\ell)^n$, and consists of all matrices b for which every $(\ell+1) \times (\ell+1)$ minor is nonzero, see (3.1). For general position arrangements, the Gauss-Manin connection was determined by Aomoto and Kita [2]. The corresponding connection 1-form is given by $\nabla = \sum \Theta_{\mathcal{T}} \otimes \Omega_{\boldsymbol{\lambda}}(\mathcal{T}, \mathcal{G})$, where the sum is over all $\ell+1$ element subsets S of $[n+1]$, $\mathcal{T} = \mathcal{T}(S, \ell+1)$, and $\Theta_{\mathcal{T}}$ is a logarithmic 1-form on $(\mathbb{CP}^\ell)^n$ with a simple pole along the divisor defined by the vanishing of the $(\ell+1) \times (\ell+1)$ minor of b with rows indexed by S. Theorem 4.6 gives:

Proposition 5.6. *Let S be an $\ell+1$ element subset of $[n]$, let $\mathcal{T} = \mathcal{T}(S, \ell+1)$, and $\boldsymbol{\lambda}$ a collection of $\mathcal{G}$-nonresonant weights satisfying $\lambda_S \neq 0$. Then the Gauss-Manin endomorphism $\Omega_{\boldsymbol{\lambda}}(\mathcal{T}, \mathcal{G})$ is diagonalizable, with eigenvalues 0 and λ_S. The dimension of the λ_S-eigenspace is 1, and the dimension of the 0-eigenspace is $\binom{n-1}{\ell} - 1$.*

5.7. **Codimension one.** If $\mathcal{T}$ is a combinatorial type for which the cardinality of $\operatorname{Dep}(\mathcal{T})_{\ell+1}$ is 1, then the moduli space $\mathsf{B}(\mathcal{T})$ is of codimension one in $(\mathbb{CP}^\ell)^n$. Write $\operatorname{Dep}(\mathcal{T})_{\ell+1} = \{K\}$. As shown by Terao [15], noted in the proof of Theorem 3.2, and illustrated in Example 3.4, the combinatorial type $\mathcal{T}$ admits three types of degeneration $\mathcal{T}' = \mathcal{T}(S, r)$. The principal dependencies of these degenerations are as follows.

I: (S, ℓ), where $|S| = \ell+1$ and $|S \cap K| \leq \ell - 1$;
II: $(S, \ell-1)$, where $S = K_p$, for each p, $1 \leq p \leq \ell+1$;
III: (S, ℓ), where $S = (m, K)$, for each $m \in [n+1] \setminus K$.

For the combinatorial type $\mathcal{T}$ and $\mathcal{T}$-nonresonant weights $\boldsymbol{\lambda}$, the Gauss-Manin connection was determined by Terao [15]. The corresponding connection 1-form is given by $\nabla = \sum \Theta_{\mathcal{T}'} \otimes \Omega_{\boldsymbol{\lambda}}(\mathcal{T}', \mathcal{T})$, where $\mathcal{T}'$ ranges over the three types of degeneration of $\mathcal{T}$ noted above. In [15], Terao also found the eigenvalues of the endomorphism $\Omega_{\boldsymbol{\lambda}}(\mathcal{T}', \mathcal{T})$ and their algebraic multiplicities. If $\boldsymbol{\lambda}$ satisfies $\lambda_S \neq 0$ for each of the principal dependence sets S recorded above, Terao's result concerning the eigenstructure of the endomorphism $\Omega_{\boldsymbol{\lambda}}(\mathcal{T}', \mathcal{T})$ may be strengthened as follows.

Proposition 5.8. *Let $\mathcal{T}$ be a combinatorial type of codimension one, let $\mathcal{T}' = \mathcal{T}(S, r)$ be a degeneration of $\mathcal{T}$, and $\boldsymbol{\lambda}$ a collection of $\mathcal{T}$-nonresonant weights satisfying $\lambda_S \neq 0$. Then the Gauss-Manin endomorphism $\Omega_{\boldsymbol{\lambda}}(\mathcal{T}', \mathcal{T})$ is diagonalizable, with eigenvalues 0 and λ_S.*

1. *If $\mathcal{T}'$ is a degeneration of type* I*, the dimension of the λ_S-eigenspace is 1, and the dimension of the 0-eigenspace is $\dim H^\ell(\mathcal{T}) - 1 = \binom{n-1}{\ell} - 2$.*
2. *If $\mathcal{T}'$ is a degeneration of type* II*, the dimension of the λ_S-eigenspace is $n-\ell-1$, and the dimension of the 0-eigenspace is $\binom{n-1}{\ell} - n + \ell$.*
3. *If $\mathcal{T}'$ is a degeneration of type* III*, the dimension of the λ_S-eigenspace is ℓ, and the dimension of the 0-eigenspace is $\binom{n-1}{\ell} - \ell - 1$.*

Proof. By Theorem 5.2, the endomorphism $\Omega_{\boldsymbol{\lambda}}(\mathcal{T}', \mathcal{T})$ is diagonalizable, with spectrum contained in $\{0, \lambda_S\}$.

Without loss, assume that $\mathrm{Dep}(\mathcal{T})_{\ell+1} = \{K\}$, where $K = [\ell+1]$. Then the **nbc** basis of $A^\ell(\mathcal{T})$ consists of monomials a_I, where $I \subset [n]$, $|I| = \ell$, and $I \neq [2, \ell+1]$. Write $F = [2, \ell+1]$. The projection $\pi : A^\ell(\mathcal{G}) \to A^\ell(\mathcal{T})$ is given by

$$\pi(e_I) = \begin{cases} a_I & \text{if } I \neq F, \\ a_1 \partial a_F & \text{if } I = F. \end{cases}$$

The β**nbc** basis for $H^\ell(\mathcal{T})$ consists of monomials ξ_I, where $I \subset [2, n]$, $|I| = \ell$, and $I \neq F$. The projection $\rho = \rho_{\mathcal{T}} : A^\ell(\mathcal{T}) \to H^\ell(\mathcal{T})$ is given by

$$\rho(a_I) = \begin{cases} (\lambda_{i_1} \cdots \lambda_{i_\ell})^{-1} \xi_I & \text{if } 1 \notin I, \\ -(\lambda_{i_1} \cdots \lambda_{i_\ell})^{-1} \sum_{j \notin I} \xi_j \xi_{I_1} & \text{if } 1 \in I, I \not\subset K, \\ -(\lambda_K \lambda_{i_1} \cdots \lambda_{i_\ell})^{-1} \sum_{j \notin K} \Big[\lambda_I \xi_j \xi_{I_1} + \xi_j \xi_p \partial \xi_{I_1} \Big] & \text{if } 1 \in I, I = K \setminus \{p\}. \end{cases}$$

If $\mathcal{T}'$ is a degeneration of type I with principal dependence (S, ℓ), then $|S \cap K| \leq \ell - 1$ and we can assume that $S \cap K \subset [3, \ell+1]$. By Theorem 4.3, the endomorphism $\omega^\ell_{\boldsymbol{\lambda}}(S, \ell)$ of $A^\ell(\mathcal{G})$ is diagonalizable, with eigenvalues 0 and λ_S. The 0-eigenspace is spanned by $\{e_J e_L \mid |J| \leq \ell - 1\} \bigcup \{\eta_S e_J \mid |J| = \ell - 1\}$, where $J \subset S$ and $L \subset [n] \setminus S$, and the λ_S-eigenspace is spanned by ∂e_S. By Theorem 5.1, the endomorphism $\Omega_{\boldsymbol{\lambda}}(\mathcal{T}', \mathcal{T})$ of $H^\ell(\mathcal{T})$ satisfies $\Omega_{\boldsymbol{\lambda}}(\mathcal{T}', \mathcal{T}) \circ \rho \circ \pi = \rho \circ \pi \circ \omega^\ell_{\boldsymbol{\lambda}}(S, \ell)$, see (5.1). Write $S = (s_1, \ldots, s_{\ell+1})$. Calculations with the projections π and ρ yield

$$\begin{aligned} \rho \circ \pi(\partial e_S) &= (\lambda_{s_1} \cdots \lambda_{s_{\ell+1}})^{-1} \partial \xi_S, \\ \rho \circ \pi(e_J e_L) &= (\lambda_{i_1} \cdots \lambda_{i_\ell})^{-1} \xi_J \xi_L, \text{ where } I = (J, L) \text{ and } 1 \notin L, \\ \rho \circ \pi(\eta_S e_J) &= \lambda_{s_1} \lambda_{s_p} (\lambda_{s_1} \cdots \lambda_{s_{\ell+1}})^{-1} (\xi_{S_p} \pm \xi_{S_1}), \text{ where } J = (s_2, \ldots, \hat{s}_p, \ldots, s_{\ell+1}). \end{aligned}$$

Checking that

$$\{\partial \xi_S)\} \bigcup \{\xi_J \xi_L \mid J \subset S, |J| \leq \ell - 1, L \subset [2, n] \setminus S\} \bigcup \{\xi_{S_p} \pm \xi_{S_1} \mid 2 \leq p \leq \ell + 1\}$$

forms a basis for $H^\ell(\mathcal{T})$, we conclude that the dimensions of the eigenspaces are as asserted for a degeneration of type I.

If $\mathcal{T}'$ is a degeneration of type II with principal dependence $(S, \ell - 1)$, we can assume that $S = K_1 = [2, \ell+1]$. By Theorem 4.3, the endomorphism $\omega^\ell_{\boldsymbol{\lambda}}(S, \ell - 1)$ of $A^\ell(\mathcal{G})$ is diagonalizable, with eigenvalues 0 and λ_S. The 0-eigenspace is spanned by $\{e_J e_L \mid |J| \leq \ell - 2\} \bigcup \{\eta_S e_J e_q \mid |J| = \ell - 2\}$, where $J \subset S$, $L \subset [n] \setminus S$, $q \notin S$, and the λ_S-eigenspace is spanned by $\{e_S\} \bigcup \{(\partial e_S) e_q \mid q \notin S\}$. Note that the λ_S-eigenspace of $\omega^\ell_{\boldsymbol{\lambda}}(S, \ell - 1)$ has dimension $n - \ell + 1$. Note also that the λ_S-eigenvectors $\partial e_K = e_S - e_1 \partial e_S$ and $e_{\boldsymbol{\lambda}} \partial e_S$ are annihilated by the projection $\rho \circ \pi$. On the other hand, it is readily checked that

$$\{\rho \circ \pi((\partial e_S) e_q) \mid \ell + 2 \leq q \leq n\} \tag{5.2}$$

is a linearly independent set of $(n-\ell-1)$ λ_S-eigenvectors for $\Omega_{\boldsymbol{\lambda}}(\mathcal{T}',\mathcal{T})$ in $H^\ell(\mathcal{T})$. Additionally, one can check that the set

$$\{\rho\circ\pi(e_J e_L) \mid J\subset S, |J|\le \ell-2\}\bigcup\{\rho\circ\pi(\eta_S e_J e_q) \mid J\subset[3,\ell+1], |J|=\ell-2\} \tag{5.3}$$

where $L\subset[n]\setminus S$ and $q\notin S$, is a linearly independent set of 0-eigenvectors for $\Omega_{\boldsymbol{\lambda}}(\mathcal{T}',\mathcal{T})$ in $H^\ell(\mathcal{T})$. Checking that the dimension of the subspace spanned by the vectors (5.3) is $\dim H^\ell(\mathcal{T})-(n-\ell-1)$, since eigenvectors associated to distinct eigenvalues are linearly independent, the vectors (5.2) and (5.3) form a basis for $H^\ell(\mathcal{T})$. Hence, the dimensions of the eigenspaces are as asserted for a degeneration of type II.

If $\mathcal{T}'$ is a degeneration of type III with principal dependence (S,ℓ), we can assume that $S=K\bigcup\{q\}$ for some $q\in[\ell+2,n]$. By Theorem 4.3, the endomorphism $\omega^\ell_{\boldsymbol{\lambda}}(S,\ell)$ of $A^\ell(\mathcal{G})$ is diagonalizable, with eigenvalues 0 and λ_S. The 0-eigenspace is spanned by $\{e_J e_L \mid |J|\le\ell-1\}\bigcup\{\eta_S e_J \mid |J|=\ell-2\}$, where $J\subset S$, $L\subset[n]\setminus S$, and the λ_S-eigenspace is spanned by $\{\partial e_J \mid J\subset S, |J|=\ell+1\}$. Note that the λ_S-eigenspace of $\omega^\ell_{\boldsymbol{\lambda}}(S,\ell)$ has dimension $\ell+1$. Note also that the λ_S-eigenvector ∂e_K is annihilated by the projection $\rho\circ\pi$. Recall that $F=[2,\ell+1]$. Let S_q denote the subspace of $H^\ell(\mathcal{T})$ spanned by $\{\xi_I \mid I\subset F\bigcup\{q\}\}$, and let $\mathsf{p}_q : H^\ell(\mathcal{T})\to S_q$ be the natural projection. For $J\subset F$, $|J|=\ell-1$, a calculation reveals that $\mathsf{p}_q\circ\rho\circ\pi(\eta_K e_J e_q)=\lambda_S(\lambda_2\cdots\lambda_{\ell+1}\lambda_q)^{-1}\xi_J\xi_q$. Consequently, the set $\{\rho\circ\pi(\eta_K e_J e_q) \mid J\subset F, |J|=\ell-1\}$ is a linearly independent set of ℓ λ_S-eigenvectors for $\Omega_{\boldsymbol{\lambda}}(\mathcal{T}',\mathcal{T})$ in $H^\ell(\mathcal{T})$. Check that the set $\{\rho\circ\pi(e_J e_L) \mid J\subset S_1, |J|\le\ell-1, L\subset[n]\setminus S\}$ is a linearly independent set of $\dim H^\ell(\mathcal{T})-\ell$ 0-eigenvectors for $\Omega_{\boldsymbol{\lambda}}(\mathcal{T}',\mathcal{T})$ in $H^\ell(\mathcal{T})$. It follows that the dimensions of the eigenspaces are as asserted for a degeneration of type III. □

5.9. **Further examples.** We present three examples of higher codimension.

Example 5.10. Let $\mathcal{S}$ be the combinatorial type of the Selberg arrangement $\mathcal{A}$ in $\mathbb{C}^2$ with defining polynomial $Q(\mathcal{A})=u_1u_2(u_1-1)(u_2-1)(u_1-u_2)$ depicted in Figure 2. See [1, 14, 10] for detailed studies of the Gauss-Manin connections arising in the context of Selberg arrangements.

FIGURE 2. A Selberg arrangement and one degeneration

Here $\mathrm{Dep}(\mathcal{S})^*=\{126,346,135,245\}$. Weights $\boldsymbol{\lambda}$ are $\mathcal{S}$-nonresonant if

$$\lambda_j\ (1\le j\le 6),\ \lambda_1+\lambda_2+\lambda_6,\ \lambda_1+\lambda_3+\lambda_5,\ \lambda_2+\lambda_4+\lambda_5,\ \lambda_3+\lambda_4+\lambda_6\notin\mathbb{Z}_{\ge 0}.$$

For $\mathcal{S}$-nonresonant weights, the β**nbc** basis for $H^2(\mathcal{S})$ is $\{\Xi_{2,4},\Xi_{2,5}\}$, where $\Xi_{2,j}=(\lambda_2a_2+\lambda_4a_4+\lambda_5a_5)\lambda_ja_j$, see Definition 4.5. Recall that $\lambda_J=\sum_{j\in J}\lambda_j$. The projection map $\tau : H^2(\mathcal{G}) \to H^2(\mathcal{S})$ is given by

$$\tau(\xi_{i,j}) = \begin{cases} -\Xi_{2,4} - \Xi_{2,5} & \text{if } (i,j) = (2,3), \\ (\lambda_{2,4}\Xi_{2,4} + \lambda_4\Xi_{2,5})/\lambda_{2,4,5} & \text{if } (i,j) = (2,4), \\ (\lambda_5\Xi_{2,4} + \lambda_{2,5}\Xi_{2,5})/\lambda_{2,4,5} & \text{if } (i,j) = (2,5), \\ 0 & \text{if } (i,j) = (3,4), \\ (-\lambda_5\Xi_{2,4} - \lambda_{3,5}\Xi_{2,5})/\lambda_{1,3,5} & \text{if } (i,j) = (3,5), \\ (-\lambda_5\Xi_{2,4} + \lambda_4\Xi_{2,5})/\lambda_{2,4,5} & \text{if } (i,j) = (4,5). \end{cases}$$

The arrangement $\mathcal{A}'$ in Figure 2 represents one degeneration type $\mathcal{S}'$ of $\mathcal{S}$. Here $\mathrm{Dep}(\mathcal{S}',\mathcal{S})^* = \{34, 35, 45, 134, 145, 234, 235, 345, 356, 456\}$. The sets 34, 35, 45, and 345 have $r = 1$, and the others $r = 2$. The principal dependence is (S, r), where $S = 345$ and $r = 1$. For $\mathcal{S}$-nonresonant weights with $\lambda_S \neq 0$, $\Omega_{\boldsymbol\lambda}(\mathcal{S}',\mathcal{S})$ is diagonalizable, with spectrum contained in $\{0, \lambda_S\}$ by Theorem 5.2. The projection τ annihilates the 0-eigenspace of $\Omega_{\boldsymbol\lambda}(S, r)$, and restricts to a surjection $E(\lambda_S) \twoheadrightarrow H^2(\mathcal{S})$, where $E(\lambda_S)$ is the λ_S-eigenspace of $\Omega_{\boldsymbol\lambda}(S, r)$, see Example 4.7. It follows that $\Omega_{\boldsymbol\lambda}(\mathcal{S}',\mathcal{S})$ has eigenvalues λ_S, λ_S. Note that 0 is not an eigenvalue of $\Omega_{\boldsymbol\lambda}(\mathcal{S}',\mathcal{S})$ in this instance.

Although the eigenvalues are determined by the principal dependence (S, r), the same principal dependence may occur for degenerations of different types. Thus the multiplicities of the eigenvalues depend on the combinatorial types as well.

Example 5.11. Consider the arrangement $\bar{\mathcal{A}}$ of type $\mathcal{T}$ obtained from the arrangement $\mathcal{A}$ in Example 5.10 by rotating line 1 by a (small) angle about the triple point 135, see Figure 3. Here, lines 1 and 2 meet in affine space, so 126 is no longer dependent. This change implies that $\dim A^2(\mathcal{T}) = 7$ and $\dim H^2(\mathcal{T}) = 3$.

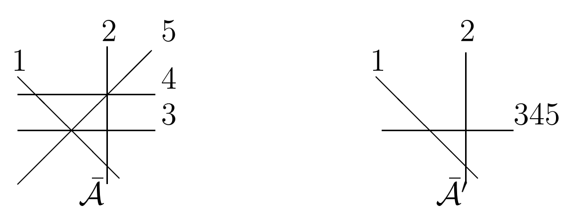

Figure 3. A line arrangement and one degeneration

Weights $\boldsymbol\lambda$ are $\mathcal{T}$-nonresonant if

$$\lambda_j\ (1 \leq j \leq 6),\ \lambda_1 + \lambda_3 + \lambda_5,\ \lambda_2 + \lambda_4 + \lambda_5,\ \lambda_3 + \lambda_4 + \lambda_6 \notin \mathbb{Z}_{\geq 0}.$$

For $\mathcal{T}$-nonresonant weights, the $\beta\mathbf{nbc}$ basis for $H^2(\mathcal{T})$ is $\{\Xi_{2,3}, \Xi_{2,4}, \Xi_{2,5}\}$, where $\Xi_{2,3} = \lambda_2\lambda_3 a_{2,3}$ and $\Xi_{2,j} = (\lambda_2 a_2 + \lambda_4 a_4 + \lambda_5 a_5)\lambda_j a_j$ for $j = 4, 5$. The projection map

$\tau : H^2(\mathcal{G}) \to H^2(\mathcal{T})$ is given by

$$\tau(\xi_{i,j}) = \begin{cases} \Xi_{2,3} & \text{if } (i,j) = (2,3), \\ (\lambda_{2,4}\Xi_{2,4} + \lambda_4\Xi_{2,5})/\lambda_{2,4,5} & \text{if } (i,j) = (2,4), \\ (\lambda_5\Xi_{2,4} + \lambda_{2,5}\Xi_{2,5})/\lambda_{2,4,5} & \text{if } (i,j) = (2,5), \\ 0 & \text{if } (i,j) = (3,4), \\ (\lambda_5\Xi_{2,3} - \lambda_3\Xi_{2,5})/\lambda_{1,3,5} & \text{if } (i,j) = (3,5), \\ (-\lambda_5\Xi_{2,4} + \lambda_4\Xi_{2,5})/\lambda_{2,4,5} & \text{if } (i,j) = (4,5). \end{cases}$$

The combinatorial type $\mathcal{T}$ has a degeneration of type $\mathcal{T}'$ similar to $\mathcal{S}'$, represented by the arrangement $\bar{\mathcal{A}}'$ in Figure 3. As in Example 5.10, the principal dependence is (S, r), where $S = 345$ and $r = 1$. For $\mathcal{T}$-nonresonant weights with $\lambda_S \neq 0$, the spectrum of $\Omega_{\boldsymbol{\lambda}}(\mathcal{T}', \mathcal{T})$ is contained in $\{0, \lambda_S\}$. Calculations with the projection τ and the eigenspace decomposition of the endomorphism $\Omega_{\boldsymbol{\lambda}}(S, r)$ of $H^2(\mathcal{G})$ given in Example 4.7 reveal that $\Omega_{\boldsymbol{\lambda}}(\mathcal{T}', \mathcal{T})$ has eigenvalues $\lambda_S, \lambda_S, 0$.

Example 5.12. The combinatorial type $\mathcal{S}$ in Example 5.10 is a degeneration of the type $\mathcal{T}$ in Example 5.11. The principal dependence of this degeneration is (S, r), where $S = 126$ and $r = 2$. For $\mathcal{T}$-nonresonant weights with $\lambda_S \neq 0$, the spectrum of $\Omega_{\boldsymbol{\lambda}}(\mathcal{S}, \mathcal{T})$ is contained in $\{0, \lambda_S\}$. A calculation shows that the eigenvalues are $\lambda_S, 0, 0$. It is interesting to note that $\lambda_S = \lambda_{1,2,6} = -\lambda_{3,4,5}$.

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